A Complete Linear Connection Induced by Berwald Connection

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Abstract

By using Berwald connection, we show that there are linear connections ∇ which are projectively equivalent and belong to the same projective structure on TM. We found a condition for the geodesics of the Berwald connection under which ∇ is complete.

Introduction

Two (torsion free) linear connections D and $\overline{\mathrm{D}}$ on a smooth manifold M are said to be projectively equivalent if there exist a 1-form ρ on M such that

$$\overline{D} = D + \rho \otimes id + id \otimes \rho,$$

where id denotes the identity (1,1)-tensor on M. Projective equivalence is an equivalence relation on the set of torsion-free linear connections on M, and an equivalence class will be called a projective equivalence class [6]. Projective equivalence can be related to the concept of a projective structure. If M has dimension n, then a projective structure on M is a principal subbundle of the bundle of 2-frames over M having as its structure group the isotropy subgroup of $PGL(n, R)$ at the origin of real projective space RP^n, [4].

According to this remark, we can introduce a projective structure on TM. Since the two (torsion free) linear connections on TM belong to the same projective structure on TM if and only if they are projectively equivalent, a projective equivalence class consists of those (torsion free) linear connections on TM which belong to the same projective structure on TM.

KeyWords. Projective structure, Finsler structure, Finsler connection, Berwald connection.

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In Finsler geometry, examples of important connections are proposed by L. Berwald [2], E. Cartan (1934), S. S. Chern [1] and Z. Shen [7]. Some of these connections are torsion free, for a list of almost all Finsler connections, one can refer to Bidabad and Tayebi [3]. So if we use a Finsler connection then we can show that there are many linear connections on TM contained in the same projective equivalence class on TM induced by this Finsler connection. For example in case of Berwald connection, we show that there is a linear connection ∇ on TM which is projectively equivalent to the Berwald connection and belong to the same projective structure on TM. We find a condition for the geodesics of the Berwald connection under which ∇ is complete (to see a similar problem in the Riemannian case, refer to Spivak [6]).

Preliminaries

Let M be an n-dimensional C^∞ manifold. Denote by $T_x M$ the tangent space at $x \in M$, and by $TM := \cup_{x \in M} T_x M$ the tangent bundle of M. Each element of TM has the form (x, y), where $x \in M$ and $y \in T_x M$. Let $TM_0 = TM - \{0\}$. The natural projection $\pi: TM \to M$ is given by $\pi(x, y) := x$.

A (globally defined) Finsler structure [1] on a manifold M is a function $F: TM \to [0, \infty)$

with the following properties:

(i) **Regularity**: F is C^∞ on the entire slit tangent bundle TM_0.

(ii) **Positive homogeneity**: $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$.

(iii) **Strong convexity**: The $n \times n$ Hessian matrix

$$(g_{ij}) := \left(\left[\frac{1}{2} F^2 \right]_{y^i y^j} \right)$$

is positive-definite at every point of TM_0.

Given a manifold M and a Finsler structure F on M, the pair (M, F) is called a Finsler manifold. F is called Riemannian if $g_{ij}(x, y)$ are independent of $y \neq 0$.

Let M be a real n-dimensional connected manifold of C^∞-class and (TM, π, M) its tangent bundle with zero section removed. Every local chart $(U, \varphi = (x^i))$ on M

induces a local chart $(\varphi^{-1}(U), \varphi = (x^i, y^i))$ on TM. The kernel of linear mapp $\pi_*: TTM \to TM$ is called the vertical distribution and is denoted by VTM. For every $u \in TM, Ker\, \pi_{*,u} = V_u TM$ is spanned by $\{\frac{\partial}{\partial y^i}|_u\}$. By a nonlinear connection on TM we mean a regular n-dimensional distribution $H: u \in TM \to H_u TM$ which is supplementary to the vertical distribution i.e.

$$T_u(TM) = H_u TM \oplus V_u TM, \qquad \forall\, u \in TM.$$

A basis for $T_u TM$ adapted to the above direct sum is $(\frac{\delta}{\delta x^i}|_u, \frac{\partial}{\partial y^i}|_u)$, where

$$\frac{\delta}{\delta x^i}|_u = \frac{\partial}{\partial x^i} - N_i^j(u)\frac{\partial}{\partial y^j}\;|_u$$

and N_j^i are coefficients of the nonlinear connection. The dual basis of $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$ is given by $(dx^i,\ dy^i + N_j^i dx^j)$. These are the Berwald bases.

A complete linear connection

Let ∇ be a linear connection on a manifold M. A curve $c: (a, b) \to M$ is an inextendible geodesic of ∇ iff c is a geodesic of ∇ and has no extension to $[0, b + \alpha)$ as a geodesic of ∇ for any $\alpha > 0$. The connection ∇ is complete iff every geodesic of ∇ defined on a subinterval of R extends to a geodesic of ∇ defined on all of R.

In what follows, by using Berwald connection, we want to construct a linear connection ∇ on TM which are projectively equivalent and belong to the same projective structure on TM. We first define notion of Berwald connection.

Let M be a real n-dimensional C^∞ manifold and $VTM = \cup_{v \in TM} V_v TM$ its vertical vector bundle. Suppose that $HTM = \cup_{v \in TM} H_v TM$ is a non-linear connection on TM. The Berwald connection induced by a nonlinear connection with local coefficients N_j^i is a linear connection with the local coefficients $\frac{\partial N_i^k}{\partial y^j}$, (see [5]).

For example, consider S as a semispray with local coefficients G^i and N the induced nonlinear connection with local coefficients $N_j^i = \frac{\partial G^i}{\partial y^j}$. Since the nonlinear connection is symmetric then the Berwald connection D induced by N is a linear connection and has the expression

$$D_{\frac{\delta}{\delta x^i}} \frac{\delta}{\delta x^j} = \frac{\partial^2 G^k}{\partial y^i \partial y^j} \frac{\delta}{\delta x^k}, \quad D_{\frac{\partial}{\partial y^i}} \frac{\delta}{\delta x^j} = 0,$$

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Theorem 3.1 Let D be a Berwald connection induced by the nonlinear connection N associated to a semispray and F a nonzero Finsler metric. Then there is a linear connection ∇ on TM defined by

$$\nabla_X Y = D_X Y + \frac{1}{2F} dF(X)Y + \frac{1}{2F} dF(Y)X, \quad \forall \mathrm{X}, \mathrm{Y} \in \chi\,(\mathrm{TM}). \tag{1}$$

Proof. With respect to the Berwald basis, ∇ has the expression

$$\nabla_{\frac{\delta}{\delta x^i}} \frac{\delta}{\delta x^j} = \frac{\partial \mathrm{N}_{\mathrm{i}}^{\mathrm{k}}}{\partial \mathrm{y}^{\mathrm{j}}} \frac{\delta}{\delta x^k} + \frac{1}{2F}\left(\frac{\delta F}{\delta x^i}\frac{\delta}{\delta x^j} + \frac{\delta F}{\delta x^j}\frac{\delta}{\delta x^i}\right)$$

$$\nabla_{\frac{\delta}{\delta x^i}} \frac{\partial}{\partial y^j} = \frac{\partial \mathrm{N}_{\mathrm{i}}^{\mathrm{k}}}{\partial \mathrm{y}^{\mathrm{j}}} \frac{\partial}{\partial y^k} + \frac{1}{2F}\left(\frac{\delta F}{\delta x^i}\frac{\partial}{\partial y^j} + \frac{\partial F}{\partial y^j}\frac{\delta}{\delta x^i}\right)$$

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It is not difficult to show that the coefficients of ∇ satisfy the transformation law for the coefficients of a linear connection on TM.

For the linear connection (1), we consider the torsion T, defined as usual

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \quad \forall\, X,Y \in \chi(TM).$$

With respect to the Berwald basis we have

$$T\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = \left(\frac{\delta N_i^k}{\delta x^j} - \frac{\delta N_j^k}{\delta x^i}\right)\frac{\partial}{\partial y^k},$$

$$T\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right) = 0,$$

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Theorem 3.2 Let D be the Berwald connection and let ∇ be the linear connection defined in theorem 3.1. If every inextendible geodesic of D, such as $c: [0, b) \to TM$, has an orientation preservation reparametrization $\sigma: [0, \infty) \to [0, b)$ such that $c\, o\sigma$ is a geodesic of ∇, then ∇ is complete.

Proof. Let $c: [0, b) \to TM$ be the inextendible geodesic of D with $c'(0) = (u, v)$. There is an orientation preserving reparametrization $\sigma: [0, \infty) \to [0, b)$ such that $\tilde{c} = c\ o\ \sigma$ is a geodesic of ∇. So we have $\tilde{c}'(0) = \lambda\, c'(0)$ for some positive constant λ. Then $\hat{c}: [0, \infty) \to TM$ given by $\hat{c}(t) = \tilde{c}(\frac{t}{\lambda})$ is also a geodesic of ∇ and $\hat{c}'(0) = (u, v)$. Thus ∇ is complete.

Let the hypotheses of 3.2 hold. If $c: (a, b) \to TM$ is a geodesic of D and $\sigma: (\alpha, \beta) \to (a, b)$ an orientation preserving reparametrization of c such that $\tilde{c} = c\ o\ \sigma$ is a geodesic of ∇, then $\overline{\nabla}_{c'(t)}\, c'(t) = 0$. Let $t = \sigma(s)$ for $s \in (\alpha, \beta)$. So $\tilde{c}'(s) = \sigma'(s) c'(\sigma(s)) = \frac{dt}{ds}\frac{dc}{dt}|_{t=\sigma(s)}$. Since

$$\nabla_{\frac{dc}{ds}} \frac{dc}{ds} = D_{\frac{dc}{ds}} \frac{dc}{ds} + \frac{1}{F} dF(\frac{dc}{ds})\ \frac{dc}{ds}$$

thus

$$\frac{dt}{ds}\left(\left(\frac{dt}{ds}\right)^{-1} \frac{d^2 t}{ds^2} + \frac{d(\ln(F))}{ds}\right)\frac{dc}{dt} = \frac{dt}{ds}\left(\frac{d}{ds}\ln\left(F\frac{dt}{ds}\right)\right)\frac{dc}{dt} = 0\ .$$

This shows that $F\frac{dt}{ds}$ is constant. As Finsler metric is positive function, so there is a constant $C_1 > 0$ such that $F\ \frac{dt}{ds} = \frac{1}{C_1}$. This differential equation can be integrated to give

$$s(t) = \sigma^{-1}(t) = C_0 + C_1 \int_{t_0}^{t} F(c(\theta)) d\theta \qquad (2)$$

where $t_0 \in (a, b), C_0 \in R$.

Theorem 3.3 Let $F(x, y)$ be a nonzero Finsler metric, If for each inextendible geodesic of D, such as $c: [0, b) \to TM$, we have

$$\int_0^b F(c(t)) dt = \infty \qquad (3)$$

then the connection ∇ defined by (1) is complete.

Proof. Suppose that the condition (3) holds. Let $c: [0, b) \to TM$ be such a curve and let $\sigma: [0, \beta) \to [0, b)$ be an orientation preserving reparametrization of c such that $\tilde{c} = c\ o\ \sigma$ is a geodesic of ∇. From (2), $\sigma^{-1}(t)$ is given by

$$\sigma^{-1}(t) = C_1 \int_{t_0}^{t} F\big(c(\theta)\big) d\theta$$

with $C_1 > 0$. But $\beta = C_1 \int_{t_0}^{b} F\big(c(\theta)\big) d\theta = \infty$, so inextendible D-geodesic c has an orientation preserving reparametrization σ: $[0, \infty) \to [0, b)$ such that $c \ o \ \sigma$ is a geodesic of ∇ and so ∇ is complete.

We showed that two linear connections introduced in this paper, the Berwald connection D and the linear connection ∇ defined in theorem 3.1, are projectively equivalent and belong to the same projective structure on TM. We have also proved that for each inextendible geodesic of the Berwald connection such that the condition (3) holds then the connection ∇ is complete.

For example, let $M = R^n - \{0\}$ and let F be a nonzero Finsler metric such that for each $x \in M, f_{|T_xM}$ is a Minkowski norm on T_xM. Consider the curve $c: [0, b) \mapsto TM$ given by $c(t) = p + t(x, y)$ where $y \neq 0$ and $b = \infty$. With respect to this curve, it can be easily shown that the equation (3) is established.

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