Contractibility and idempotents in Banach algebras

R. Alizadeh:Amirkabir University of Technology

Abstract

Let A be a Banach algebra. It is shown that a contractible ideal of a Banach algebra is complemented by its annihilator. Then, it is proved the existence of minimal central idempotents in a contractible Banach algebra with a nonzero character. Moreover, the notion of b-contractibility and one of its equivalent forms are introduced. Throgh an example, it is shown that b-contractibility is strictly weaker than contractibility.

Introduction

Taylor in [13, Theorem 5.11] showed that a contractible Banach algebra with bounded approximation property is finite dimensional. Johnson in [6, Proposition 8.1] showed that a contractible commutative semisimple Banach algebra is finite dimensional. Curtis and Loy [1, Theorem 6.2] extended this result by dropping the semisimplicity assumption. But the question for noncommutative case has remained open. For more results of this type see [4],[5], [8], [10], [13].

This paper is organized as follows. In the second section, we show that a contractible ideal of a Banach algebra is controlled by its commutant and annihilator. Then, we prove the existence of minimal central idempotents in a contractible Banach algebra with a nonzero character. In the third section, we introduce a weaker version of contractibility which we call b-contractibility. We give a characterization of b-contractibility analog to that of contractibility given by Taylor. Also, we show that b-contractibility is strictly weaker than contractibility.

2000 Mathematics Subject Classification, 46H10, 46H20.

Key words:Contractible Banach algebras, Minimal idempotent.

First we recall some terminology. Throughout this paper, A is a Banach algebra and A-module means Banach A-bimodule. For a subset E of A, E' is the commutant of E. If for every A-bimodule X every bounded derivation from A into X is inner, then A is called *contractible*. Also, the term "semisimple" means *Jacobson semisimple*. An idempotent $e \in A$ is called *minimial* if eAe is a division ring. If e and f are idempotents in A, we write $e \le f$ if $fe = ef = e$ holds. A nonzero idempotent $e \in A$ is called *primitive* if $0 \le f \le e$ imlies that $f = 0$ or $f = e$. Also, two idempotents e and f are said to be *orthogonal* if they satisfy $ef = fe = 0$. Let S be a subset of A. The *right annihilator* of S in A which we denote by $ran(S)$ is the set

$$ran(S) = \{a \in A : ba = 0 \text{ for } b \in S\}.$$

The left annihilator $lan(S)$ is defined semilarly. The *annihilator* of S is the set $Ann(S) = ran(S) \cap lan(S)$.

Contractibility

Theorem 2.1. Let A be a contractible Banach algebra which is an ideal in a Banach algebra B. Then $A + A' = B$.

Proof. If $A + A' \neq B$, then we can choose $b \in B - (A + A')$. Now define

$$D : A \to A, x \mapsto xb - bx.$$

Clearly D is a derivation on A. By assumption there exists an $a \in A$ such that $D(x) = xa - ax$ for all $x \in A$. The latter result implies that $b - a \in A'$ or equivalently $b \in A + A'$ which contradicts the selection of b. Therefore $A + A' = B$.

Theorem 2.2. Let A be a contractible Banach algebra which is an ideal in a Banach algebra B. Then $B = A \oplus \text{Ann}(A)$.

Proof. Since A is contractible then $M_2(A)$ with l^1-norm is contarctible, where $M_2(A)$ is the algebra of 2×2 matrices with the enteries from A. On the other hand $M_2(A)$ is an ideal in $M_2(B)$ and by Theorem 2.1 we have the equality $M_2(B) = M_2(A) + M_2(A)'$. One can easily observe that

$$M_2(A)' = \begin{bmatrix} A' & Ann(A) \\ Ann(A) & A' \end{bmatrix}.$$

Thus $B = A + Ann(A)$. But $A \cap Ann(A) = 0$, because A is unital. Therefore the identity $B = A \oplus Ann(A)$ holds.

Remark. In Theorems 2.1 and 2.2, A and B are related only algebrically. Indeed if there exists an infinite dimensional contractible Banach algebra A which is an ideal in a Banach algebra B, then the norm topology of A could be different from the relative norm topology of A which inherits from B.

Theorem 2.3. Let A be a contractible Banach algebra which admits a nonzero multiplicative linear functional f. Then A contains a central minimal idempotent.

Proof. Let $d = \sum_{n=1}^{\infty} a_n \otimes b_n$ be a diagonal for A and define

$$T : A, \to a \mapsto \sum_{n=1}^{\infty} < f, aa_n > b_n.$$

Since $\sum_n a_n b_n = 1$, then

$$< f, T(1) >=< f, \sum_n < f, a_n > b_n >= \sum_n < f, a_n >< f, b_n >$$
$$= \sum_n < f, a_n b_n >=< f, \sum_n a_n b_n >=< f, 1 >= 1.$$

Thus T(1) $\neq 0$. Moreover for every $a \in A$ and $g, h \in A^*$ we have

$$\begin{aligned} < h, \sum_n < g, aa_n > b_n >&= \sum_n < g, aa_n >< h, b_n > \\ &=< g \otimes h, \sum_n aa_n \otimes b_n > \\ &=< g \otimes h, \sum_n a_n \otimes b_n a > \\ &= \sum_n < g, a_n >< h, b_n a > \\ &=< h, \sum_n < g, a_n > b_n a > . \end{aligned}$$

This implies that

$$\sum_n < g, aa_n > b_n = \sum_n < g, a_n > b_n a.$$

Thus we assume that

T(1)=e, then we have $T(a) = \sum_n < f, aa_n > b_n = \sum_n < f, a_n > b_n a = ea.$ On the other hand we have $T(a) = \sum_n < f, aa_n > b_n =< f, a > \sum_n < f, a_n > b_n =< f, a > e.$ Hence T is an operator of rank one and $e^2 = T(e) =< f, e > e = e.$ Now define

$$T_1 : A \to A, a \mapsto \sum_n a_n < f, aa_n >.$$

With a similar argument we can show that

$$T_1(a) = ae' = <f, a> e' \quad a \in A$$

where $e' = T_1(1)$. Also we have $e'^2 = e'$ and $<f, e'> = 1$. Now the identities

$$ee' = <f, e'> e = e, \qquad ee' = <f, e> e' = e'$$

imply that $e = e'$ and for every $a \in A$ we have

$$ea = <f, a> e = <f, a> e' = ae' = ae.$$

Therefore e is a central idempotent. In addition since T is a rank one operator and $ranT = eAe$, then $eA = eAe = Ce$ is a division ring. Therefore e is a minimal idempotent.

b-Contractibility

Definition. Let A be a Banach algebra and π be the natural map,

$$\pi : A \otimes A \longrightarrow A, \qquad \pi(\sum_n a_n \otimes b_n) \to \sum_n a_n b_n.$$

Let $b \in A$ and X be an A-module. We say that a derivation $\delta\, A \longrightarrow X$ is a *b-derivation* if there exists another derivation $\delta'\, A \longrightarrow X$ such that $\delta = b\delta'$, where $(b\delta')(a) = b\delta'(a)$. Also we say that A is *b-contractible* if for every A-module X, every bounded b-derivation from A into X is inner. We call $d \in A \hat{\otimes} A$ a *b-diagonal* if $\pi(d) = b$ and $a.d = d.a$ for all $a \in A$.

Theorem 3.1. Let A be a unital Banach algebra and $b \in A' - \{0\}$. Then A is b-contractible if and only if A has a b-diagonal.

Proof. First suppose A is b-contractible and π is defined as above. Clearly $\ker \pi$ is an A-module and if we define

$$\delta : A \to \ker \pi, a \mapsto ab \otimes 1 - b \otimes a$$

then it is easy to see that δ is a b-derivation. Indeed $\delta = b\delta'$ where

$$\delta' : A \to \ker \pi, a \mapsto a \otimes 1 - 1 \otimes a$$

ince A is b-contractible, then threre exists an element $\sum_n c_n \otimes d_n \in \ker \pi$ such that

$$\delta(a) = \sum_n ac_n \otimes d_n - \sum_n c_n \otimes d_n a \quad a \in A.$$

Let $d = b \otimes 1 - \sum_n c_n \otimes d_n$. The above identities show that $\pi(d) = b$ and $a.d = d.a$ for all $a \in A$. Therefore, d is a b-diagonal for A.

Conversely suppose $d = \sum_n a_n \otimes b_n$ is a b-diagonal for A, X is an A-module and $\delta : A \longrightarrow X$ is a bounded derivation. Clearly the map

$$\psi : A \times A \to X, (a, c) \mapsto a\delta(c)$$

is a bounded bilinear map. So by the universal property of projective tensor product there is a linear map $\varphi : A \hat{\otimes} A \longrightarrow X$ such that $\varphi \circ \otimes = \psi$ that is $\varphi(a \otimes c) = a\delta(c)$. In particular using the fact that d is a b-diagonal for A, we get

$$\sum_n aa_n\delta(b_n) = \varphi(a.d) = \varphi(d.a) = \sum_n a_n\delta(b_n a), \quad a \in A.$$

Now if $x = \sum_n a_n\delta(b_n)$, then for every $a \in A$ we have

$$ax - xa = \sum_n aa_n\delta(b_n) - \sum_n a_n\delta(b_n)a = \sum_n aa_n\delta(b_n) + b\delta(a) - \sum_n a_n\delta(b_n a).$$

Thus the identity $ax - xa = b\delta(a)$ holds for every $a \in A$. Therefore every b-derivation is inner.

Example 3.2. Let A be the Banach algebra $l_1(N)$ with pointwise multiplication and $\{e_n\}$ be the standard basis for A.Then for every positive integer n, A is e_n-contractible. Indeed $e_n \otimes e_n$ is an e_n-diagonal for A. But A is not contractible, since it is not unital. Therefore b-contractibility dose not imply contractibility.

Remark. If A is contractible, then it is unital and one can easily observe that A is b-contractible for every $b \in A - \{0\}$. However the above example shows that for non-unital Banach algebras the converse is not true. We do not know whether this is true for unital Banach algebras or not.

Problem. Does there exist a unital Banach algebra which is b-contactible for some nonzero central idempotent b, but is not contractible?

References

1. P. C. Curtis, Jr and R. J. Loy, The structure of amenable Banach algebras, J. London Math. Soc. 40 (1989) 89-104.

2. G. H. Esslamzadeh Banach, algebra structure and amenability of a class of matrix algebras with applications, J. Funct. Anal. 161 (1999) 364–383 .

3. G. H. Esslamzadeh and T. Esslamzadeh, contractibility of ℓ^1-Munn algebras with applications, Semigroup Forum 63 (1) (2001) 1–10.

4. J. E. Gale, T. J. Ransford and M. C. White, Weakly compact homomorphisms, Trans. Amer. Math. Soc.331 (1992) 815–824.

5. N. Gronbaek, Various notions of amenability, a survey of problems, Proceedings of 13th International Conference on Banach Algebras in Blaubeuren, 1997, Walter de Gruyter, Berlin (1998) 535–547.

6. B. E. Johnson, Cohomology in Banach algebras, Mem. Amer. Math. Soc. 127 (1972).

7. B. E. Johnson, Approximate diagonals and cohomology of certain annihilator Banach algebras, Amer. J. Math. 94 (1972) 685–698.

8. A. Y. Helemskii, Flat Banach modules and amenable algebras, Trans. Moscow Math. Soc. (1984), Amer. Math. Soc. Translations (1985) 199–244 .

9. A. Y. Helemskii, The homology of Banach and topological algebras, (Translated from Russian), Klauwer (989)

10. A. J. Lazar, S. K. Tsui and S. Wright, A cohomological characterization of finite dimensional C^*-algebras, J. Operator Theory 14 (1985) 239–247.

11. T. W. Palmer, The Banach algebras and the general theory of $*$-algebras, Vol. I, algebras and Banach algebras, Cambridge Univ. Press(1994)

12. V. Runde, The structure of contractible and amenable Banach algebras, Proceedings of 13th International Conference on Banach Algebras in Blaubeuren, 1997, Walter de Gruyter, Berlin (1998) 415-430.

13. J. L. Taylor, Homology and cohomology for topological algebras, Adv. in Math. 9 (1972) 137–182.

14. Y. Zhang, Maximal ideals and the structure of contractible and amenable Banach algebras, Bull. Aust. Math. Soc. (2) 62 (2000) 221–226.