An Augmented Galerkin Algorithm for First Kind Integral Equations of Hammerstein Type

S. Abbasbandy
and
E. Babolian
Institute of Mathematics,
University of Teacher Education,
Tehran 15614, IRAN

Abstract

Recent papers, [1],[2] & [3], describe some algorithms for linear first kind integral equations. These algorithms are based on augmented Galerkin method and Cross-validation scheme [5]. The results show that, these algorithms work well for linear equations.

In this paper we apply algorithms of [1] & [2] on non-linear first kind integral equations of Hammerstein type with bounded solution. In order to obtain a posteriori error estimate, we apply fifteen-point Gauss-Kronrod quadrature rule [4]. Finally, we give a number of numerical examples showing that the algorithms work well in practice.

AMS subject classifications: 65R20, 45L10

1. Introduction

We consider numerical solution of nonlinear first kind integral equation of Hammerstein type

$$\int_{-1}^{1} k(x,y)\, F(y)\, dy = g(x), \quad -1 \leq x \leq 1, \tag{1}$$

where

$$F(y) = \varphi(y, f(y)), \tag{2}$$

and k, g and φ are known functions and f is the unknown function which is assumed to be bounded, i.e.,

$$f_d \leq f(x) \leq f_u, \quad -1 \leq x \leq 1.$$

We can approximate F by

$$F_N(x) = \sum_{i=0}^{N} a_i T_i(x), \tag{3}$$

where $T_i(x)$ is the Chebyshev polynomial of the first kind of degree i. We obtain $a_0, \ldots, a_N$ by one of the algorithms in [1] or [2].

$$\begin{aligned} \underset{\mathbf{a}}{Minimize} \quad & \|\mathbf{Ba} - \mathbf{g}\| \\ s.t. \quad & |a_i| \leq \delta_i = C_f \hat{i}^{-r}, \\ & i = 0, 1, \ldots, N, \\ & \hat{i} = max(1, i), \end{aligned}$$

where the elements of the coefficient matrix $\mathbf{B}$ are given by

$$B_{ij} = \int_{-1}^{1}\int_{-1}^{1} \frac{k(x,y) T_i(x) T_j(y)}{(1-x^2)^{1/2}}\, dx\, dy,$$
$$i, j = 0, 1, \ldots, N,$$

and the elements of the vector $\mathbf{g}$ are

$$g_i = \int_{-1}^{1} g(x) T_i(x)/(1-x^2)^{1/2}\, dx,$$
$$i = 0, 1, \ldots, N,$$
$$\mathbf{a} = (a_0, a_1, \ldots, a_N)^t,$$

and C_f and r are regualrization parameters. Here we impose $r > 0.5$ and set heuristically, and for C_f,

$$C_f = \lambda \|\mathbf{g}\|_\infty / \|\mathbf{B}\|_\infty,$$

where λ must be set heuristically. There are some strategies in [1] for choosing r and λ.

Suppose P is a partition of $[-1, 1]$,

$$-1 = x_0 < x_1 < \ldots < x_{n-1} < x_n = 1,$$

where $x_i = -1 + i/50$ for $i \in I = \{0, 1, \ldots, 100\}$ and $n = 100$.

Let Q be the partition of $[-1, 1]$ corresponding to fifteen points of Gauss-Kronrod nodes [4]. we denoted Q by

$$q_j, \quad j \in J = \{0, 1, \ldots, 14\}.$$

2. The basic method

This method is pointwise, let $\overline{x}$ be an arbitrary point in $[-1, 1]$, we approximate $f(\overline{x})$. Obviously, if we approximate f at Gauss-Chebyshev points, we would be able to approximate f by expansion method [1] & [2]. Here, we approximate f on Q, i.e., we evaluate approximate values of $f_j =$

$f(q_j)$ for $j \in J$. Sometimes, to obtain more accuracy, we must approximate f on more points, see example 4.

From (2) and (3)

$$\varphi(q_j, f_j) - F_N(q_j) \approx 0, \quad j \in J.$$

If $\varphi(x, y)$ has inverse with respect to y, we can compute f_j by

$$f_j = \varphi^{-1}(q_j, F_N(q_j)), \quad j \in J,$$

otherwise by knowing lower and upper bounds of f we continue as follows. For computing f_j, we may solve the following optimization problem

$$\begin{aligned} &Minimize \quad \phi_j(x), \\ &f_d \le x \le f_u \end{aligned} \qquad (4)$$

where

$$\phi_j(x) = (\varphi(q_j, x) - F_N(q_j))^2, \quad j \in J.$$

To solve (4), we apply Brent method which uses a combination of the golden section search and successive parabolic interpolation [4].

3. A posteriori error

By approximating f on Q, we can compute

$$\int_{-1}^{1} k(x_i, y)\, \varphi(y, f(y))\, dy, \quad i \in I,$$

by Gauss-Kronrod quadrature rule, u_i, which has a posteriori error estimate, v_i, [4].

Let

$$e_1 = \left[\sum_{i=0}^{100} \{u_i - g(x_i)\}^2 / 101\right]^{\frac{1}{2}},$$

$$e_2 = \left[\sum_{i=0}^{100} v_i^2 / 101\right]^{\frac{1}{2}}.$$

Usually, it is expected that $e_1 \le e_2$.

Let z_j be the optimum value of (4), $j \in J$. If (1) has any solution, it is expected to have $z_j \approx 0$. Let

$$e_3 = \left[\sum_{j=0}^{14} z_j^2 / 15\right]^{\frac{1}{2}}.$$

It is expected that if $e_1 \not\le e_2$ for almost all N, and if e_3 is not negligible, (1) has no solutions. Obviously, if we use composite Gauss-Kronrod quadrature rule, the definition of e_1, e_2 and e_3 will change and their values become smaller.

4. Numerical examples and results.

We consider a set of six examples. All computations were carried out on an **IBM-PC** using **C** language and long double precision.

4.1. Examples

1)

$$\int_{-\pi}^{\pi} (sin(x) + cos(y))(1 + f(y))^2\, dy$$

$$= 3\pi\, sin(x), \quad -\pi \le x \le \pi,$$

with solution $f(x) = sin(x)$.

2)

$$\int_0^1 k(x,y)f^2(y)\,dy = \frac{x^4 - x}{12}, \quad 0 \le x \le 1,$$

where

$$k(x,y) = \begin{cases} y(x-1), & y < x, \\ x(y-1), & x \le y, \end{cases}$$

with solutions $f(x) = \pm x$.

3)

$$\int_0^x cos(x-y)(1+f(y))^2\,dy = sin(x),$$

$$0 \le x \le 2,$$

with solution $f(x) = 1$.

4)

$$\int_{-1}^x ln(x-y)f^2(y)\,dy = g(x), \ -1 \le x \le 1,$$

where

$$g(x) = ln(1+x)\frac{1+x^3}{3} - \frac{11x^3 + 6x^2 - 3x + 2}{18},$$

with solutions $f(x) = \pm x$.

5)

$$\int_0^1 e^{xy}\sqrt{f(y)}\,dy = \frac{2}{x^3} - \frac{2e^x}{x^3} + \frac{2e^x}{x^2} - \frac{e^x}{x},$$

$$0 \le x \le 1,$$

with no solutions.

6)

$$\int_{-1}^0 e^{xy}\sqrt{f(y)}\,dy = \frac{(x+1)e^{-x} - 1}{x^2},$$

$$-1 \le x \le 0,$$

with no solutions.

Results for the above examples are presented in tables 1-6, respectively.

Table 1 (Example 1)

$r = 5, \lambda = 4$

N	e_1	e_2	e_3
2	3.4E-12	3.3E-8	1.1E-23
3	3.4E-12	3.3E-8	1.1E-23
4	1.0E-9	3.3E-8	5.7E-21
5	3.4E-12	3.3E-8	1.1E-23
6	3.4E-12	3.3E-8	1.1E-23
7	3.4E-12	3.3E-8	1.1E-23
8	3.4E-12	3.3E-8	1.1E-23
9	3.4E-12	3.3E-8	1.1E-23
10	3.4E-12	3.3E-8	1.1E-23

Table 2 (Example 2)

$r = 5, \lambda = 4$

N	e_1	e_2	e_3
2	1.1E-10	4.0E-10	7.1E-21
3	1.1E-10	4.0E-10	7.1E-21
4	1.1E-10	4.0E-10	7.1E-21
5	1.1E-10	4.0E-10	7.1E-21
6	1.1E-10	4.0E-10	7.1E-21
7	1.1E-10	4.0E-10	7.1E-21
8	1.1E-10	4.0E-10	7.1E-21
9	1.1E-10	4.0E-10	7.1E-21
10	1.1E-10	4.0E-10	7.1E-21

Table 3 (Example 3)

$r = 4, \lambda = 4$

N	e_1	e_2	e_3
2	1.0E-3	2.5E-4	1.9E-7
3	1.1E-6	1.3E-7	3.3E-10
4	1.2E-8	5.9E-9	1.8E-16
5	1.6E-11	5.9E-9	2.6E-20
6	9.1E-12	5.9E-9	1.4E-22
7	9.1E-12	5.9E-9	1.4E-22
8	9.1E-12	5.9E-9	1.4E-22
9	9.1E-12	5.9E-9	1.4E-22
10	9.1E-12	5.9E-9	1.4E-22

Table 5 (Example 5)

$r = 2, \lambda = 5$

N	e_1	e_2	e_3
2	1.028	4.5E-14	4.4E-1
3	1.028	1.2E-6	4.4E-1
4	1.028	4.5E-14	4.4E-1
5	1.028	4.5E-14	4.4E-1
6	1.028	4.5E-14	4.4E-1
7	1.028	1.9E-4	4.4E-1
8	1.028	4.5E-14	4.4E-1
9	1.030	3.0E-3	4.5E-1
10	1.030	3.0E-3	4.5E-1

Table 4 (Example 4)

$r = 4, \lambda = 4$

N	e_1	e_2	e_3
2	2.4E-2	4.5E-2	3.4E-21
3	1.2E-2	4.9E-2	9.0E-5
4	2.0E-4	4.7E-2	3.2E-21
5	2.0E-4	4.7E-2	3.2E-21
6	2.0E-4	4.7E-2	3.2E-21
7	2.0E-4	4.7E-2	3.2E-21
8	2.0E-4	4.7E-2	3.2E-21
9	2.0E-4	4.7E-2	3.2E-21
10	2.0E-4	4.7E-2	3.2E-21

Table 6 (Example 6)

$r = 2, \lambda = 5$

N	e_1	e_2	e_3
2	1.470	4.5E-14	5.2E-1
3	1.470	4.5E-14	5.2E-1
4	1.470	4.5E-14	5.2E-1
5	1.470	4.5E-14	5.2E-1
6	1.470	4.5E-14	5.2E-1
7	1.470	4.5E-14	5.2E-1
8	1.470	4.5E-14	5.2E-1
9	1.470	4.5E-14	5.2E-1
10	1.470	4.5E-14	5.2E-1

4.2. Comments

In all examples, it is assumed $f_d = -1$, $f_u = 1$. The results in tables 1-4 show that, approximate solutions are very accurate, values of e_3 are very small and $e_1 \leq e_2$ for almost all N, but tables 5 and 6 show that, these examples have no solutions.

Example 4 is a Volterra singular integral equation and to obtain accurate result, we used a composite Gauss-Kronrod quadrature rule with 5 panels for each interval.

4.3. Conclusion

From the results, we conclude that automatic augmented Galerkin method work well for linear and Hammerstein first kind integral equations. The cost of operations is not high. We can also compute a posteriori error estimate and use it as an indication of accuracy of approximate solution. Also with knowing the inverse of φ, we can apply this method without knowing lower and upper bounds of f.

Acknowledgment. We thank Prof. Du Jin-Yuan for thoroughly reading first draft of this paper and suggesting valuable comments.

References

[1] Babolian E. & Delves L.M. "An augmented Galerkin method for first kind Fredholm equations", J. Inst. Maths. Applics. 24 (1979), 157-174.

[2] Delves L.M. & Abd-Elal L.F. & Hendry J.A. "A Fast Galerkin Algorithm for Singular Integral Equations", J. Inst. Maths. Applics. 23 (1979), 139-166.

[3] Essah W.A. & Delves L.M. "The Numerical Solution of First Kind Integral Equations", J. Comput. Appl. Math. 27 (1989),363-387.

[4] Kahaner D. & Moler C. & Nash S. " Numerical Methods and and Software", Prentice-Hall, London, 1989.

[5] Wahba G. "Practical Approximate Solutions to Linear Operator Equations When the Data are Noisy", SIAM J. Numer. Anal. 14 (1977) 651-667.