

Some Aspects of Hypergroup Algebras

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Abstract

For a locally compact Hausdorff space X , $M(X)$, the space of bounded regular Borel measures on X , is a Banach space which can be made into a Banach algebra by defining the convolution $(\mu, \nu) \rightarrow \mu \star \nu$ by $\mu \star \nu = \iint \lambda_{(x,y)} d\mu(x) d\nu(y)$ where $\lambda_{(x,y)}$ is a probability measure. We define $L(X)$ to be the set of all measures μ on X for which the function $x \rightarrow |\mu| \star \delta_x$ is weakly continuous. We shall study some aspects of $L(X)$.⁽¹⁾

1. introduction.

The purpose of this paper is to study some aspects of hypergroups. The idea of convolution operator on a locally compact space goes back a long time ago, and the historical points can be found in Ross [10].

In early seventies three mathematicians have been concerned with this theory. Dunkl [3], and Spector [2] use the term hypergroup for their systems while Jewett [6] calls them convos. We shall adopt Dunkl's axioms.

Throughout X will be a locally compact Hausdorff space. We recall some standard notations of [3]. The spaces $C_c(X)$, $C_b(X)$ and $C_0(X)$ will denote the space of continuous complex valued functions with compact support, the space of continuous complex bounded functions and the space of complex continuous functions vanishing at infinity on X each of them with uniform norm. The dual space of $C_0(X)$ is just $M(X)$, the space of finite regular Borel measures on X and $M_p(X)$ will denote the set of positive probability measures on X , i. e.

$\{\mu \mid \mu \in M(X), \mu(X) = 1\}$. The value of a functional μ at an element f will be denoted by $\mu(f)$ and thus $\mu(f) = \int f d\mu$. We use the familiar symbol δ_x for the special functional $f \rightarrow f(x)$. The support of a measure μ will be denoted by $\text{supp}\mu$.

On $M(X)$ we define the convolution by $\mu * \nu = \iint \lambda_{(x,y)} d\mu(X) d\nu(y)$, where $\lambda_{(x,y)}$ is a probability measure on X and $\lambda_{(x,y)} = \lambda_{(x,y)}$. With this definition $M(X)$ becomes a convolution Banach algebra and X is called a commutative

hypergroup. Our definition is based on [3] and [8].

Proposition 1.1.

With the addition, the above convolution and measure norm, $M(X)$ is a commutative Banach algebra. Moreover, $M_p(X) * M_p(X) \subseteq M_p(X)$ [3].

The last part of this proposition states that $M_p(X)$ is a semigroup. We note that for $x, y \in X$ we have $\delta_x, \delta_y \in M_p(X)$, and write $\lambda_{(x,y)} = \delta_x * \delta_y \in M_p(X)$ [3].

definition 1.2.

For $\psi \in C_c(X)$, $x \in X$, $\mu \in M(X)$ define the function $x \rightarrow R(x)\psi$ by $R(x)\psi(y) = \lambda_{(x,y)}(\psi)$ and the function $\mu \rightarrow R(\mu)\psi$ by

$$R(\mu)\psi(y) = \int R(z)\psi(y) d\mu(z) \quad (y \in X)$$

Lemma 1.4.

If $(\mu_\alpha)_\alpha$ is a net such that $\mu_\alpha \rightarrow \mu$ in the weak*-topology, then $\mu_\alpha * \nu \rightarrow \mu * \nu$ in the weak*-topology for all $\nu \in M(X)$.

The proof of the above lemma is a direct consequence of the foregoing proposition and is therefore omitted.

Our main interest is to isolate a subalgebra of $M(X)$ which reduces to $L^1(G)$ when $X=G$, is a locally compact group.

Definition 1.5.

We define $L(X)$ to be the set of all measures $\mu \in M(X)$ such that the function $x \rightarrow |\mu| * \delta_x$ is weakly continuous from X into $M(X)$.

6.

not only a subalgebra of $M(X)$ but also $M(X)$.

$$S = \overline{\bigcup_{\mu \in L(X)} \text{supp } \mu}$$

1.7.

space S has a hypergroup structure.

$\mu \in S$, and let $(U_\alpha), (V_\beta)$ be two nets of neighbourhoods of x, y respectively. Then the nets of positive measures $(\mu_\alpha), (\nu_\beta)$ in $L(X)$ such that $\text{supp } \mu_\alpha \in U_\alpha, \text{supp } \nu_\beta \in V_\beta$ and $\|\mu_\alpha\| = \|\nu_\beta\| = 1$. Clearly $\lambda_{(x,y)}$ is equal to the limit of $(\mu_\alpha \star \nu_\beta)$. Since $\mu_\alpha, \nu_\beta \in L(X)$, $\mu_\alpha \star \nu_\beta \in L(X)$ and therefore $\text{supp } \lambda_{(x,y)} \in S$. Thus $\lambda_{(x,y)}(t) = 1$. Now, let $\psi \in C_c(S)$ we must show the function $(x,y) \rightarrow \int_S \psi d\lambda_{(x,y)}$ is in $C_b(S \times S)$. $\rightarrow \int_S \psi d\lambda_{(x,y)}$ is in $C_c(S)$ for every $y \in S$. Applying the Tietze extension theorem to the point compactification of X , we can find an extension ψ_I of ψ to X with $\psi \in C_c(X)$. Hence $\lambda_{(x,y)}(\psi_I)$ and thus $(x,y) \rightarrow \lambda_{(x,y)}(\psi_I)$ is in $C_b(S \times S)$ for every $y \in S$. This completes the proof.

When S is not the whole space X , however, now on we restrict our attention to the "hypergroup" S or in other words we assume that

Theorem 1.8.

There exists a net in $L(X)$ which is positive and is a weak approximate identity of norm 1 for $L(X)$.

Proof.

Let U be a compact neighbourhood of e . Let A be the collection of all compact neighbourhoods of e contained in U which forms a directed set with set inclusion order. Let $(\nu_\alpha)_\alpha$ be a net of positive normalized measures in $L(X)$ which is supported in $(U_\alpha)_\alpha$ and thus $\nu_\alpha(X \setminus U_\alpha) = 0$. Let $h \in M(X)$ and $\varepsilon > 0$. Since $x \rightarrow |\mu| \star \delta_x$ is weak-continuous $(\mu \in L(X))$, $N = \{x \mid |h(\mu \star \delta_x) - h(\mu)| < \varepsilon\}$ is a neighbourhood of e . So there is an $\alpha_0 \in A$ such that $U_{\alpha_0} \subset N$. Let $\alpha > \alpha_0$. Then

$$\begin{aligned} |h(\mu \star \nu_\alpha) - h(\mu)| &= \\ \left| \int h(\mu \star \delta_\alpha) d\nu_\alpha(x) - \int h(\mu) d\nu_\alpha(x) \right| \\ &\leq \int_{U_\alpha} |h(\mu \star \delta_\alpha) - h(\mu)| d\nu_\alpha(x) < \varepsilon \end{aligned}$$

Theorem 1.9.

There exists a net in $L(X)$ which is a positive approximate identity of norm 1 for $L(X)$.

Proof.

The result is a direct consequence of the last theorem and ([2], Chapter 10).

The most important property of $L(X)$ is the norm continuity of $x \rightarrow |\mu| \star \delta_x(\psi)$ into $L(X)$. For this we need some elementary and also basic results.

Lemma 1.10.

Let $\psi \in C_0(X)$ and $\mu \in M(X)$. Then the function $x \rightarrow |\mu| \star \delta_x(\psi)$ is continuous on X [3].

Proposition 1.11.

Let τ be a topology on $M(X)$ finer than the weak*-topology. Then the following are equivalent.

(i) The function $x \rightarrow |\mu| \star \delta_x$ is τ -continuous.

(ii) $A = \{|\mu| \star \delta_x \in K\}$ is τ -compact for each compact set $K \subseteq X$.

Proof.

Clearly (i) implies (ii). Now let (ii) hold. Since the τ -topology is finer than the weak-topology and A is τ -compact, the topologies coincide on $\{|\mu| \star \delta_x \mid x \in K\}$. On the other hand the function $x \rightarrow |\mu| \star \delta_x$ is weak*-continuous [6] into $M(X)$, so it is τ -continuous when regarded as a map from K into $M(X)$ for each compact set K , as X is locally compact, the function is τ -continuous on X . This completes the proof.

A Banach space A is said to have the Dunford - Pettis property if for each Banach space B and each W -compact operator T from A to B , $T(K)$ is compact in B whenever K is W -compact in A [7]. It is well known that $M(X)$ possesses this property [12].

Theorem.1.12.

Let $\mu \in L(X)$. Then the operator $R_\mu: M(K) \rightarrow M(X)$ defined by $R_\mu(\nu) = \mu \star \nu$ is a W -compact

mapping, where K is a compact subset of X .

Proof.

By lemma 1.9 the set $\{\mu \star \delta_x \mid x \in K\}$ is weakly compact. Thus $W - \overline{CO} \{\mu \star \delta_x \mid x \in K\}$, the weak-closed convex hull of $\{\mu \star \delta_x \mid x \in K\}$ is weak-compact. Also on this set the weak and weak*-topologies coincide. Now, we know that the closed-convex hull of $\{c\delta_x \mid c \mid x \in K\}$ is the unit ball of $M(X)$ [6], also R_μ is weak*-continuous. Therefore, the image under R_μ of the unit ball of $M(K)$ is just $W - \overline{CO} \{\mu \star \delta_x \mid x \in K\}$ which is weakly compact.

Lemma 1.13.

Let $\nu_1, \nu_2 \in L(X)$, and K be compact. Then $\{\nu_1 \star \nu_2 \star \delta_x \mid x \in K\}$ is norm-compact.

Proof.

Since the set of measures with compact support is dense in $L(X)$, we may assume without loss of generality that $\text{supp} \nu_2 = F$ is compact thus $F \star K$ is compact and $R_{\nu_2}: M(K) \rightarrow M(F \star K)$, $R_{\nu_2}: M(F \star K) \rightarrow M(X)$ are weakly - compact operators. Therefore $K_1 = \{\nu_2 \star \delta_x \mid x \in K\}$ is weakly compact and thus, by the Dunford-Pettis property $R_{\nu_1}(K_2) = \{\nu_1 \star \nu_2 \star \delta_x \mid x \in K\}$ is norm-compact.

Now, We establish a main result.

Theorem 1.14.

Let $\mu \in L(X)$. Then the function $x \rightarrow \mu \star \delta_x$ is

norm continuous.

Proof.

It is clear from 1.11 and 1.12 that $\overline{L(X) \star L(X)} \subseteq L_N(X)$ the space of all measures μ such that $x \rightarrow \mu \star \delta_x$ is norm-continuous. Since $L(X)$ has a bounded approximate identity, we have $L(X) = \overline{L(X) \star L(X)} \subseteq L_N(X)$, it is also obvious that $L_N(X) \subseteq L(X)$. Thus $L(X) = L_N(X)$, i.e. $x \rightarrow \mu \star \delta_x$ is norm-continuous.

Finally, we mention a result which has an exact parallel in the semigroup case.

Proposition 1.15.

The algebra $L(X)$ has an identity if and only if X is discrete, in which case the identity element of $L(X)$ is δ_e .

We have also to mention that there is a one to one correspondence between the characters and complex homomorphisms of semigroup algebras. A modification of the proof 3.1 of [1] shows that there is a similar relation for $L(X)$, namely for each character ϕ there is a complex homomorphism h such that $\phi(x) = \frac{h(\mu \star \delta_x)}{h(\mu)}$ where $\mu \in L(X)$, $h(\mu) \neq 0$.

It is true that hypergroups are the extension of semigroup or group algebras to this type of algebras over locally compact spaces. However we also consider some different type of algebras with different approaches. For this we wait until the end of section 2.

2. Multipliers and isomorphisms.

This section is devoted to demonstrating results for $L(X)$ similar to those for group algebras. First of all it should be noted that, since $L(X)$ has an approximate identity of norm 1, $M(L(X))$ the multiplier algebra of $L(X)$ is isometrically isomorphic to $M(X)$ via the correspondence: $\mu \in L(X)$ corresponds to the multiplier T if and only if $Tv = v \star \mu$.

A multiplier is called unitary if T is onto and an isometry. First, we characterize the unitary multipliers on $L(X)$. Now, we need some elementary results.

Lemma 2.1.

Let $(v_\alpha)_\alpha$ be a bounded net in $L(X)$ which tends to v in the strong operator topology, (i.e. $v_\alpha \star \mu \rightarrow v \star \mu$ for every $\mu \in L(X)$). Then $(v_\alpha)_\alpha$ tends to v in the weak*-topology $\sigma(M(X), C_0(X))$.

The proof is not too difficult so it is omitted.

Lemma 2.2.

Let $SE_X = \{ k\delta_x \mid x \in X, |k| = 1 \}$. Then $\overline{CO}[SE_X, SO] = \overline{CO}[SE_X, \sigma] =$ the unit ball of $M(X)$.

Note that we mean by $\overline{CO}[SE_X, SO]$ the closed convex hull of SE_X with strong operator topology, and the latter is the same with weak*-topology.

Lemma 2.3.

Let T be a unitary multiplier on $L(X)$. Then it can be extended to a unitary multiplier on $M(X)$.

Proof.

Because T is a unitary multiplier on $L(X)$ it has an inverse T^{-1} which is also a unitary multiplier. Thus, there are measures $\tau, \tau_1 \in M(X)$ such that $T\mu = \tau \star \mu, T^{-1}\mu = \tau_1 \star \mu$ for $\mu \in L(X)$. Then $\delta_e \star \mu = \mu = T^{-1}T\mu$, so that $\tau_1 \star \tau = \delta_e$, and similarly, $\tau \star \tau_1 = \delta_e$. Also $\|\tau\| = \|\tau_1\| = 1$. Define \bar{T} on $M(X)$ by $\bar{T}\nu = \tau \star \nu$. Then \bar{T} has an inverse, because $S\nu = \tau_1 \star \nu$ is clearly an inverse to \bar{T} and is an isometry because $\|\nu\| = \|\tau_1 \star \tau \star \nu\| \leq \|\tau \star \nu\| \leq \|\nu\| \nu \in M(X)$ for $\nu \in M(X)$. Hence T is unitary on $M(X)$.

Theorem 2.4.

Let T be a unitary multiplier on $L(X)$. Then there exists a constant k ($|k| = 1$) and a homeomorphism α on X such that

$$T\mu(\psi) = k\mu(\psi \circ \alpha) \quad (\psi \in C_o(X), \mu \in L(X)) \quad (1)$$

and

$$\lambda_{(x,y)}(\psi \circ \alpha) = \lambda_{(\alpha(x),y)}(\psi). \quad (2)$$

Conversely, if (2) holds then (1) defines a unitary multiplier T on $L(X)$.

Proof.

Since $\|T(\delta_x)\| = \|\delta_x\| = 1$, $T(\delta_x)$ is an extreme point of the unit ball of $M(X)$, also we have by lemma 2.3, $T(\delta_x) = \delta_x \star \tau$. Thus $T(\delta_x) = \delta_x \star \tau = k(x)\delta_{\alpha(x)}$ where k is a complex function on X with ($|k| = 1$) and α is a function from X to X . Therefore,

$$T(\delta_x) = T(\delta_x \star \delta_e) = T(\delta_e) \star \delta_x \quad (3)$$

$$= k(e)\delta_{\alpha(e)} \star \delta_x = k(e)\lambda_{(\alpha(e),x)} \quad (4)$$

So

$$k(x)\delta_{\alpha(x)} = k(e)\lambda_{(\alpha(e),x)}$$

Now, by integration over X we obtain $k(x) = k(e)$, $x \in X$. Thus $x \rightarrow k(x)$ is a constant k , say. In the same way, T^{-1} gives rise to a function β and it is easy to see that $\beta = \alpha^{-1}$. Since

$$x \rightarrow \delta_x \star \tau = k\delta_{\alpha(x)}$$

is weakly continuous, α is continuous. Similarly, α^{-1} is continuous, so that α is a homeomorphism. Because T is weak*-continuous and linear on $M(X)$ we may extend the formula $T(\delta_x)(\psi) = k\delta_{\alpha(x)}(\psi) = k\delta_{\alpha(x)}(\psi \circ \alpha)$ ($\psi \in C_o(X)$) from measures of the form δ_x to every μ in $M(X)$ to obtain formula (1). The formula (2) comes from the relationship $T(\delta_x \star \delta_y)(\psi) = T(\delta_x) \star \delta_y(\psi)$ and (1).

Note 1.

We also have $T\mu = T(\delta_x \star \mu) = k\delta_{\alpha(e)} \star \mu$. This is a generalization of wendel's theorem [14]. It also shows that for every unitary multiplier T , there is a point $x \in X$ such that $T\mu = k\delta_x \star \mu$ and it is easy to observe that the point is uniquely determined by $\alpha(e)$.

Note 2.

It can be shown that $I(L(X)$, the group of isometric multipliers on the commutative algebra $L(X)$ is a topological group in the so-topology when X is compact we have the following.

Theorem 2.5.

If X is compact, then $I(L(X)$ is compact in the strong operator topology.

It is easy to show that for each $\mu \in L(X)$, the set $\{T \in I(L(X)) \mid |T| = 1\}$ is compact. But $B = \{k \mid |k| = 1\}$ is a homeo. on X . Since the map $x \rightarrow k\mu\delta_x$ is continuous, $\{k\mu\delta_{\alpha(e)} \mid \alpha \text{ is a homeo. on } X\}$ is compact. Now as the set $\{k\mu\delta_x \mid x \in X\}$ is compact and $x \rightarrow \mu\delta_x \rightarrow k\mu\delta_x$ is continuous, $\{k\mu\delta_{\alpha(e)} \mid \alpha \text{ is a homeo. on } X\}$ is compact. This completes the proof of the theorem.

We have $\int \psi(\alpha(t)) d\lambda_{(x,y)}(t) = \int \psi(t) d\lambda_{(\alpha(x),\alpha(y))}(t)$. Putting $x=e$, we get

$$\lambda_{(\alpha(x),y)} = \delta_{\alpha(y)}.$$

For $\alpha(e) \in Z(X)$ where $Z(X)$, the center of X , is the set $\{x \in X \mid \text{supp } \lambda_{(x,y)} \text{ is a single point } y \in X\}$. It follows that the group of multipliers is contained in $Z(X)$.

Examples.

In this section we consider some examples. It is well known and it is noted that the most famous examples of hypergroups are the coset space and double coset space of a locally compact group and there are many other examples in [9].

We shall consider some other examples.

Let $X = [0,1]$, and define $xoy = \min\{x+y, 1\}$, then (X, \cdot) is a foundation semigroup and $L(X) = \{K \delta_1 \mid K \in C\}$, thus with $\lambda_{(x,y)} = \delta_{xoy}$ is a hypergroup. Infact $L(X)$ is the group of multipliers of the quotient space $L^1[0,1]$ with the

hypergroup structure with $\lambda_{(x,y)} = \delta_{x+y}$ if $x+y < 1$, $\lambda_{(x,y)} = 0$ otherwise. The unitary multiplier group is just the unit circle.

II. In [6] another type of convolution was defined on a locally compact space X , namely, let $\psi \in C_0(X)$ and define

$$\mu * \nu(\psi) = \iint T\psi_{(x,y)} d\mu(x) d\nu(y)$$

where $T\psi \in C_b(X \times X)$. With our condition on continuity, if we define $\lambda_{(x,y)}(\psi) = T\psi_{(x,y)}$ ($x, y \in X$), with the additional condition that $T\psi_{(x,e)} = \psi(x)$ for some element e of X , then X has a hypergroup structure. With this modification the remaining property of [6] is still valid. More precisely, if T is a lattice isomorphism on compact space X , and $\lambda_{(x,y)}$ is defined as before, then there exists a continuous map m of $X \times X$ into X and μ such that $\lambda_{(x,y)}(\psi) = \psi(m(x,y))\mu(x,y)$ where μ is a strictly positive function on $X \times X$. If we further assume this multiplication is associative, we conclude that

$$\psi(m(x, m(y, z))) = \psi(m(m(x, y), z))$$

for all $\psi \in C(X)$. Thus

$$m(x, m(y, z)) = m(m(x, y), z).$$

That is $(x, y) \rightarrow m(x, y)$ is an associative multiplication which makes X a compact semigroup. The group of unitary multipliers on X is isomorphic to $C \times G$ where C is the unit circle and G is the group

generated by translations on X .

III. It is clear that if X is a locally compact semigroup with continuous multiplication, then $L(X) = L^1(G) \oplus \{K\delta_e \mid K \in \mathbb{C}\}$, thus every commutative locally compact group is a foundation hypergroup.

IV. In this final example we adapt an interesting example of [4] which is very suitable to our purpose in a very general sense. For notation we refer to [4], however, we explain very briefly some essential points. Let $I = D^2 - q$ be the Sturm-Liouville operator acting on R and q be a bounded variation function such that the function $p(x) = (1 + |x|)q(x)$ is integrable on R . Define $M_w(R)$ to be the Banach space of measures on R with norm $\|\mu\|_\omega = \int_R w(t) d|\mu|(t)$, where w is a positive continuous increasing function on $[0, \infty]$. For each ψ , $T\psi$ is the unique solution of the system

$$(L \oplus I)u = (I \oplus L)u,$$

$$u(x, 0) = \psi(x), \left(\frac{\partial u}{\partial \nu}\right)(x, 0) = B\psi(x)$$

where I is an identity and B is the given operator which defines the boundary conditions. Since $T\psi$ is the unique solution of the above equation, $T\psi(x, 0) = \psi(x)$. We put the additional restriction $T\psi > 0$ whenever $\psi > 0$. Now we define $\lambda_{(x,y)}(\psi) = T\psi(x,y)$ and the convolution is defined by $\mu * \nu(\psi) = (\mu \oplus \nu)(T\psi)$. Thus R has a hypergroup structure with identity zero and weight norm defined by $\|\mu\| = \int$

$\omega(t) d|\mu|(t)$.

The algebra $L(R)$ is $\psi^1(R) \oplus \{k\delta_o \mid k \in \mathbb{C}\}$ where $\psi^1(R)$ consists of the absolutely continuous measures in $M_w(R)$. By our Theorem $L(R)$ has a bounded approximate identity and in this case $L(R)$ is a semi-simple algebra.

V. The most interesting hypergroups have invariant measures [3]. In this case $L(X)$ is $L^1(X, m) \oplus \{k\delta_e \mid k \in \mathbb{C}\}$ where e is the identity of X . Thus all of our results carry over to $L^1(X, m)$. The interesting question is: Under what conditions is $L^1(X, m)$ "is" a group algebra?

In [9] we have studied the second dual algebra of a hypergroup, and in other paper we have developed some isomorphism-problems on $L(X)$ the second dual of $L(X)$, which has not been appeared so far.

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