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On pointwise inner automorphisms of nilpotent groups of class2

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**Abstract** 

An automorphism  $\theta$  of a group G is pointwise inner if  $\theta(x)$  is conjugate to x for any

 $x \in G$ . The set of all pointwise inner automorphisms of group G, denoted by  $Aut_{pwi}(G)$ 

form a subgroups of Aut(G) containing Inn(G). In this paper, we find a necessary and

sufficient condition in certain finitely generated nilpotent groups of class 2 for which

 $Aut_{pwi}(G) \simeq Inn(G)$ . We also prove that in a nilpotent group of class 2 with cyclic

commutator subgroup  $Aut_{pwi}(G) \simeq Inn(G)$  and the quotient  $Aut_{pwi}(G)/Inn(G)$  is torsion.

In particular if G' is a finite cyclic group then  $Aut_{pwi}(G) = Inn(G)$ .

MSC: Primary 20D45; Secondary 20E36

Introduction

By definition, a pointwise inner automorphism of a group G is an automorphism

 $\theta: G \to G$  such that t and  $\theta(t)$  are conjugate for any  $t \in G$ . This notion appears in the

famous book of Burnside [1, Note B, p 463]. Denote by Autpwi(G) the set of all

pointwise inner automorphisms of G.

Obviously, Aut<sub>pwi</sub>(G) contains Inn(G), the group of all inner automorphisms of G.

These groups can coincide, for instance when G is  $S_n$ ,  $A_n$ ,  $SL_n(D)$  and  $GL_n(D)$  where D

is an Euclidean domain (see [7], [10], [11]).

By a result of Grossman [5], it turns out that  $Aut_{pwi}(G) = Inn(G)$  when G is a free

group. Endimioni in [4] proved that this property remains true in a free nilpotent group.

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Also Yadav in [12] gave a sufficient condition for a finite p-group G of nilpotent class 2 to be such that  $Aut_{pwi}(G) = Inn(G)$ . But the equality does not hold in general.

In fact, in 1911, Burnside posed the following question: Does there exist any finite group G such that G has a non-inner and pointwise inner automorphism? In 1913, Burnside himself gave an affirmative answer to this question [3]. Indeed, there are many examples of groups admitting a pointwise inner automorphism which is not inner (see, for instance [3], [4], [8], [9], [12] where these groups are besides nilpotent).

Segal also gave a subtle example. He constructed a finitely generated torsion-free nilpotent group G, in which Aut<sub>pwi</sub>(G)/Inn(G) contains an element of infinite order (see [9]).

In this paper we study the pointwise inner automorphisms of a finitely generated nilpotent group of class 2 with cyclic commutator subgroup.

We introduce the following definition:

**Definition.** Let G be a finitely generated nilpotent group of class 2. Then G/Z(G) is finitely generated abelian group and thus  $G/Z(G) = \langle x_1 Z(G) \rangle \times ... \times \langle x_k Z(G) \rangle$  for some  $x_1, ..., x_k \in G$ . The group G is called **d**-group if the following distributive law holds in G.

$$[x_1^{\alpha_1} \dots x_k^{\alpha_k}, G] = [x_1, G]^{\alpha_1} \dots [x_k, G]^{\alpha_k}$$

where  $\alpha_i \in \mathbb{Z}$  and  $1 \le i \le k$ .

Let G be a 2-generator nilpotent group of class 2. It is straightforward to show that G is a d-group.

To give an example of an infinite d-group, consider the group G with the following presentation

$$G = \langle x_1, x_2, x_3, x_4, x : [x_i, x_j] = x^{m_{ij}}, [x_i, x] = 1; 1 \le i \le 4 \text{ and } i < j \rangle,$$
 where  $m_{ii+1} = 1$  for all  $1 \le i < 4$  and  $m_{ij} = 0$  for all  $i+1 < j$ . Then  $G' = Z(G) = \langle x \rangle \simeq \mathbb{Z}$  and  $G/Z(G) = \langle \overline{x_1}, \overline{x_2}, \overline{x_3}, \overline{x_4} \rangle \simeq \mathbb{Z}^4$ . A quick calculation shows that 
$$[x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} x_4^{\alpha_4}, G] = [x_1, G]^{\alpha_1} [x_2, G]^{\alpha_2} [x_3, G]^{\alpha_3} [x_4, G]^{\alpha_4} = \langle x^{\alpha} \rangle,$$

Where  $\alpha_i \in \mathbb{Z}$  for all  $1 \le i \le 4$  and  $\alpha = \gcd(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ . Therefore G is an infinite d-

group.

Now we give a nilpotent group G of class 2 which is not a d-group.

Let G be a free nilpotent group of class 2 on 4 generators  $a_1, a_2, a_3$  and  $a_4$ . If  $c_{ij} = [a_i, a_j]$  for  $1 \le i < j \le 4$ , then the relations in G are  $[c_{ij}, a_k] = 1$  for  $1 \le i < j \le 4$  and  $1 \le k \le 4$ , and their consequences. Macdonald in [6] proved that  $c_{13}c_{24}$  is not a commutator. Therefore G is not a d-group.

**Theorem 1.** Let G be a finitely generated nilpotent group of class 2 and

$$G/Z(G) = \langle \overline{x_1} \rangle \times ... \times \langle \overline{x_k} \rangle$$
.

- (i) There exists a monomorphism  $\operatorname{Aut}_{pwi}(G) \hookrightarrow \prod_{i=1}^k \operatorname{Hom}(\langle \overline{x_i} \rangle, [x_i, G])$ .
- (ii) If  $[x_i, G]$  is cyclic for all  $1 \le i \le k$ , then there exists a monomorphism  $\operatorname{Aut}_{\mathrm{pwi}}(G) \hookrightarrow \operatorname{Inn}(G)$ .

In particular if G is a d-group of class 2 then the monomorphisms in (i) and (ii) are isomorphism. Furthermore  $\operatorname{Aut}_{pwi}(G) \simeq \operatorname{Inn}(G)$  if and only if  $[x_i, G]$  is cyclic for all  $1 \le i \le k$ .

Notice that if G is a finite group then, as consequence of this result, we derive Theorem 3.5 and Corollary 3.6 of Yadav in [12].

In particular, we derive the following consequences of Theorem 1.

**Corollary 1.** Let G be a finitely generated nilpotent group of class 2 in which G' is cyclic, then  $Aut_{pwi}(G) \simeq Inn(G)$ . In particular if G' is finite, then  $Aut_{pwi}(G) = Inn(G)$ .

Recall that by Corollary 3.6 in [12], in a finite nilpotent group of class 2, if G' is cyclic then  $\operatorname{Aut}_{\operatorname{pwi}}(G) = \operatorname{Inn}(G)$ . But we cannot hope for a similar conclusion when G is not finite. We will provide an example in the section 2. However, in a finitely generated nilpotent group of class 2, by Corollary 1 we have  $\operatorname{Aut}_{\operatorname{pwi}}(G) \simeq \operatorname{Inn}(G)$ . So the structure of  $\operatorname{Aut}_{\operatorname{pwi}}(G)$  is determined.

**Corollary 2.** Let G be a finitely generated nilpotent group of class 2. If the commutator subgroup of G is cyclic, then  $Aut_{pwi}(G)/Inn(G)$  is torsion.

Let G be a group and N be a non-trivial proper normal subgroup of G. The pair

(G, N) is called a Camina pair if  $xN \subseteq x^G$  for all  $x \in G \setminus N$ . A group G is called a Camina group if (G, G') is a Camina pair.

Clearly, if G is a Camina group of class 2 then it is a d-group. So, as an immediate consequence of Theorem 1, one readily gets the following corollary.

**Corollary 3.** Let G be a finitely generated nilpotent group of class 2. If G is a Camina group then  $\operatorname{Aut}_{pwi}(G) \simeq \operatorname{Inn}(G)$  if and only if G' is cyclic. Particularly, if G/Z(G) is finite, then  $\operatorname{Aut}_{pwi}(G) = \operatorname{Inn}(G)$  if and only if G' is cyclic.

# **Preliminary results**

Our notation is standard. Let G be a group, by  $C_m$ , G' and Z(G), we denote the cyclic group of order m, the commutator subgroup and the center of G, respectively.

If  $x, y \in G$ , then  $x^y$  denotes the conjugate element  $y^{-1}xy \in G$ . For  $x \in G$ ,  $x^G$  denotes the conjugacy class of x in G. The commutator of two elements  $x, y \in G$  is defined by  $[x, y] = x^{-1}y^{-1}xy$  and more generally, the left-normed commutator of n elements  $x_1, \ldots, x_n$  is defined inductively by

$$[x_1, ..., x_{n-1}, x_n] = [x_1, ..., x_{n-1}]^{-1} x_n^{-1} [x_1, ..., x_{n-1}] x_n.$$

If  $H \le G$ , [x, H] denotes the set of all [x, h] for  $h \in H$ , this is a subgroup of G when G is of class 2. For any group H and abelian group K, Hom(H, K) denotes the group of all homomorphisms from H to K. Also  $C^*$  is the set of all central automorphisms of G fixing Z(G) elementwise.

Yadav in [12] shows that in a finite nilpotent group of class 2, there exists a monomorphism from  $Aut_{pwi}(G)$  into Hom(G/Z(G), G'). It turns out that this result remains true when G is an infinite nilpotent group of class 2.

For that, let G be a nilpotent group (finite or infinite) of class 2. Let  $\alpha \in \operatorname{Aut}_{pwi}(G)$ . Then the map  $g \mapsto g^{-1}\alpha(g)$  is a homomorphism from G into G'. This homomorphism sends Z(G) to 1. So it induces a homomorphism  $f_{\alpha}: G/Z(G) \to G'$ , sending  $\overline{g} = gZ(G)$  to  $g^{-1}\alpha(g)$ , for any  $g \in G$ . Define

$$\operatorname{Hom}_{\operatorname{pwi}}(G/\operatorname{Z}(G),G')=\{f\in\operatorname{Hom}\left(\frac{G}{\operatorname{Z}(G)},G'\right)\colon f(\overline{g})\in[g,G]\text{ for all }g\in G\}.$$

To prove  $\operatorname{Aut}_{pwi}(G) \simeq \operatorname{Hom}_{pwi}(G/Z(G), G')$ , we use the following well-known result.

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**Lemma 1.1** Let N be a normal subgroup of a group G. Let  $\theta$  be an endomorphism of G such that  $\theta(N) \leq N$ . Denote by  $\overline{\theta}$  and  $\theta_0$  the endomorphisms induced by  $\theta$  in G/N and N, respectively. If  $\overline{\theta}$  and  $\theta_0$  are surjective (injective), then so is  $\theta$ .

**Proposition 1.2** Let G be a nilpotent group of class 2. Then the above map  $\varphi: \alpha \mapsto f_{\alpha}$  is an isomorphism from  $Aut_{pwi}(G)$  into  $Hom_{pwi}(G/Z(G), G')$ .

Proof. Since for any  $\alpha \in \operatorname{Aut}_{pwi}(G)$ , by the definition  $f_{\alpha} \in \operatorname{Hom}_{pwi}(G/Z(G), G')$ ,  $\varphi$  is well defined. Let  $\alpha_1, \alpha_2 \in \operatorname{Aut}_{pwi}(G)$  and  $g \in G$ . We have  $\alpha_1(g^{-1}\alpha_2(g)) = g^{-1}\alpha_2(g)$ , since  $g^{-1}\alpha_2(g) \in G' \leq Z(G)$ . This implies that

$$\begin{split} f_{\alpha_1 \alpha_2}(\overline{g}) &= g^{-1} \alpha_1(\alpha_2(g)) = g^{-1} \alpha_1(gg^{-1} \alpha_2(g)) \\ &= g^{-1} \alpha_1(g). \, g^{-1} \alpha_2(g) = f_{\alpha_1}(\overline{g}). \, f_{\alpha_2}(\overline{g}). \end{split}$$

Hence  $\varphi$  is a homomorphism. Clearly,  $\varphi$  is injective. Now it suffices to show that  $\varphi$  is surjective.

Let f be any element of  $\operatorname{Hom}_{\operatorname{pwi}}(G/\operatorname{Z}(G),G')$ . By Lemma 1.1 a quick calculation shows that  $\varphi(\alpha) = f$ , where  $\alpha$  is an element of  $\operatorname{Aut}_{\operatorname{pwi}}(G)$ , sending  $g \in G$  to  $\operatorname{gf}(g\operatorname{Z}(G))$ . Then we have  $\operatorname{Aut}_{\operatorname{pwi}}(G) \simeq \operatorname{Hom}_{\operatorname{pwi}}(G/\operatorname{Z}(G),G')$ .

\* Note that if G is a nilpotent group of class 2 then  $\operatorname{Aut}_{pwi}(G) \simeq \operatorname{Hom}_{pwi}(G/Z(G), G')$ . It is easy to see that in a Camina nilpotent group of class 2,  $\operatorname{Hom}_{pwi}(G/Z(G), G') = \operatorname{Hom}(G/Z(G), G')$ . Hence if G is a Camina group of class 2, then  $\operatorname{Aut}_{pwi}(G) \simeq \operatorname{Hom}(G/Z(G), G')$ .

The following well-known facts will be used repeatedly.

## **Lemma 1.3** Let A, B and C be abelian groups.

- (i)  $\operatorname{Hom}(A \times B, C) \simeq \operatorname{Hom}(A, C) \times \operatorname{Hom}(B, C)$ .
- (ii)  $\operatorname{Hom}(A, B \times C) \simeq \operatorname{Hom}(A, B) \times \operatorname{Hom}(A, C)$ .
- (iii)  $Hom(C_m, C_n) \simeq C_d$  where d = gcd(m, n).
- (iv)  $\text{Hom}(\mathbb{Z}, A) \simeq A$ .
- (v) If A is torsion group and B is torsion-free group, then Hom(A, B) = 1.
- (vi) If  $gcd(|A|, |B|) \neq 1$ , then  $Hom(A, B) \neq 1$ .

## **Main Result**

Let G be a finite abelian group. We denote by  $G_p$ , the p-primary component of G. Hence  $G = \prod_{p \in \pi(G)} G_p$  where  $\pi(G)$  denotes the set of all primes p dividing |G|. To prove Theorem 1, we need the following Lemma.

**Lemma 2.1** ([1, Corollary 1.4]) Let A and B be two finite abelian groups and  $\exp(A)|\exp(B)$ . Then  $\operatorname{Hom}(A,B) \simeq A$  if and only if  $B \simeq C_m \times H$  in which  $C_m \simeq \Pi_{p \in \pi(A)} B_p$  and  $H \simeq \Pi_{p \notin \pi(A)} B_p$ . In particular, if  $\pi(A) = \pi(B)$  then this is equivalent to B is a cyclic group.

Let G be a finitely generated nilpotent group of class 2. Then G/Z(G) is finitely generated abelian group and thus  $G/Z(G) = \langle x_1 Z(G) \rangle \times ... \times \langle x_k Z(G) \rangle$  for some  $x_1, ..., x_k \in G$ .

Let  $f \in \text{Hom}_{pwi}(G/Z(G), G')$ . So  $f(gZ(G)) \in [g, G]$  for all  $g \in G$ . In particular, for all  $1 \le i \le k$  we have  $f(x_iZ(G)) \in [x_i, G]$ . Now we prove Theorem 1.

## **Proof of Theorem 1.**

- (i) By Proposition 1.2, we have  $\operatorname{Aut}_{\operatorname{pwi}}(G) \simeq \operatorname{Hom}_{\operatorname{pwi}}(G/Z(G), G')$ . It suffices to show that there exists a monomorphism from  $\operatorname{Hom}_{\operatorname{pwi}}(G/Z(G), G')$  into  $\prod_{i=1}^k \operatorname{Hom}(\langle \overline{x_i} \rangle, [x_i, G])$ . Let  $f \in \operatorname{Hom}_{\operatorname{pwi}}(G/Z(G), G')$ . Denote by  $f_i$ , the homomorphism induced by f in  $\langle \overline{x_i} \rangle$ , for all  $1 \leq i \leq k$ . Since G is a nilpotent group of class 2, we have  $[a^m, b] = [a, b]^m = [a, b^m]$  for each  $a, b \in G$  and  $m \in \mathbb{Z}$ . Consequently,  $[x_i^m, G] \leq [x_i, G]$  for all  $m \in \mathbb{Z}$  and  $1 \leq i \leq k$ . Therefore  $f_i \in \operatorname{Hom}(\langle \overline{x_i} \rangle, [x_i, G])$ . Thus the map  $\alpha$  sending any  $f \in \operatorname{Hom}_{\operatorname{pwi}}(G/Z(G), G')$  to  $\alpha(f) = (f_1, \dots, f_k) \in \prod_{i=1}^k \operatorname{Hom}(\langle \overline{x_i} \rangle, [x_i, G])$  is well defined. Now we prove that this map is a monomorphism. Since  $(fg)_i = f_ig_i$  for each  $f, g \in \operatorname{Hom}_{\operatorname{pwi}}(G/Z(G), G')$  and  $1 \leq i \leq k$ ,  $\alpha$  is homomorphism. Clearly,  $\ker \alpha$  is trivial, this implies that  $\alpha$  is monomorphism. Hence the proof of (i) is complete.
- (ii) First we show that  $[x_i, G]$  is finite if and only if  $\langle \overline{x_i} \rangle$  is finite, and further

 $\exp([x_i,G]) = \exp(\langle \overline{x_i} \rangle) = |\overline{x_i}|$ . For this, let  $|[x_i,G]| = n$ . Since G is a nilpotent group of class 2, we have  $[x_i^n,g] = [x_i,g]^n = 1$  for all  $g \in G$  and so  $x_i^n \in Z(G)$ . Hence  $\langle \overline{x_i} \rangle$  is finite and  $|\overline{x_i}||n$ . Conversely if  $|\overline{x_i}| = m$  then  $x_i^m \in Z(G)$  and  $[x_i,G]^m = [x_i^m,G] = 1$ . Consequently  $[x_i,G]$  is finite and  $\exp([x_i,G]) = n|m$ . Therefore in this case, m = n. Hence by Lemma 2.1, for all  $1 \le i \le k$  we have  $Hom(\langle \overline{x_i} \rangle, [x_i,G]) \simeq \langle \overline{x_i} \rangle$  if and only if  $[x_i,G]$  is cyclic.

Now from (i), we have a monomorphism from  $\operatorname{Aut}_{pwi}(G)$  into  $\prod_{i=1}^k \operatorname{Hom}(\langle \overline{x_i} \rangle, [x_i, G])$  and therefore we conclude that there exists a monomorphism  $\operatorname{Aut}_{pwi}(G) \hookrightarrow G/Z(G)$ , this completes the proof of (ii).

If G is a d-group, then it is easy to see that the monomorphism defined in (i) is an isomorphism from  $\operatorname{Aut}_{pwi}(G)$  into  $\prod_{i=1}^k \operatorname{Hom}(\langle \overline{x_i} \rangle, [x_i, G])$ .

Finally to complete the proof, it is sufficient to show that if  $\operatorname{Aut}_{pwi}(G) \simeq \operatorname{Inn}(G)$ , then  $[x_i, G]$  is cyclic for all  $1 \le i \le k$ . Since  $\operatorname{Aut}_{pwi}(G) \simeq \operatorname{Inn}(G)$ , by Proposition 1.2 we have  $G/Z(G) \simeq \operatorname{Hom}_{pwi}(G/Z(G), G')$ . On the other hand, G is a d-group and hence

$$\operatorname{Hom}_{\operatorname{pwi}}(G/\operatorname{Z}(G),G')\simeq\prod_{i=1}^k\operatorname{Hom}(\langle\overline{x_i}\rangle,[x_i,G]).$$

It follows that

$$G/Z(G) = \langle \overline{x_1} \rangle \times ... \times \langle \overline{x_k} \rangle \simeq \prod_{i=1}^k Hom(\langle \overline{x_i} \rangle, [x_i, G]).$$

Now we may assume that  $\langle \overline{x_1} \rangle \times ... \times \langle \overline{x_n} \rangle$  is the torsion part and  $\langle \overline{x_{n+1}} \rangle \times ... \times \langle \overline{x_k} \rangle$  is the torsion-free part of G/Z(G). Since for all  $1 \le i \le n$ ,  $\exp([x_i, G]) = \exp(\overline{x_i}) = |\overline{x_i}|$  and  $\prod_{i=1}^n \operatorname{Hom}(\langle \overline{x_i} \rangle, [x_i, G]) \simeq \langle \overline{x_1} \rangle \times ... \times \langle \overline{x_n} \rangle$ ,  $\operatorname{Hom}(\langle \overline{x_i} \rangle, [x_i, G]) \simeq \langle \overline{x_i} \rangle$  for all  $1 \le i \le n$  and hence  $[x_i, G]$  is cyclic. Furthermore, we have

$$\prod_{i=n+1}^{k} \operatorname{Hom}(\langle \overline{x_{i}} \rangle, [x_{i}, G]) \simeq \langle \overline{x_{n+1}} \rangle \times ... \times \langle \overline{x_{k}} \rangle \simeq \mathbb{Z}^{k-n}.$$

Now we have  $\operatorname{Hom}(\langle \overline{x_i} \rangle, [x_i, G]) \simeq [x_i, G]$ , since  $\langle \overline{x_i} \rangle \simeq \mathbb{Z}$  and hence  $\prod_{i=n+1}^m [x_i, G] \simeq \mathbb{Z}$  and hence  $\prod_{i=n+1}^m [x_i, G] \simeq \mathbb{Z}$  and hence  $\prod_{i=n+1}^m [x_i, G] \simeq \mathbb{Z}$  for all  $n+1 \le i \le k$ . This implies that  $[x_i, G]$  is cyclic for all  $1 \le i \le k$ , as required.

\*Notice that if G is a finite group then, as a consequence of this result, we derive Theorem 3.5 and Corollary 3.6 of Yadav in [12].

The following corollary is an easy consequence of the above theorem.

**Corollary 2.2** Let G be a finitely generated nilpotent group of class 2 with cyclic commutator subgroup. Then there exists a monomorphism from  $Aut_{pwi}(G)$  into Inn(G) or equivalently  $Aut_{pwi}(G)$  is isomorphic to a subgroup of G/Z(G).

## **Remark 2.3** We keep here the notation used in Theorem 1.

- (i) By the discussion of (ii) in Theorem 1, if G' is finite cyclic, then G/Z(G) is finite and  $|\operatorname{Aut}_{pwi}(G)| \leq |\operatorname{Inn}(G)| = |G/Z(G)|$ . On the other hand,  $\operatorname{Inn}(G) \leq \operatorname{Aut}_{pwi}(G)$  conclude that  $\operatorname{Aut}_{pwi}(G) = \operatorname{Inn}(G)$ . Note that in this case, G is not necessarily finite.
- (ii) If G' is infinite cyclic, it follows from the discussion of (ii) in Theorem 1, that G/Z(G) is a free abelian group of finite rank, say r(G/Z(G)) = k. We certainly have  $Inn(G) \leq Aut_{pwi}(G)$  and thus  $r(Inn(G)) \leq r(Aut_{pwi}(G))$ . Also  $r(Aut_{pwi}(G)) \leq r(Inn(G))$ , since  $Aut_{pwi}(G)$  is isomorphic to a subgroup of Inn(G). Therefore  $Aut_{pwi}(G)$  and Inn(G) have the same rank and hence  $Aut_{pwi}(G) \simeq Inn(G)$ .

Now it is easy to deduce Corollary 1 from Remark 2.3.

**Remark 2.4** It is known that in a nilpotent groups of class 2,  $Inn(G) \leq Aut_{pwi}(G) \leq C^*$ . So  $Inn(G) = Aut_{pwi}(G)$  when  $Inn(G) = C^*$ . In [1] we characterized all non torsion-free finitely generated groups in which  $Inn(G) = C^*$ . We proved that  $Inn(G) = C^*$  if and only if G is an abelian group or nilpotent of class 2 and  $Z(G) \simeq C_m \times H \times \Box^r$  in which  $C_m \simeq \prod_{p \in \pi(G/Z(G))} Z(G)_p$ ,  $H \simeq \prod_{p \notin \pi(G/Z(G))} Z(G)_p$  and  $r \geq 0$  is the torsion-free rank of Z(G) and G/Z(G) has finite exponent.

Hence if G is nilpotent group of class 2,  $Z(G) \simeq C_m \times H \times \Box^r$  and G/Z(G) has finite exponent then we have  $Inn(G) = Aut_{pwi}(G)$ . Notice that in this case, G' is cyclic and the equality  $Inn(G) = Aut_{pwi}(G)$  also follows from Corollary 1.

Recall that by Corollary 3.6 in [12], in a finite nilpotent group of class 2, if G' is cyclic then  $Aut_{pwi}(G) = Inn(G)$ . But we cannot hope for a similar conclusion when G is not finite.

For example, consider countably infinitely many copies  $H_1, H_2, ...$  of a given nilpotent group H of class 2 with cyclic commutator subgroup. Let G (respectively,  $\overline{G}$ ) be the direct product (the cartesian product) of the family  $(H_i)_{i>0}$ . Clearly, G and  $\overline{G}$  are nilpotent of class 2. For each integer i > 0, choose an element  $a_i \in H_i$  which is not in the center of  $H_i$ . Then the inner automorphism of  $\overline{G}$  defined by  $\overline{\alpha}((t_i)_{i>0}) = (a_i^{-1}t_ia_i)_{i>0}$  induces in G a pointwise inner automorphism  $\alpha$  which is not inner (see [4]).

However, in a finitely generated nilpotent group of class 2 with cyclic commutator subgroup, we have  $\operatorname{Aut}_{pwi}(G) \simeq \operatorname{Inn}(G)$ , by Corollary 1. So the structure of  $\operatorname{Aut}_{pwi}(G)$  is determined.

Furthermore it is fairly easy to deduce Corollary 2 from Remark 2.3.

We end this part of the paper with some examples of infinite groups G satisfying the conditions of Corollary 1 and therefore  $\operatorname{Aut}_{pwi}(G) \simeq \operatorname{Inn}(G)$ .

**Example 2.5** Let  $G = \langle x_1, x_2, y_1, y_2 : x_1^p = x_2^p = y_1^p = 1, [x_1, x_2] = y_1, [y_1, y_2] = [x_i, y_j] = 1; 1 \le i, j \le 2 \rangle$ . Then G satisfies the condition of Corollary 1. We have  $G' = \langle y_1 \rangle \simeq C_p$ ,  $Z(G) = \langle y_1, y_2 \rangle \simeq C_p \times \mathbb{Z}$  and  $G/Z(G) = \langle \overline{x_1}, \overline{x_2} \rangle \simeq C_p \times C_p$  and hence  $Aut_{pwi}(G) = Inn(G)$ .

**Example 2.6** Let  $G = \langle x_1, x_2, x: [x_1, x_2] = x, [x_i, x] = 1; 1 \le i \le 2 \rangle$ . Then G satisfies the condition of Corollary 1. We have  $G' = Z(G) = \langle x \rangle \simeq \mathbb{Z}$  and  $\frac{G}{Z(G)} = \langle \overline{x_1}, \overline{x_2} \rangle \simeq \mathbb{Z} \times \mathbb{Z}$ . Hence  $\operatorname{Aut}_{pwi}(G) \simeq \operatorname{Inn}(G)$ . It is easy to see that in this case every pointwise inner automorphism is inner and so  $\operatorname{Aut}_{pwi}(G) = \operatorname{Inn}(G)$  (see [1, Example 3.4]).

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