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Formal Local Cohomology Modules and Serre Subcategories

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Abstract

Let (R, \mathbf{m}) be a Noetherian local ring, \mathbf{a} an ideal of R and M a finitely generated Rmodule. We investigate some properties of formal local cohomology modules with respect to a Serre subcategory. We provide a common language to indicate some properties of formal local cohomology modules.

1. Introduction

Throughout this paper (R, \mathbf{m}) is a commutative Noetherian local ring, \mathbf{a} an ideal of R and M is a finitely generated R-module. For an integer $i \in \mathbb{N}_0$, $H_a^i(N)$ denotes the ith local cohomology module of M with respect to a as introduced by Grothendieck (cf. [1], [2].

We shall consider the family of local cohomology modules $\{H_m^i\left(\frac{M}{a^nM}\right)\}_{n\in\mathbb{N}}$ for a non-negative integer $i \in \mathbb{N}_0$. With natural homomorphisms; this family forms an inverse system. Schenzel introduced the i-th formal local cohomology of M with respect to ain the form of $f_a^i(M) \coloneqq \frac{\lim}{n \in \mathbb{N}} H_m^i\left(\frac{M}{a^n M}\right)$, which is the *i-th* cohomology module of the **a**adic completion of the Čech complex $\check{c}_x \otimes_R M$, where \underline{x} denotes a system of elements of R such that $Rad(\underline{x}, R) = m$ (see [3, Definition 3.1]). He defines the formal grade as $f.grade(\boldsymbol{a},M) = \inf \{i \in \mathbb{N}_0 \mid f_{\boldsymbol{a}}^i(M) \neq 0\}$. For any ideal \boldsymbol{a} of R and finitely generated *R*- module *M* the following statements hold:

(i) (See [3, Theorem 3.11]). If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of finitely generated R- modules, then there is the following long exact sequence:

$$\cdots \to f_a^i(M') \to f_a^i(M) \to f_a^i(M'') \to \cdots$$

 $\cdots \to f_a^i(M') \to f_a^i(M) \to f_a^i(M'') \to \cdots$. **Keywords:** Local cohomology, Formal local cohomology, Serre subcategory, Formal grade, Formal cohomological dimension.

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*Corresponding author: taheri@khu.ac.ir (ii) (See [3, Theorem 1.3]). $f. grade(\mathbf{a}, M) \le \dim(M) - cd(\mathbf{a}, M)$; some properties of formal local cohomology have been presented in [3, 4, 5 and 6].

Throughout this paper S denotes a Serre subcategory of the category of R- modules and R – homomorphisms (we recall that a class S of R- modules is a Serre subcategory of the category of R- modules and R-homomorphisms if S is closed under taking submodules, quotients and extensions).

Our paper contains three sections. In Section 2, we shall define the formal grade of \boldsymbol{a} with respect to M in \mathcal{S} as the infimum of the integers i such that $f_a^i(M) \notin \mathcal{S}$ and is denoted by $f.grade_{\mathcal{S}}(\boldsymbol{a},M)$. (See definition 2.1). Then we shall obtain some properties of this notion. We show that if $\Gamma_a(M)$ is a pure submodule of M, then $Hom_R(\frac{R}{m},f_a^t(\Gamma_a(M)))$ and $Hom_R(\frac{R}{m},f_a^{t-1}(\frac{M}{\Gamma_a(M)}))$ belong to \mathcal{S} ,where $t=f.grade_{\mathcal{S}}(\boldsymbol{a},M)$.

In Section 3, we shall define the formal cohomological dimension of \boldsymbol{a} with respect to M in \mathcal{S} as the supremum of the integers i such that $f_a^i(M) \notin \mathcal{S}$ and is denoted by $f.cd_{\mathcal{S}}(\boldsymbol{a},M)$. (See definition 3.1). The main result of this section is that if $f_a^i(M) \in \mathcal{S}$ and $H_m^i(M) \in \mathcal{S}$ for all i > t, then $\frac{R}{a} \bigotimes_R f_a^t(M)$ belongs to \mathcal{S} .

2. The formal grade of a module in a Serre subcategory

Definition 2.1. The formal grade of a with respect to M in S is the infimum of the integers i such that $f_a^i(M) \notin S$ and is denoted by $f.grade_S(a, M)$.

Proposition 2.2. Let (R, m) be a local ring and a be an ideal of R. If $0 \to L \to M \to N \to 0$ is an exact sequence of finitely generated R-modules, then the following statements hold.

- (a) $f. grade_{\mathcal{S}}(\mathbf{a}, M) \ge \min\{f. grade_{\mathcal{S}}(\mathbf{a}, L), f. grade_{\mathcal{S}}(\mathbf{a}, N)\}$.
- (b) $f. grade_{\mathcal{S}}(\mathbf{a}, L) \ge \min\{f. grade_{\mathcal{S}}(\mathbf{a}, M), f. grade_{\mathcal{S}}(\mathbf{a}, N) + 1\}.$
- (c) $f. grade_{\mathcal{S}}(\mathbf{a}, N) \ge \min\{f. grade_{\mathcal{S}}(\mathbf{a}, L) 1, f. grade_{\mathcal{S}}(\mathbf{a}, M)\}$

Proof. According to [3, Theorem 3.11], the above short exact sequence induces the following long exact sequence.

$$\cdots \to f_a^{i-1}(N) \to f_a^i(L) \to f_a^i(M) \to f_a^i(N) \to f_a^{i+1}(L) \to \cdots.$$

So, the result follows.

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Corollary 2.3. If $\underline{x} = x_1, ..., x_n$ is a regular M-sequence, then $f. grade_{\mathcal{S}}\left(\boldsymbol{a}, \frac{M}{\underline{x}M}\right) \geq f. grade_{\mathcal{S}}\left(\boldsymbol{a}, M\right) - n$.

Proof. Consider the following exact sequence $(n \in \mathbb{N})$

$$0 \to \frac{M}{(x_1, \dots, x_{n-1})M} \xrightarrow{x_n} \frac{M}{(x_1, \dots, x_{n-1})M} \xrightarrow{nat.} \frac{M}{(x_1, \dots, x_n)M} \to 0$$

whenever n = 1 by $(x_1, ..., x_{n-1})M$ we means 0.

Corollary 2.4. Let a and b be ideals of R. Then

- (a) $f. grade_{\mathcal{S}}(\mathbf{a} \cap \mathbf{b}, M) \ge \min\{f. grade_{\mathcal{S}}(\mathbf{a}, M), f. grade_{\mathcal{S}}(\mathbf{b}, M), f. grade_{\mathcal{S}}((\mathbf{a}, \mathbf{b}), M) + 1\}\}.$
- (b) $f. grade_{\mathcal{S}}((\boldsymbol{a}, \boldsymbol{b}), M) \ge \min\{f. grade_{\mathcal{S}}(\boldsymbol{a} \cap \boldsymbol{b}, M) 1, f. grade_{\mathcal{S}}(\boldsymbol{a}, M), f. grade_{\mathcal{S}}(\boldsymbol{b}, M)\}.$

Proof. For all $n \in \mathbb{N}$ there is a short exact sequence as follows:

$$0 \to \frac{M}{\boldsymbol{a}^n M \cap \boldsymbol{b}^n M} \to \frac{M}{\boldsymbol{a}^n M} \oplus \frac{M}{\boldsymbol{b}^n M} \to \frac{M}{(\boldsymbol{a}^n, \boldsymbol{b}^n) M} \to 0.$$

By using [3,Theorem 5.1], the above exact sequence induces the following long exact sequence.

$$\cdots \to \frac{\lim}{n \in \mathbb{N}} H^{i}_{\boldsymbol{m}} \left(\frac{M}{(\boldsymbol{a} \cap \boldsymbol{b})^{n} M} \right) \to \frac{\lim}{n \in \mathbb{N}} H^{i}_{\boldsymbol{m}} \left(\frac{M}{\boldsymbol{a}^{n} M} \right) \bigoplus \frac{\lim}{n \in \mathbb{N}} H^{i}_{\boldsymbol{m}} \left(\frac{M}{\boldsymbol{b}^{n} M} \right) \to \frac{\lim}{n \in \mathbb{N}} H^{i}_{\boldsymbol{m}} \left(\frac{M}{(\boldsymbol{a}, \boldsymbol{b})^{n} M} \right) \to \cdots$$

So by using an argument similar to that of Proposition 2.2, the result follows.

Corollary 2.5. Assume that M is a finitely generated R- module and N_1 and N_2 are submodules of M. Then considering the exact sequence $0 \to \frac{M}{N_1 \cap N_2} \to \frac{M}{N_1 \cap N_2}$

$$\frac{M}{N_1} \oplus \frac{M}{N_2} \to \frac{M}{N_1 + N_2} \to 0$$
 we shall have

(a)
$$f. grade_{\mathcal{S}}\left(\boldsymbol{a}, \frac{M}{N_1 \cap N_2}\right) \ge \min\{f. grade_{\mathcal{S}}\left(\boldsymbol{a}, \frac{M}{N_1}\right), f. grade_{\mathcal{S}}\left(\boldsymbol{a}, \frac{M}{N_1}\right)\}$$

MN2, f.gradeS **a**, MN1+N2+1}.

(b)
$$f. \operatorname{grade}_{\mathcal{S}}\left(\boldsymbol{a}, \frac{M}{N_1 + N_2}\right) \geq \min\left\{f. \operatorname{grade}_{\mathcal{S}}\left(\frac{M}{N_1 \cap N_2}\right) - 1, f. \operatorname{grade}_{\mathcal{S}}\left(\boldsymbol{a}, \frac{M}{N_1}\right)\right\}$$

$$MN1, f. \operatorname{grade}_{\mathcal{S}}\boldsymbol{a}, MN2.$$

Theorem 2.6. Let \boldsymbol{a} be an ideal of a local ring (R, \boldsymbol{m}) , M be a finitely generated R-module and L be a pure submodule of M. Then $f.grade_{\mathcal{S}}(\boldsymbol{a},L) \geq f.grade_{\mathcal{S}}(\boldsymbol{a},M)$ where \mathcal{S} is a Serre subcategory of the category of R-modules and R-homomorphisms. In particular, inf $\{i|H^i_{\boldsymbol{m}}(L) \notin \mathcal{S}\} \geq \inf\{i|H^i_{\boldsymbol{m}}(M) \notin \mathcal{S}\}$.

Proof. Let L be a pure submodule of M. So $\frac{L}{a^{n_L}} \to \frac{M}{a^{n_M}}$ is pure for each $n \in \mathbb{N}$. Now according to [8, Corollary 3.2 (a)], $H^i_m\left(\frac{L}{a^{n_L}}\right) \to H^i_m\left(\frac{M}{a^{n_M}}\right)$ is injective. Since inverse limit is a left exact functor, $f^i_a(L)$ is isomorphic to a submodule of $f^i_a(M)$. Consequently, $f. \operatorname{grade}_{\mathcal{S}}(a, L) \geq f. \operatorname{grade}_{\mathcal{S}}(a, M)$. If a = 0 then, $f. \operatorname{grade}_{\mathcal{S}}(0, M) = \inf \{i | H^i_m(M) \notin \mathcal{S}\}$ and the result follows.

Corollary 2.7. If $0 \to L \to M \to N \to 0$ is a pure exact sequence of finitely generated R-modules, then min $\{f. grade_S(\boldsymbol{a}, L), f. grade_S(\boldsymbol{a}, N) + 1\} \ge f. grade_S(\boldsymbol{a}, M)$.

Proof. Since L is a pure submodules of M, as a result of the previous theorem, $f.grade_{\mathcal{S}}(\boldsymbol{a},L) \geq f.grade_{\mathcal{S}}(\boldsymbol{a},M)$. Hence we must prove that $f.grade_{\mathcal{S}}(\boldsymbol{a},N) + 1 \geq f.grade_{\mathcal{S}}(\boldsymbol{a},M)$. We assume that $i < f.grade_{\mathcal{S}}(\boldsymbol{a},M)$ and we show that $i < f.grade_{\mathcal{S}}(\boldsymbol{a},N) + 1$. Consider the following long exact sequence.

$$\cdots \to f_a^{i-1}(M) \to f_a^{i-1}(N) \to f_a^i(L) \to f_a^i(M) \to f_a^i(N) \to \cdots . \ (**)$$

If $i < f. grade_{\mathcal{S}}(\boldsymbol{a}, M)$, then $f_a^0(M), f_a^1(M), ..., f_a^{i-1}(M), f_a^i(M) \in \mathcal{S}$. On the other hand, since $i < f. grade_{\mathcal{S}}(\boldsymbol{a}, M) \leq f. grade_{\mathcal{S}}(\boldsymbol{a}, L), f_a^0(L), ..., f_a^i(L) \in \mathcal{S}$. Hence, it follows from (**) that $f_a^0(N), ..., f_a^{i-1}(N) \in \mathcal{S}$ and so $i-1 < f. grade_{\mathcal{S}}(\boldsymbol{a}, N)$.

Theorem 2.8. Let (R, \mathbf{m}) be a local ring, \mathbf{a} be an ideal of R, S be a Serre subcategory of the category of R-modules and R_homomorphisms and $M \in S$ be a finitely generated R-module such that $\Gamma_{\mathbf{a}}(M)$ is a pure submodule of M. Then $Hom_R\left(\frac{R}{\mathbf{a}}, f_a^t(\Gamma_{\mathbf{a}}(M))\right) \in S$, where $t = f.grade_S(\mathbf{a}, M)$.

Proof. Due to the previous theorem, $f.grade_{\mathcal{S}}(\boldsymbol{a},\Gamma_{\boldsymbol{a}}(M)) \geq f.grade_{\mathcal{S}}(\boldsymbol{a},M)$. If $f.grade_{\mathcal{S}}(\boldsymbol{a},\Gamma_{\boldsymbol{a}}(M)) > f.grade_{\mathcal{S}}(\boldsymbol{a},M)$, then the result is obvious. Accordingly, we assume that $f.grade_{\mathcal{S}}(\boldsymbol{a},\Gamma_{\boldsymbol{a}}(M)) = f.grade_{\mathcal{S}}(\boldsymbol{a},M)$. We know that $Supp(\Gamma_{\boldsymbol{a}}(M)) \subseteq Var(\boldsymbol{a})$. By using [4, Lemma 2.3], $f_{\boldsymbol{a}}^i(\Gamma_{\boldsymbol{a}}(M)) \cong H_{\boldsymbol{m}}^i(\Gamma_{\boldsymbol{a}}(M))$ for all $i \geq 0$. So, if $j < f.grade_{\mathcal{S}}(\boldsymbol{a},M)$, then $f_{\boldsymbol{a}}^j(\Gamma_{\boldsymbol{a}}(M)) \cong H_{\boldsymbol{m}}^j(\Gamma_{\boldsymbol{a}}(M)) \in \mathcal{S}$ and $Ext_R^k(\frac{R}{m},H_{\boldsymbol{m}}^j(\Gamma_{\boldsymbol{a}}(M)) \in \mathcal{S}$ for all $k \geq 0$ and $j < f.grade_{\mathcal{S}}(\boldsymbol{a},M)$. Moreover $Ext_R^t(\frac{R}{m},\Gamma_{\boldsymbol{a}}(M)) \in \mathcal{S}$, because $\Gamma_{\boldsymbol{a}}(M) \in \mathcal{S}$. Consequently, according to [7, Theorem 2.2],

$$Hom_R(\frac{R}{m}, H_m^t(\Gamma_a(M)) \in \mathcal{S}, \text{ where } t = f. grade_{\mathcal{S}}(a, M).$$

Corollary 2.9 With the same notations as Theorem 2.8, let $X \in \mathcal{S}$ be a submodule of $f_a^t(\Gamma_a(M))$, where $t = f. grade_{\mathcal{S}}(a, M)$. Then $Hom_R(\frac{R}{m}, \frac{f_a^t(\Gamma_a(M))}{X}) \in \mathcal{S}$.

Proof. Consider the long exact sequence:

 $Hom_R\left(\frac{R}{m}, f_a^t(\Gamma_a(M))\right) \to Hom_R\left(\frac{R}{m}, \frac{f_a^t(\Gamma_a(M))}{X}\right) \to Ext_R^1\left(\frac{R}{m}, X\right).$ (*)

In accordance with the previous theorem $Hom_R(\frac{R}{m}, f_a^t(\Gamma_a(M))) \in \mathcal{S}$. Moreover $Ext_R^1(\frac{R}{m}, X) \in \mathcal{S}$. It follows from the exact sequence (*) that $Hom_R(\frac{R}{m}, \frac{f_a^t(\Gamma_a(M))}{X}) \in \mathcal{S}$.

Theorem 2.10. Suppose that \boldsymbol{a} is an ideal of (R, \boldsymbol{m}) and $M \in \mathcal{S}$ is a finitely generated R-module such that $\Gamma_{\boldsymbol{a}}(M)$ is a pure submodule of M. Then $Hom_R\left(\frac{R}{\boldsymbol{m}}, f_{\boldsymbol{a}}^{t-1}\left(\frac{M}{\Gamma_{\boldsymbol{a}}(M)}\right)\right) \in \mathcal{S}$, where $t = f. \ grade_{\mathcal{S}}(\boldsymbol{a}, M)$.

Proof. One has $f.grade_{\mathcal{S}}(\boldsymbol{a}, \Gamma_{\boldsymbol{a}}(M)) \geq f.grade_{\mathcal{S}}(\boldsymbol{a}, M)$, by Theorem 2.6. Now, the exact sequence $0 \to \Gamma_{\boldsymbol{a}}(M) \to M \to \frac{M}{\Gamma_{\boldsymbol{a}}(M)} \to 0$ induces the following long exact sequence:

$$\cdots \xrightarrow{\alpha} f_a^{t-1}(\Gamma_a(M)) \xrightarrow{\beta} f_a^{t-1}(M) \xrightarrow{\gamma} f_a^{t-1}(\frac{M}{\Gamma_a(M)}) \xrightarrow{\xi} f_a^t(\Gamma_a(M)) \xrightarrow{\varphi} \cdots (*)$$

Using the exact sequence (*), we obtain the short exact sequence $0 \to \operatorname{Im}(\beta) \to f_a^{t-1}(M) \to \operatorname{Im}(\gamma) \to 0$. Since $f_a^{t-1}(M) \in \mathcal{S}$, $\operatorname{Im}(\beta) \in \mathcal{S}$ and $\operatorname{Im}(\gamma) \in \mathcal{S}$. Furthermore, we have the exact sequence $0 \to \operatorname{Im}(\xi) \to H_m^t(\Gamma_a(M)) \to \operatorname{Im}(\varphi) \to 0$ which induces the following long exact sequence:

$$0 \to Hom_R(\frac{R}{m}, Im(\xi)) \to Hom_R(\frac{R}{m}, H^t_m(\Gamma_a(M))) \to \cdots$$

Thus $Hom_R(\frac{R}{m}, Im(\xi)) \in \mathcal{S}$. Finally, by considering the short exact sequence $0 \to Im(\gamma) \to f_a^{t-1}\left(\frac{M}{\Gamma_a(M)}\right) \to Im(\xi) \to 0$ we can conclude that $Hom_R\left(\frac{R}{m}, f_a^{t-1}\left(\frac{M}{\Gamma_a(M)}\right)\right) \in \mathcal{S}$.

Theorem 2.11. Suppose that R is complete with respect to the <u>a</u>-adic topology and $M \in \mathcal{S}$ be a finitely generated R-module and t a positive integer such that $f_a^i(M) \in \mathcal{S}$ for all i < t. Then $Hom_R\left(\frac{R}{m}, f_a^t(M)\right) \in \mathcal{S}$.

Proof.We use induction on t. Let t=0. Consider the following isomorphisms.

$$Hom_{R}(\frac{R}{\underline{m}}, f_{\underline{a}}^{0}(M)) \cong \lim_{\stackrel{\longleftarrow}{n \in \mathbb{N}}} Hom_{R}(\frac{R}{\underline{m}}, H_{\underline{m}}^{0}(\frac{M}{\underline{a}^{n}M})) \cong \lim_{\stackrel{\longleftarrow}{n \in \mathbb{N}}} Hom_{R}(\frac{R}{\underline{m}}, \frac{M}{\underline{a}^{n}M})$$

$$\cong Hom_{R}(\frac{R}{\underline{m}}, \lim_{\stackrel{\longleftarrow}{n \in \mathbb{N}}} (\frac{M}{\underline{a}^{n}M})) \cong Hom_{R}(\frac{R}{\underline{m}}, \hat{M}^{\underline{a}}) \cong Hom_{R}(\frac{R}{\underline{m}}, M)$$

It is clear that $Hom_R(\frac{R}{\underline{m}}, M) \in S$. So by the above isomorphisms, we deduce that $Hom_R(\frac{R}{\underline{m}}, f_{\underline{a}}^{\,0}(M)) \in S$.

Suppose that t>0 and the result is true for all integer i less than t. Set $N:=\Gamma_{\mathbf{m}}(M)$. Then $f_a^i(M)\cong f_a^i\left(\frac{M}{N}\right)$ for all i>0, and so we may assume that $depth_R(M)>0$. There is an M-regular element $x\in \mathbf{m}$. The exact sequence $0\to M\xrightarrow{x} M\to 0$ induces the following long exact sequence:

$$\cdots \to f_a^{t-2}(M) \xrightarrow{x} f_a^{t-2}(M) \xrightarrow{f} f_a^{t-2}\left(\frac{M}{xM}\right)$$

$$\to f_a^{t-1}(M) \xrightarrow{x} f_a^{t-1}(M) \xrightarrow{g} f_a^{t-1}\left(\frac{M}{xM}\right)$$

$$\to f_a^t(M) \xrightarrow{x} f_a^t(M) \xrightarrow{h} \cdots (*)$$

Using the exact sequence (*) we obtain the short exact sequence

$$0 \to \frac{f_a^{t-1}(M)}{xf_a^{t-1}(M)} \to f_a^{t-1}\left(\frac{M}{xM}\right) \to \left(0:x\right) \to 0.$$

Now, this exact sequence induces the following long exact sequence:

$$0 \to Hom_{R}\left(\frac{R}{m}, \frac{f_{a}^{t-1}(M)}{xf_{a}^{t-1}(M)}\right) \to Hom_{R}\left(\frac{R}{m}, f_{a}^{t-1}\left(\frac{M}{xM}\right)\right) \to Hom_{R}\left(\frac{R}{m}, \left(0:x\right)\right) \to Ext_{R}^{1}\left(\frac{R}{m}, \frac{f_{a}^{t-1}(M)}{xf_{a}^{t-1}(M)}\right) \to \cdots . (**)$$

By using (*), $f_a^i\left(\frac{M}{xM}\right) \in \mathcal{S}$ for all i < t-1. Therefore by the induction hypothesis $Hom_R\left(\frac{R}{m}, f_a^{t-1}\left(\frac{M}{xM}\right)\right) \in \mathcal{S}$. Furthermore $Ext_R^1\left(\frac{R}{m}, \frac{f_a^{t-1}(M)}{xf_a^{t-1}(M)}\right) \in \mathcal{S}$ because $f_a^{t-1}(M) \in \mathcal{S}$. Thus in accordance with (**), $Hom_R\left(\frac{R}{m}, (0:x)\right) \in \mathcal{S}$. Since $x \in m$ according to [9,10.86] we have the following isomorphisms.

$$Hom_R\left(\frac{R}{m}, (0:x)\right) \cong Hom_R\left(\frac{R}{m}, Hom_R\left(\frac{R}{xR}, f_a^t(M)\right)\right) \cong Hom_R\left(\frac{R}{m} \otimes_R \frac{R}{xR}, f_a^t(M)\right) \cong Hom_R\left(\frac{R}{m}, f_a^t(M)\right).$$

Consequently $Hom_R\left(\frac{R}{m}, f_a^t(M)\right) \in \mathcal{S}$.

3. The formal cohomological dimension in a Serre subcategory

We recall from [3,Theorem 1.1] that for a finitely generated R-module M, $\sup\{i \in \mathbb{N}_0 \mid f_a^i(M) \neq 0\} = \dim\left(\frac{M}{aM}\right)$.

Definition 3.1. The formal cohomological dimension of M with respect to \underline{a} in S is The supremum of the integers i such that $f_a^i(M) \notin S$ and is denoted by $f \cdot cd_S(a, M)$.

Theorem 3.2. Suppose that S is a Serre subcategory of the category of R- modules and R – homomorphisms and L and N are two finitely generated R- modules such that $Supp_R(L) \subseteq Supp_R(N)$. Then $f \cdot cd_S(\boldsymbol{a}, L) \leq f \cdot cd_S(\boldsymbol{a}, N)$.

Proof. It is enough to prove that $f_a^i(L) \in \mathcal{S}$ for all $i > f.cd_{\mathcal{S}}(\boldsymbol{a},N)$ and all finitely generated R- module L such that $Supp_R(L) \subseteq Supp_R(N)$. We use descending induction on i.For all $i > \dim(\frac{L}{aL}) + f.cd_{\mathcal{S}}(\boldsymbol{a},N)$, $f_a^i(L) = 0 \in \mathcal{S}$. Let $i > f.cd_{\mathcal{S}}(\boldsymbol{a},N)$ and the result is proved for i + l. By Gruson's theorem, there is a chain $0 = L_0 \subset L_l \subset \cdots \subset L_l = L$ of submodules of L such that $\frac{L_i}{L_{i-1}}$ is a homomorfic image of a direct sum of finitely many copies of N. Consider the exact sequence $0 \to L_{i-1} \to L_i \to \frac{L_i}{L_{i-1}} \to 0$ (i = 0, l, ..., l). We may assume that l = l. The exact sequence $0 \to K \to \bigoplus_{j=1}^t N \to L \to 0$ where K is a finitely generated R- module iduces the following long exact sequence:

$$\cdots \to f_a^i \bigl(\bigoplus_{j=1}^t N \bigr) \to f_a^i(L) \to f_a^{i+1}(K) \to \cdots . \, (*)$$

Based on the induction hypothesis $f_a^{i+1}(K) \in \mathcal{S}$. Moreover $f_a^i(\bigoplus_{j=1}^t N) = \bigoplus_{j=1}^t f_a^i(N) \in \mathcal{S}$ for all i > f. $cd_{\mathcal{S}}(a, N)$. Hence it follows from the exact sequence (*) that $f_a^i(L) \in \mathcal{S}$.

The next example shows that even if $Supp_R(M) = Supp_R(N)$, then it may not true that $f.grade_S(\boldsymbol{a}, M) = f.grade_S(\boldsymbol{a}, N)$.

Example 3.3. (See [4, Example 4.3 (i)]) Let (R, \mathbf{m}) be a 2 dimensional complete regular local ring, S = 0 and \mathbf{a} be an ideal of R with $\dim \left(\frac{R}{a}\right) = I$. Then by using [5,Theorem 1.1], $f.\operatorname{grade}_{S}(\mathbf{a},R) = 1$ and $f.\operatorname{grade}_{S}\left(\mathbf{a},\frac{R}{m}\right) = 0$. Set $M := R \oplus \frac{R}{m}$.

Then $Supp_R(M) = Supp_R(R)$. But

$$f. grade_{\mathcal{S}}(\boldsymbol{a}, M) = \inf \left\{ f. grade_{\mathcal{S}}(\boldsymbol{a}, R), f. grade_{\mathcal{S}}(\boldsymbol{a}, \frac{R}{m}) \right\} = 0.$$

Corollary3.4. For all $x \in a$, $f.cd_{\mathcal{S}}(a, M) \ge f.cd_{\mathcal{S}}(a, \frac{M}{xM})$.

Corollary3.5. Suppose that $0 \to L \to M \to N \to 0$ is an exact sequence of finitely generated R- modules. Then $f \cdot cd_{\mathcal{S}}(\boldsymbol{a}, M) = \max\{f \cdot cd_{\mathcal{S}}(\boldsymbol{a}, L), f \cdot cd_{\mathcal{S}}(\boldsymbol{a}, N)\}$.

Proof. Since $Supp_R(M) = Supp_R(L) \cup Supp_R(N)$ by referring to Theorem 3.2 we deduce that $f.cd_{\mathcal{S}}(\boldsymbol{a}, M) \geq f.cd_{\mathcal{S}}(\boldsymbol{a}, L)$ and $f.cd_{\mathcal{S}}(\boldsymbol{a}, M) \geq f.cd_{\mathcal{S}}(\boldsymbol{a}, N)$. Therefore $f.cd_{\mathcal{S}}(\boldsymbol{a}, M) \geq max\{f.cd_{\mathcal{S}}(\boldsymbol{a}, L), f.cd_{\mathcal{S}}(\boldsymbol{a}, N)\}$.

Next we prove that $max \{f. cd_{\mathcal{S}}(\boldsymbol{a}, L), f. cd_{\mathcal{S}}(\boldsymbol{a}, N)\} \geq f. cd_{\mathcal{S}}(\boldsymbol{a}, M)$.

Let $i > max \{f. cd_{\mathcal{S}}(\boldsymbol{a}, L), f. cd_{\mathcal{S}}(\boldsymbol{a}, N)\}$. Then $f_a^i(N), f_a^i(L) \in \mathcal{S}$ and from the exact sequence $f_a^i(L) \to f_a^i(M) \to f_a^i(N)$ we conclude that $f_a^i(M) \in \mathcal{S}$. Thus,

 $max\{f.cd_{\mathcal{S}}(\boldsymbol{a},L),f.cd_{\mathcal{S}}(\boldsymbol{a},N)\} \geq f.cd_{\mathcal{S}}(\boldsymbol{a},M).$

We recall that the cohomological dimension of an R-module M with respect to an ideal α of R in S is defind as

$$cd_{\mathcal{S}}(\boldsymbol{a}, M) := \sup \{i \in \mathbb{N}_{0} | H_{\boldsymbol{a}}^{i}(M) \notin \mathcal{S} \}.$$

The following lemma shows that when we considering the Artinianness of $f_a^i(M)$, we can assume that M is a-torsion-free.

Lemma 3.6. Suppose that a is an ideal of a local ring (R, m) and t be a non-negative integer. If $H_m^i(M) \in S$ for all $i \ge t$, then the following are equivalent:

(a)
$$f_a^i(M) \in \mathcal{S} \text{ for all } i \geq t.$$

(b)
$$f_a^i\left(\frac{M}{\Gamma_a(M)}\right) \in \mathcal{S} \text{ for all } i \geq t.$$

Proof. According to the hypothesis $t > cd_{\mathcal{S}}(\boldsymbol{m}, M)$. On the other hand $Supp_{R}(\Gamma_{\boldsymbol{a}}(M)) \subseteq Supp_{R}(M)$. So by referring to [7,Theorem 3.5], $cd_{\mathcal{S}}(\boldsymbol{m}, \Gamma_{\boldsymbol{a}}(M)) \leq cd_{\mathcal{S}}(\boldsymbol{m}, M)$. Thus, $t > cd_{\mathcal{S}}(\boldsymbol{m}, \Gamma_{\boldsymbol{a}}(M))$ and $H^{i}_{\boldsymbol{m}}(\Gamma_{\boldsymbol{a}}(M)) \in \mathcal{S}$ for all $i \geq t$. Now, consider the following long exact sequence:

$$\cdots \to f_a^i(\Gamma_a(M)) \to f_a^i(M) \to f_a^i(\frac{M}{\Gamma_a(M)}) \to f_a^{i+l}(\Gamma_a(M)) \to \cdots (*)$$

According to [4,Lemma 2.3] $f_a^i(\Gamma_a(M)) \cong H_m^i(\Gamma_a(M))$. By using the hypothesis $f_a^i(\Gamma_a(M)) \in \mathcal{S}$ for all $i \geq t$. So it follows from the exact sequence (*) that $f_a^i(M) \in \mathcal{S}$ if and only if $f_a^i(\frac{M}{\Gamma_a(M)}) \in \mathcal{S}$ for all $i \geq t$.

Theorem 3.7. Let (R, \mathbf{m}) be a local ring and $M \in \mathcal{S}$ be a finitely generated R-module of dimension d such that $cd_{\mathcal{S}}(\mathbf{m}, M) \leq f. cd_{\mathcal{S}}(\mathbf{a}, M)$. Then $\frac{f_a^t(M)}{af_a^t(M)} \in \mathcal{S}$ where $t = f. cd_{\mathcal{S}}(\mathbf{a}, M)$.

Proof. We use induction on $d = \dim(M)$. If d = 0, then $\dim\left(\frac{M}{aM}\right) = 0$. Accordingly to [3, Theorem 1.1], $f_a^i(M) = 0$ for all i > 0.

Moreover $f_a^0(M) \cong M \in \mathcal{S}$. By definition $H_m^i(M) \in \mathcal{S}$ for all i > t. Therefore from the above lemma we can assume that M is **a**-torsion-free and there is an M-regular element $x \in a$. Consider the long exact sequence :

$$\cdots \to f_a^i(M) \xrightarrow{x} f_a^i(M) \xrightarrow{f} f_a^i\left(\frac{M}{xM}\right) \xrightarrow{g} f_a^{i+l}(M) \xrightarrow{h} \cdots (*)$$

By using the hypothesis $f_a^i(M) \in \mathcal{S}$ for all i > t (because $t = f.cd_{\mathcal{S}}(\boldsymbol{a}, M)$). So using the above long exact sequence $f_a^i\left(\frac{M}{xM}\right) \in \mathcal{S}$ for all i > t. By induction hypothesis, $\frac{f_a^t\left(\frac{M}{xM}\right)}{af_a^t\left(\frac{M}{xM}\right)} \in \mathcal{S}$ because $\dim\left(\frac{M}{xM}\right) = \dim(M) - I$.

Afterwards from the exact sequence (*) we get the following short exact sequence.

$$0 \to Im(f) \to f_a^t \left(\frac{M}{xM}\right) \to Im(g) \to 0$$

So we obtain the following long exact sequence.

...
$$\rightarrow Tor_{I}^{R}\left(\frac{R}{a}, Im(g)\right) \rightarrow \frac{Im(f)}{aIm(f)} \rightarrow \frac{f_{a}^{t}\left(\frac{M}{xM}\right)}{af_{a}^{t}\left(\frac{M}{xM}\right)} \rightarrow \frac{Im(g)}{aIm(g)} \rightarrow 0.$$

Since $f_a^t(M) \in \mathcal{S}$ and Im(g) is a submodule of $f_a^{t+1}(M)$, we deduce that $Tor_l^R(\frac{R}{a}, Im(g)) \in \mathcal{S}$. On the other hand, $\frac{f_a^t(\frac{M}{xM})}{af_a^t(\frac{M}{xM})} \in \mathcal{S}$. Therefore, $\frac{Im(f)}{aIm(f)} \in \mathcal{S}$ by the

above long exact sequence.

Now, consider the following long exact sequence.

$$\frac{f_a^t(M)}{af_a^t(M)} \xrightarrow{x} \frac{f_a^t(M)}{af_a^t(M)} \xrightarrow{lm(f)} \frac{lm(f)}{alm(f)} \to 0.$$

So, $\frac{f_a^t(M)}{af_a^t(M)} \cong \frac{Im(f)}{aIm(f)}$ because $x \in a$. Consequently, $\frac{f_a^t(M)}{af_a^t(M)} \in \mathcal{S}$.

Proposition 3.8. For a finitely generated *R*-module *M*,

$$f.cd_{\mathcal{S}}(\boldsymbol{a}, M) = max \{f.cd_{\mathcal{S}}(\boldsymbol{a}, \frac{R}{P}) | P \in Ass_{R}(M) \}.$$

Proof. Set $N := \bigoplus_{P \in Ass_R(M)} \frac{R}{P}$. Then $Supp_R(M) = Supp_R(N)$. So, by Theorem 3.2 and Corollary 3.5, $f \cdot cd_{\mathcal{S}}(\boldsymbol{a}, M) = f \cdot cd_{\mathcal{S}}(\boldsymbol{a}, N) = max \{ f \cdot cd_{\mathcal{S}}(\boldsymbol{a}, \frac{R}{P}) | P \in Ass_R(M) \}$.

Proposition 3.9. Assume that \boldsymbol{a} is an ideal of the local ring (R, \boldsymbol{m}) . Then $Hom_R(\frac{R}{m}, f_a^0(M)) \in \mathcal{S}$ if and only if. $Hom_R(\frac{R}{m}, \widehat{M}^a) \in \mathcal{S}$.

Proof. It is enough to consider the following isomorphisms

$$Hom_{R}\left(\frac{R}{\boldsymbol{m}},f_{\boldsymbol{a}}^{0}(M)\right) \cong \frac{\lim_{n \in \mathbb{N}} Hom_{R}\left(\frac{R}{\boldsymbol{m}},H_{\boldsymbol{m}}^{0}\left(\frac{M}{\boldsymbol{a}^{n}M}\right)\right) \cong \frac{\lim_{n \in \mathbb{N}} Hom_{R}\left(\frac{R}{\boldsymbol{m}},\frac{M}{\boldsymbol{a}^{n}M}\right)$$
$$\cong Hom_{R}\left(\frac{R}{\boldsymbol{m}},\frac{\lim_{n \in \mathbb{N}} \frac{M}{\boldsymbol{a}^{n}M}\right) \cong Hom_{R}\left(\frac{R}{\boldsymbol{m}},\widehat{M}^{\boldsymbol{a}}\right).$$

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