Characterization of Filters Preserving Reciprocality

By
EINOLLAH PASHA

Institute of Mathematics
University for Teacher Education
599 Taleghani Ave,
15614 Tehran, IRAN.

ABSTRACT

In this paper we characterize the system function of a linear filter that its output will be a reciprocal process whenever its input is a reciprocal one.

Introduction

Let $X = \{X(t), -\infty < t < \infty\}$ be a process defined on some complete probability space $(\Omega, \mathcal{F}, P)$. The notion of reciprocality was first defined by Jamison [1], and studied in some extend by pasha [2] and [3]. The process X has reciprocal property on $(-\infty, \infty)$ if for each $n \in \mathbf{N}$, and for each reals $u < v$, and for each reals $t_1, \ldots, t_n$ in the complement of interval (u, v), and finally for each $t \in (u, v)$, the conditional distribution of X_t given $X_u, X_v, X_{t_1}, \ldots, X_{t_n}$ is the same as the conditional distribution of X_t given X_u and X_v.

In [2] a martingale representation of Gaussian stationary reciprocal processes is given. In [3] the notion of reciprocality is generalized. Jamison [1] proved that the covariance function of Gaussian stationary reciprocal processes with zero mean is of the following form

$$C_X(t) = E(X(s)X(t+s)) = be^{-a|t|} \quad t \in \mathbb{R},$$

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for some positive numbers a and b. It is clear that $\sigma^2(X(t)) = b$.

In this paper we make the following assumptions:

Assumption A: we assume that the process X satisfies the following conditions:

(i) X is Gaussion,

(ii) X is stationary,

(iii) The mean of X_t is zero

(iv) The covariance function of X_t is continuous,

(v) X has reciprocal property on $(-\infty, \infty)$.

Linear filters

Let X be the input of a linear filter with quasi system function h, i.e.

$$h(t) = 0, \quad t \leq 0.$$

Let $Y = \{y(t), -\infty < t < \infty\}$ be the output of the system, i.e.

$$Y(t) = \int_0^\infty h(t)X(t-s)ds.$$

It is well known that if the process X is Gaussian and stationary then the out-put process Y also is Gaussian and stationary. In the following we want to determine the function h so that if X statisfies assumption A, then Y satisfies the assumption A, specifically it has reciprocal property.

If X is stationary then the covariance function of Y is given by

$$\begin{aligned} C_Y(t) &= E(y(t+s)y(s)) \\ &= \int_0^\infty h(s)C_X(s+t)ds \\ &= C_X(t) * h(-s) \end{aligned}$$

where $*$ stands for the convolution of the function $C_X(t)$ and $h_1(t) = h(-t)$.

We will use the following notions in the sequel:

$$\begin{aligned} C_X(t) &= E(X(t+s)X(s)), \\ C_Y(t) &= E(Y(t+s)Y(s)) \\ C_{XY}(t) &= E(X(t+s)Y(s)) \\ S_X(w) &= \int_{-\infty}^{\infty} e^{-itw}C_X(t)dt, \\ H(w) &= \int_0^\infty e^{-itw}h(t)dt. \end{aligned}$$

$S_Y(w)$,$S_{XY}(w)$will be defined similarly.

Lemma. Let X satisfies assumption A ((i)-(iv)), then

$$\begin{aligned} S_{XY}(w) &= S_X(w)H(-w) \\ S_Y(w) &= S_{XY}(w)H(w). \end{aligned}$$

Proof. We have

$$\begin{aligned} C_{XY} &= \int_0^\infty h(s)C_X(s+t)ds \\ &= \int_{-\infty}^\infty h(s)C_X(s+t)ds \\ &= \int_{-\infty}^\infty h(-s)C_X(t-s)ds \\ &= (C_X * h_1)(t) \end{aligned}$$

where $h_1(t) = h(-t)$. Therefore by taking the Fourier transform we will have

$$\begin{aligned} S_{XY}(w) &= S_X(w).H_1(w) \\ &= S_X(w)H(-w). \end{aligned}$$

Where $H_1(w)$ is the fourier transform of h_1, which is equal to $H(-w)$. A similar computation will prove the second equality.

Now we have the following theorem.

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Theorem 1. Let X satisfies assumption A ((i)-(v)) and $C_X(t) = be^{-a|t|}$. Let Y be the output of the linear quasi system with system function h. Then Y is reciprocal if and only if

$$H(w)H(-w) = \frac{a'b'(a^2+w^2)}{ab(a'^2+w^2)}$$

for some positive numbers a, b, a', b'.

Proof. Assume that the input and output of the system satisfies assumption A. From

$$C_X(t) = be^{-a|t|}$$

we get

$$S_X(w) = \frac{2ab}{a^2+w^2}.$$

Similarly, for some $a' > 0$, $b' > 0$, we have

$$C_Y(t) = b'e^{-a'|t|},$$

Therefore

$$S_Y(w) = \frac{2a'b'}{a'^2+w^2}$$

But, from lemma 1, we have

$$\begin{aligned} S_Y(w) &= S_{XY}(w)H(w) \\ &= S_{XY}(w).H(-w)H(w) \end{aligned}$$

Therefore

$$\frac{2a'b'}{a'^2+w^2} = \frac{2ab}{a^2+w^2}H(w)H(-w)$$

From here we get

$$H(w)H(-w) = \frac{a'b'(a^2+w^2)}{ab(a'^2+w^2)}.$$

Now assume that H satisfies the above relation and the input process satisfies assumption A ((i)-(v)), we prove that Y satisfies assumption A ((i)-(v)). The only property that we have to prove is the reciprocal property of Y. From lemma 1 and the given condition on H we have

$$\begin{aligned} S_Y(w) &= S_X(w)H(w).H(-w) \\ &= \frac{2ab}{a^2+w^2}.\frac{a'b'(a^2+w^2)}{ab(a'^2+w^2)}. \end{aligned}$$

Thus,

$$S_Y(w) = \frac{2a'b'}{a'^2+w^2}.$$

This is the Fourier transform of a function of the following form

$$C_Y(t) = b'e^{-a'|t|}$$

Now from the Jamison result in [1] we conclude that Y has reciprocal propertly.

Example: An example of this kind of filters is

$$h(t) = \frac{b'}{b't^2 + b\pi^2},$$

The Fourier transform of h is

$$H(w) = \sqrt{\frac{b'}{b}}e^{-\pi w\sqrt{\frac{b}{b'}}}$$

Therefore, for any $a > 0$ we have

$$\begin{aligned} H(w)H(-w) &= \frac{b'}{b} \\ &= \frac{b'a(a^2+w^2)}{ba(a^2+w^2)}. \end{aligned}$$

This filter will take an input with covariance function

$$C_X(t) = be^{-a|t|}$$

to an output with covariance function

$$C_Y(t) = b'e^{-a|t|}.$$

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This filter gives more weight to the most recent input than to the most far inputs.

References

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