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Smoothness and Rotundity in Banach Spaces

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Abstract

The concept of rotundity is not far from differentiability. There are several papers in the literature devoted to the study of relations between rotundity and smoothness in Banach spaces. In this paper, we study new relations between some kinds of rotundity and smoothness in Banach spaces. In particular, we investigate relations between one kind of rotundity, which is called strongly very rotund, and very smoothness, in Banach spaces. A Banach space is rotund if the midpoint of every two distinct points of unit sphere is in the open unit ball of the Banach space. A Banach space X is smooth if its norm is Gateaux differentiable at every non-zero point of the space and it is very smooth if the norm is very Gateaux differentiable, that is, the second dual norm in the second dual of X is Gateaux differentiable at every non zero point of $\widehat{X}$.

Introduction

Let X be a real Banach space and $\|.\|$ be a norm on X. For $0 \neq x \epsilon X$ we define $D_x = \{f \epsilon X^*: f(x) = \|x\| \|f\|^*, \|f\|^* = \|x\|\}$, which is non-empty by the Hahn-Banach Theorem.

Definition1. The norm of X is Gateaux differentiable at $0 \neq x_0 \epsilon$ X if there exists $f \epsilon X^*$ such that

$$\lim_{t\to 0} \frac{\| x_0 + th \| - \| x_0 \|}{t} = f(h)$$

for all $h \epsilon X$.

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Let $0 \neq x\epsilon X$. It is obviouse that the norm is Gateaux differentiable at x if and only if the norm is Gateaux differentiable at $\frac{x}{\|x\|}$. It is proved in [1, page 5] that the norm is Gateaux differentiable at $\frac{x}{\|x\|}$ if and only if there is a unique fϵ S_{X^*} such that $f\left(\frac{x}{\|x\|}\right) = 1$ and hence the norm is Gateaux differentiable at $0 \neq x$ if and only if D_x contains only one point.

We say that the norm is smooth if the norm is Gateaux differentiable at all $0 \neq x\epsilon X$.

Definition2. The norm of X is very Gateaux differentiable at $0 \neq x\epsilon X$ if the norm of X^{**} is Gateaux differentiable at $\hat{x}$, where $\hat{x}(f) = f(x)$ for every $f\epsilon X^*$.

We say that the norm is very smooth if the norm is very Gateaux differentiable at all $0 \neq x\epsilon X$.

Definition3. The norm of X is Frechet differentiable at $0 \neq x_0\epsilon X$ if the norm is Gateaux differentiable at x_0 and the above limit is uniform for $h\epsilon S_X$, or equivalently (Smulyan Lemma, [2, Lemma 8.4]), if $\{f_n\}_{n\epsilon\mathbb{N}}$ and $\{g_n\}_{n\epsilon\mathbb{N}}$ are two sequences in X^* such that $lim_{n\to\infty} \| f_n \|^* = lim_{n\to\infty} \| g_n \|^* = \| x_0 \|$ and

$$lim_{n\to\infty} f_n(x_0) = lim_{n\to\infty} g_n(x_0) = \| x_0 \|^2, \text{ then } lim_{n\to\infty} \| f_n - g_n \|^* = 0.$$

Obviously very Gateaux differentiability implies Gateaux differentiability. It is shown in [6, Theorem3] that Frechet differentiability implies very Gateaux differentiability.

Definition4. The norm of X is rotund (R) if for $x, y\epsilon S_X$ such that $\| \frac{x+y}{2} \| = 1$, we have $x = y$. Geometrically this means that S_X has no non-trivial line segment. A Banach space with this property is called rotund.

Definition5. The norm of X is very rotund (VR) if for $0 \neq x\epsilon X, x^{**}\epsilon X^{**}$, and $f\epsilon D_x$ such that $\| x^{**} \|^{**} = \| x \|$, $x^{**}(f) = \| x \|^2$ we have $x^{**} = \hat{x}$.

If we divide x, x^{**}, f *by* $\| x \|$ we can see that this definition is equivalent to say that for $x\epsilon S_X, x^{**}\epsilon S_{X^{**}}$ and $f\epsilon D_x$, relation $x^{**}(f) = 1$, implies $x^{**} = \hat{x}$.

Definition6. The norm of X is strongly very rotund (SVR) if for $0 \neq x\epsilon X, x^{(4)}\epsilon X^{(4)}$ and $f\epsilon D_x$ such that $\| x^{(4)} \|^{****} = \| x \|, f(x) = x^{(4)}(\hat{f}) = \| x \|^2$ we have $x^{(4)} = \hat{\hat{x}}$, where $X^{(4)}$ is the fourth dual of X and $\hat{f}(x^{**}) = x^{**}(f)$ for every $x^{**}\epsilon X^{**}$.

If we devide $x, x^{(4)}$, f by $\| x \|$ we can see that this definition is equivalent to say that for $x \epsilon S_X, x^{(4)} \epsilon S_{X^{(4)}}$ and $f \epsilon D_x$, relation $x^{(4)}(\hat{f}) = 1$, implies $x^{(4)} = \hat{\hat{x}}$.

Definition7. The norm of X is locally uniformly rotund (LUR) if for $x \epsilon S_X$ and any sequence $\{x_n\}_{n \epsilon \mathbb{N}}$ in B_X such that $lim_{n\to\infty} \| \frac{x_n+x}{2} \| = 1$, we have $lim_{n\to\infty} \| x_n - x \| = 0$. In this definition $lim_{n\to\infty} \| \frac{x_n+x}{2} \| = 1$ implies that $\| x_n \| \to 1$. Hence if $lim_{n\to\infty} \| \frac{x_n+x}{2} \| = 1$, we have $\| \frac{\frac{x_n}{\|x_n\|}+x}{2} \| \to 1$ and if $\| \frac{x_n}{\|x_n\|} - x \| \to 0$ we have $lim_{n\to\infty} \| x_n - x \| = 0$. Therefore, this definition is equivalent to say that for $x \epsilon S_X$ and any sequence $\{x_n\}_{n \epsilon \mathbb{N}}$ in S_X such that $lim_{n\to\infty} \| \frac{x_n+x}{2} \| = 1$, we have $lim_{n\to\infty} \| x_n - x \| = 0$.

Example8. Let $X = \ell_2$ and $\|.\|_2$ be the canonical norm on X. By parallelogram law in the Hilbert space ℓ_2, we have $\| x+y \|_2^2 + \| x-y \|_2^2 = 2 \| x \|_2^2 + 2 \| y \|_2^2$ for every $x, y \epsilon X$. Let $x \epsilon S_X$ and $\{x_n\}_{n \epsilon \mathbb{N}} \subseteq S_X$ such that $lim_{n\to\infty} \| \frac{x_n+x}{2} \|_2 = 1$. Since $\| x_n - x \|_2^2 = 2 \| x_n \|_2^2 + 2 \| x \|_2^2 - \| x_n + x \|_2^2 = 4 - \| x_n + x \|_2^2$, we have $\| x_n - x \|_2 \to 0$. Therefore, $\|.\|_2$ is LUR on X.

Definition9. The norm of X is weakly locally uniformly rotund (WLUR) if for $x \epsilon S_X$ and any sequence $\{x_n\}_{n \epsilon \mathbb{N}}$ in B_X such that $lim_{n\to\infty} \| \frac{x_n+x}{2} \| = 1$, we have $lim_{n\to\infty} f(x_n - x) = 0$, for each $f \epsilon X^*$.

Definition10. The norm of X is weakly uniformly rotund (WUR) if for any two sequence $\{x_n\}_{n \epsilon \mathbb{N}}, \{y_n\}_{n \epsilon \mathbb{N}}$ in B_X such that $lim_{n\to\infty} \| \frac{x_n+y_n}{2} \| = 1$, we have $lim_{n\to\infty} f(x_n - y_n) = 0$, for each $f \epsilon X^*$.

By the same method applied after Definition 7, it is easy to prove that B_X can be replaced by S_X in the Definitions 9 and 10.

Let the norm of X be SVR and $x \epsilon S_X, x^{**} \epsilon S_{X^{**}}, f \epsilon D_x$ such that $x^{**}(f) = 1$. We have $\widehat{x^{**}} \epsilon X^{(4)}$ where $\widehat{x^{**}}(F) = F(x^{**})$ for every $F \epsilon X^{***}$, and $\| \widehat{x^{**}} \|^{****} = \| x \| = 1$, and $f(x) = \widehat{x^{**}}(\hat{f}) = \| x \|^2 = 1$, then $\widehat{x^{**}} = \hat{\hat{x}}$ which implies that $x^{**} = \hat{x}$. Therefore, the norm of X is VR. Let the norm of X be VR and $x, y \epsilon S_X$ such that $\| \frac{x+y}{2} \| = 1$. By the Hahn-Banach Theorem, there exists $f \epsilon S_{X^*}$ such that $f\left(\frac{x+y}{2}\right) = 1$, then we have $f(x) = f(y) = 1$. Since $f \epsilon D_x$, $\hat{y} \epsilon S_{X^{**}}$ and $\hat{y}(f) = f(y) = f(x) = 1$, we have $\hat{y} = \hat{x}$ and so $x = y$. Hence the norm of X is R. Consequently we have:

$$\text{SVR} \rightarrow \text{VR} \rightarrow \text{R}.$$

We will prove in the next section that in reflexive spaces the properties R, VR and SVR are equivalent.

Obviously we have the following implications:

$$\text{LUR} \rightarrow \text{WLU R} \rightarrow \text{R},$$

$$\text{WUR} \rightarrow \text{WLU R} \rightarrow \text{R}.$$

The converse of the above implications is not true in general. To see this, consider the following example:

Example11. Let $X = \ell_2$, $x = (x_1, x_2, \dots) \epsilon X$, $x' = (0, x_2, x_3, \dots)$ and $\| x \|_F = |x_1| + \| x' \|_2$. This is an equivalent norm on X. Let $\{\alpha_n\}_{n\epsilon\mathbb{N}}$ be a decreasing sequence of positive numbers converging to zero. Let $T: \ell_2 \longrightarrow \ell_2$, where $T(x_1, x_2, \dots) = (\alpha_1 x_1, \alpha_2 x_2, \dots)$. This is a continuous linear map. For $x \epsilon \ell_2$ define:

$$\| x \|_A^2 = \| x \|_F^2 + \| T(x) \|_2^2.$$

It is easy to check that this is an equivalent norm on ℓ_2. Moreover, it is shown in [5, Example 3] that $\| x \|_A$ is R norm on ℓ_2. Since ℓ_2 is reflexive, this norm is VR and SVR but it is shown in [7, Example] that this norm is not WLUR.

In [2], [6], [7] and [8] relations between rotundity and smoothness are investigated. We recall some of them:

Theorem12. [6, Lemma 1(2)] If $\|.\|^*$ is smooth on X^* then $\|.\|$ is VR on X.

Theorem13. [7, Theorem 1(3)] If $\|.\|$ is LUR on X then $\|.\|^*$ is Frechet differentiable on $D_X = \cup_{x\epsilon X} D_x$.

In the next section we investigate some other relations between rotundity and smoothness.

Results

Theorem1. If X is a reflexive space then R, VR and SVR are equivalent properties in X.

Proof. It suffices to show that R implies SVR. Let $\|.\|$ be R norm on X and $x \epsilon X, x^{(4)} \epsilon X^{(4)}$

and $f\epsilon X^*$ such that $\| x^{(4)} \|^{****}=\| x \|=\| f \|^*, f(x) = x^{(4)}(\hat{f}) = \| x \|^2$. Since X is reflexive we have $\hat{\hat{X}} = X^{(4)}$. Hence there exists $y\epsilon X$ such that $\hat{\hat{y}} = x^{(4)}$ and $\| x^{(4)} \|^{****}=\| y \|=\| x \|$. Since $f(y) = \hat{\hat{y}}(\hat{f}) = x^{(4)}(\hat{f}) = \| x \|^2= f(x)$ we have $f\left(\frac{x+y}{2}\right) =\| x \|^2$ which implies that $\| \frac{x+y}{2} \|=\| x \|$ and hence $\| \frac{\frac{x}{\|x\|}+\frac{y}{\|y\|}}{2} \|= 1$. Sine $\|.\|$ is R we have $\frac{x}{\|x\|} = \frac{y}{\|y\|}$, which implies that $x = y$ and so $\hat{\hat{x}} = \hat{\hat{y}} = x^{(4)}$. Consequently $\|.\|$ is SVR.

Theorem2. The norm $\|.\|$ is VR on X if and only if $\|.\|^*$ is Gateaux differentiable on $D_X = \cup_{x\epsilon X} D_x$

Proof. Let $\|.\|$ on X be VR and $f\epsilon D_X$. There exists $x\epsilon X$ such that $f\epsilon D_x$. Since $\hat{x}(f) =\| x \|\| f \|^*, \| \hat{x} \|^{**}=\| f \|^*$ we have $\hat{x}\epsilon D_f$. If $F\epsilon D_f$ then $F(f) = \hat{x}(f) =\| x \|^2$ and so $F = \hat{x}$. Consequently D_f contains only $\hat{x}$ and hence $\|.\|^*$ is Gateaux differentiable at f.

Conversely let $\|.\|^*$ be Gateaux differentiable on $D_X = \cup_{x\epsilon X} D_x$ and $f\epsilon D_x$ for some $x\epsilon X$. Let X^{**} *such that* $f(x) = F(f), \| F \|^{**}=\| x \|$. By Gateaux differentiability $\hat{x} = F$. Then $\|.\|$ on X is VR.

Let $\|.\|$ on X be R and consider $x\epsilon X$. If $f_x\epsilon D_x$ then $\hat{x}(f_x) =\| x \|\| f_x \|^*, \| \hat{x} \|^{**}=\| f_x \|^*$ and consequently $\hat{x}\epsilon D_{f_x}$. If $\hat{y}\epsilon \hat{X}$ such that $\hat{y}(f_x) =\| \hat{y} \|^{**}\| f_x \|^*$ *and* $\| \hat{y} \|^{**}=\| f_x \|^*$ then $\|x+y\|=\| \hat{x} + \hat{y} \|^{**}= 2 \| f_x \|^*$ that implies $\hat{x} = \hat{y}$ since $\|.\|$ is R. Therefore, $D_{f_x} \cap \hat{X} = \{\hat{x}\}$. However, we show in the next example that $D_{f_x} \cap X^{**}$ may contain more than one point, in fact, $\|.\|^*$ may not be Gateaux differentiable at f_x. Hence we cannot replace VR by R in Theorem2.

Example3. Let $X = \ell_\infty$ and $Y = c_0$ be a subset of X consisting of all sequences converging to zero. Then we have $Y^* = \ell_1$ [2, Proposition 2.14] and $X = Y^{**}$ [2, Proposition 2.15]. Let $\{f_n\}_{n\epsilon \mathbb{N}}$ be a dense subset of S_{Y^*}. *For* $x\epsilon X$ define $|x|^2 =\| x \|_\infty^2+ \sum_{i=1}^{\infty} 2^{-i} f_i^2(x)$. By use of induction we can easily prove that $|.|$ is a norm, since $|.|_i$ defined by $|x|_i^2 = 2^{-i} f_i^2(x)$ is a norm for each $i\epsilon \mathbb{N}$. The norm $|.|$ is an equivalent norm on Y *since* $\| x \|_\infty^2 \leq |x|^2 \leq 2 \| x \|_\infty^2$. It is proved in [1, page 66 Corollary 6.9(ii)] that $|.|$ is WUR norm on Y and we will prove in next theorem that if the norm $|.|$ is WUR on Y then $|.|^{**}$ is R on Y^{**}, but it is proved in [3] that there exists $f \in S_{Y^*}$ such that $|.|^{***}$ is not Gateaux differentiable at $\hat{f} \epsilon S_{Y^{***}}$. If $F \epsilon X$ such that $F \epsilon D_f$ then $\hat{f} \epsilon D_F$, which implies that $\hat{f} \epsilon D_X$. Hence $|.|^{***}$ is not Gateaux differentiable on D_X .

Remark4. The norm $|.|^{**}$ is R on ℓ_∞ but it is not VR, by Theorem2 and the last assertion in Example3.

Theorem5. If $\|.\|$ is WUR on X then $\|.\|^{**}$ is R on X^{**}.

Proof. Let $x^{**}, y^{**} \epsilon S_{X^{**}}$ such that $\| \frac{x^{**}+y^{**}}{2} \|^{**} = 1$. By Goldstine Theorem [2, Theorem 3.27] there exist a directed set Γ and two nets $\{x_\alpha\}_{\alpha \epsilon \Gamma}, \{y_\alpha\}_{\alpha \epsilon \Gamma} \subseteq B_X$ such that $\hat{x}_\alpha \to x^{**}$ and $\hat{y}_\alpha \to y^{**}$ in w^*-topology. Since $\|.\|^{**}$ is w^*-lower semicontinuous [2, Lemma 8.8], for each $\varepsilon > 0$ there exists $\alpha_\varepsilon \epsilon \Gamma$ such that

$$\| \hat{x}_\alpha + \hat{y}_\alpha \|^{**} > \| x^{**} + y^{**} \|^{**} - \varepsilon = 2 - \varepsilon$$

for every $\alpha > \alpha_\varepsilon$. From $\| \hat{x}_\alpha + \hat{y}_\alpha \|^{**} \leq 2$, it follows that $\| x_\alpha + y_\alpha \| = \| \hat{x}_\alpha + \hat{y}_\alpha \|^{**} \to 2$. Since each net contains a sequence as a subnet, there exists a sequence $\{x_{\alpha_n} + y_{\alpha_n}\}_{n \epsilon \mathbb{N}}$ such that $\| x_{\alpha_n} + y_{\alpha_n} \| \to 2$ and since the norm $\|.\|$ is WUR on X, we have $f(x_{\alpha_n} - y_{\alpha_n}) \to 0$ for every $f \epsilon X^*$, which implies that $\hat{x}_{\alpha_n} - \hat{y}_{\alpha_n} \to 0$ in w^*-topology. Therefore, $x^{**} = y^{**}$ and hence the norm on X^{**} is R.

Theorem6. The norm $\|.\|$ on X is SVR if and only if $\|.\|^*$ is very Gateaux differentiable on $D_X = \cup_{x \epsilon X} D_x$.

Proof. Let $\|.\|$ on X be SVR and $f \epsilon D_X$. Then there exists $x \epsilon X$ such that $f \epsilon D_x$. Hence $f(x) = \| f \|^* \| x \| = \| x \|^2$ since $\| f \|^* = \| x \|$. Since

$$\hat{\hat{x}}(\hat{f}) = f(x) = \| \hat{\hat{x}} \|^{****} \| \hat{f} \|^{***}, \| \hat{\hat{x}} \|^{****} = \| \hat{f} \|^{***}$$

it follows that $\hat{\hat{x}} \epsilon D_{\hat{f}}$. Let $F \epsilon X^{(4)}$ such that $F \epsilon D_{\hat{f}}$. Then

$$F(\hat{f}) = \| \hat{f} \|^{***} \| F \|^{****}, \| F \|^{****} = \| \hat{f} \|^{***} = \| f \|^*$$

Which implies that $F(\hat{f}) = \| x \|^2 = f(x)$ and so $F = \hat{\hat{x}}$ by the hypothesis. In fact, $D_{\hat{f}}$ contains only $\hat{\hat{x}}$ and consequently $\|.\|^*$ is very Gateaux differentiable at f. Conversely, let $\|.\|^*$ be very Gateaux differentiable on $D_X = \cup_{x \epsilon X} D_x$ and $f \epsilon D_x$ for some $x \epsilon X$.

Let $F\epsilon\, X^{(4)}$ such that $F(\hat{f}) = f(x), \| F \|^{****} = \| \hat{f} \|^{***} = \| f \|^{*}$. Then $F(\hat{f}) = \| \hat{f} \|^{***2}$ or $F\epsilon\, D_{\hat{f}}$. Since $\hat{\hat{x}}\epsilon\, D_{\hat{f}}$ and $D_{\hat{f}}$ consists of one point by the hypothesis, it follows that $F = \hat{\hat{x}}$, which implies that $\|.\|$ is SVR on X.

Theorem7. Let $X = c_0, x = \{x_i\}_{i\epsilon\, \mathbb{N}} \epsilon\, X$ and $\| x \|_r = \| x \|_\infty + (\sum_{i=1}^{\infty} 2^{-i} x_i^2)^{1/2}$. Then $\|.\|_r$ is an equivalent norm on X which is WLUR but it is not LUR.

Proof. It is easy to see that $\|.\|_r$ is a norm on X which is equivalent to $\| \|_\infty$. Let $x\epsilon\, S_X, \{x_n\}_{n\epsilon\mathbb{N}} \subseteq S_X$ such that $lim_{n\to\infty} \| \frac{x_n+x}{2} \|_r = 1$. For every $n\epsilon\, \mathbb{N}$, let $x_n' = \{2^{-\frac{i}{2}} (x_n)_i\}_{i\epsilon\, \mathbb{N}}$, $x' = \{2^{-\frac{i}{2}} x_i\}_{i\epsilon\, \mathbb{N}}$, then $x_n', x' \epsilon\, \ell_2$. For every $n\epsilon\, \mathbb{N}$, let

$$t_n = \| x_n \|_\infty + \| x \|_\infty - \| x_n + x \|_\infty,$$

$$t'_n = \| x_n \|_r + \| x \|_r - \| x_n + x \|_r = 2 - \| x_n + x \|_r,$$

$$t''_n = \| x'_n \|_2 + \| x' \|_2 - \| x'_n + x' \|_2.$$

By the hypothesis $t'_n \to 0$ and it is obvious that $t_n \geq 0$ for every $n\epsilon\, \mathbb{N}$.

We have

$$\| x_n' + x' \|_2 \leq \| x'_n \|_2 + \| x' \|_2 = 1 - \| x_n \|_\infty + 1 - \| x \|_\infty = 2 - \| x_n \|_\infty - \| x \|_\infty,$$

then

$$\| x_n + x \|_\infty = \| x_n + x \|_r - \| x'_n + x' \|_2 = 2 - t'_n - \| x'_n + x' \|_2$$

$$\geq 2 - t'_n - 2 + \| x_n \|_\infty + \| x \|_\infty = \| x_n \|_\infty + \| x \|_\infty - t'_n.$$

Hence $t_n \leq t'_n$ which implies that $t_n \to 0$. Since $t''_n = t'_n - t_n$, we have $t''_n \to 0$.

For $\varepsilon_0 = 1 - \| x \|_\infty > 0$ there exists $m_0 \epsilon\, \mathbb{N}$ such that $2 - \varepsilon_0 < \| x_n + x \|_r$ for every $n > m_0$. Since $\| x_n + x \|_r \leq 2 \| x_n + x \|_\infty$ for every $n\epsilon\, \mathbb{N}$, it follows that

$$\| x_n + x \|_\infty > 1 - \frac{\varepsilon_0}{2} \tag{1}$$

for every $n > m_0$. Since $x\epsilon\, c_0$ there exists $i_0 \epsilon\, \mathbb{N}$ such that $|x_i| < \frac{\|x\|_\infty}{2}$ for every $i > i_0$, which implies that

$$|(x_n + x)_i| = |(x_n)_i + x_i| \leq \| x_n \|_\infty + \frac{\| x \|_\infty}{2} \tag{2}$$

for every $i > i_0, n\epsilon\, \mathbb{N}$. Since $t_n \to 0$, there exists $m_1 \epsilon\, \mathbb{N}$ such that

$$\| x_n \|_\infty + \| x \|_\infty - \| x_n + x \|_\infty < \frac{\| x \|_\infty}{2} \tag{3}$$

for every $n > m_1$. Set $m = max\{m_0, m_1\}$ and let $n > m$. If there exists $i > i_0$ such that $\| x_n + x \|_\infty = |(x_n + x)_i|$, it follows from (2) and (3) that

$$\frac{\| x \|_\infty}{2} \leq \| x_n \|_\infty + \| x \|_\infty - \| x_n + x \|_\infty < \frac{\| x \|_\infty}{2},$$

which is a contradiction, hene there exists $i \leq i_0$ such that

$$\| x_n + x \|_\infty = |(x_n + x)_i|. \tag{4}$$

By (1) we have $|(x_n + x)_i| > 1 - \frac{\varepsilon_0}{2}$, which implies that

$$2^{\frac{i_0}{2}} \| x'_n \|_2 \geq 2^{\frac{i}{2}} \| x'_n \|_2 \geq |(x_n)_i| > 1 - \frac{\varepsilon_0}{2} - |x_i| \geq 1 - \frac{\varepsilon_0}{2} - \| \mathrm{x} \|_\infty = \frac{\varepsilon_0}{2}.$$

Hence $\| x'_n \|_2 \geq 2^{\frac{-i_0}{2}} \frac{\varepsilon_0}{2}$ for every $n > m$. If

$$a = min\,\{2^{\frac{-i_0}{2}} \frac{\varepsilon_0}{2}, \| x'_1 \|_2, \| x'_2 \|_2, \ldots, \| x'_m \|_2\}$$

then $\| x'_n \|_2 \geq a$ for every $n \epsilon\, \mathbb{N}$. For $n \epsilon\, \mathbb{N}$, if $\| x'_n \|_2 \leq \| x' \|_2$ then

$$\begin{aligned}2 \geq \left\| \frac{x'_n}{\| x'_n \|_2} + \frac{x'}{\| x' \|_2} \right\|_2 \geq \left\| \frac{x'_n}{\| x'_n \|_2} + \frac{x'}{\| x'_n \|_2} \right\|_2 \\ - \left\| \frac{x'}{\| x'_n \|_2} - \frac{x'}{\| x' \|_2} \right\|_2 = \frac{1}{\| x'_n \|_2} \| x'_n + x' \|_2 \\ - \| x' \|_2 \left(\frac{1}{\| x'_n \|_2} - \frac{1}{\| x' \|_2} \right) = \frac{1}{\| x'_n \|_2} (\| x'_n \|_2 + \| x' \|_2 - t''_n) \\ - \| x' \|_2 \left(\frac{1}{\| x'_n \|_2} - \frac{1}{\| x' \|_2} \right) = 2 - \frac{t''_n}{\| x'_n \|_2}\end{aligned}$$

and if $\| x' \|_2 < \| x'_n \|_2$ then

$$\begin{aligned}2 \geq \left\| \frac{x'_n}{\| x'_n \|_2} + \frac{x'}{\| x' \|_2} \right\|_2 \geq \left\| \frac{x'_n}{\| x' \|_2} + \frac{x'}{\| x' \|_2} \right\|_2 \\ - \left\| \frac{x'_n}{\| x'_n \|_2} - \frac{x'_n}{\| x' \|_2} \right\|_2 = \frac{1}{\| x' \|_2} \| x'_n + x' \|_2 \\ - \| x'_n \|_2 \left(\frac{1}{\| x' \|_2} - \frac{1}{\| x'_n \|_2} \right) = \frac{1}{\| x' \|_2} (\| x'_n \|_2 + \| x' \|_2 - t''_n) \\ - \| x'_n \|_2 \left(\frac{1}{\| \mathrm{x}' \|_2} - \frac{1}{\| x'_n \|_2} \right) = 2 - \frac{t''_n}{\| x' \|_2}.\end{aligned}$$

Therefore, we have $\left\| \frac{x'_n}{\|x'_n\|_2} + \frac{x'}{\|x'\|_2} \right\|_2 \to 2$ since $t''_n \to 0$. As we shown in Example8, the norm $\|.\|_2$ is LUR, then $\left\| \frac{x'_n}{\|x'_n\|_2} - \frac{x'}{\|x'\|_2} \right\|_2 \to 0$ or $\left\| x'_n - \frac{\|x'_n\|_2\, x'}{\|x'\|_2} \right\|_2 \to 0$. Suppose, without loss of generality, that there exists $t > 0$ such that $\| x'_n \|_2 \to t$, and hence $\| x'_n \|_2 \to \lambda \| x' \|_2$, where $\lambda = \frac{t}{\|x'\|_2}$. For every $n \epsilon\, \mathbb{N}$ if $\lambda_n = \frac{\|x'_n\|_2}{\|x'\|_2}$, then $\lambda_n \to \lambda$ and $\sum_{i=1}^{\infty} 2^{-i} (x_n - \lambda_n x)_i^2 \to 0$, which implies that $(x_n)_i \to \lambda x_i$ for every $i \epsilon\, \mathbb{N}$, and hence by (4) we have $\| x_n \|_\infty \to \lambda \| x \|_\infty$.

Therefore,

$$\| x_n \|_r = \| x_n \|_\infty + \| x'_n \|_2 \to \lambda \| x \|_\infty + \lambda \| x' \|_2 = \lambda \| x \|_r$$

which implies that $\lambda = 1$, since $\| x_n \|_r = \| x \|_r = 1$. Therefore, $lim_{n\to\infty}(x_n)_i - x_i = 0$ for every $i \epsilon \mathbb{N}$. Moreover, $|(x_n)_i - x_i| \leq 2$ for every $n, i \epsilon \mathbb{N}$ since $x, x_n \epsilon S_X$.

Let $0 \neq f \epsilon X^* = \ell_1$. Then $f = \{y_i\}_{i\epsilon \mathbb{N}}$, where $\sum_{i=1}^{\infty} |y_i| < \infty$. For $\varepsilon > 0$, there exists $j_0 \epsilon \mathbb{N}$ such that $\sum_{i=j_0+1}^{\infty} |y_i| < \frac{\varepsilon}{4}$ and $\alpha = \sum_{i=1}^{j_0} |y_i| > 0$. On the other hand there exists $n_0 \epsilon \mathbb{N}$ such that $|(x_n)_i - x_i| < \frac{\varepsilon}{2\alpha}$ for every $n > n_0$ and for each $i \leq j_0$. For $n > n_0$ we have

$$|f(x_n - x)| = |\sum_{i=1}^{\infty} y_i\,((x_n)_i - x_i)| \leq$$
$$\sum_{i=1}^{j_0} |y_i|\,|(x_n)_i - x_i| + \sum_{i=j_0+1}^{\infty} |y_i|\,|(x_n)_i - x_i| \leq \sum_{i=1}^{j_0} |y_i| \frac{\varepsilon}{2\alpha} + \frac{\varepsilon}{4} \times 2 \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, $f(x_n - x) \to 0$ and hence $\|.\|_r$ is WLUR on X.

To show that $\| \|_r$ is not LUR, let $e_n = (t_1, t_2, \dots) \epsilon X, n \epsilon \mathbb{N}$ such that $t_n = 1$ and $t_i = 0$ for $i \neq n$. Let $x = e_1, x_n = x + e_n, f \epsilon D_x$. Then

$$lim_{n\to\infty} \| \hat{x}_n \|_r^{**} = lim_{n\to\infty} \| x_n \|_r = \| x \|_r = \| \hat{x} \|_r^{**} = \| f \|_r^{*} = 1 + \frac{1}{\sqrt{2}}.$$

Since $f \epsilon X^* = \ell_1$, it has the form $f = (y_i)_{i\epsilon \mathbb{N}}$, where $\sum_{i=1}^{\infty} |y_i| < \infty$. Hence $f = \sum_{i=1}^{\infty} y_i \ \ e'_i$, where $(e'_i)_{i\epsilon \mathbb{N}}$ is the standard base of ℓ_1. It follows that $f(e_n) = \sum_{i=1}^{\infty} y_i e'_i(e_n) = y_n$ and so $f(e_n) \to 0$. Then

$$lim_{n\to\infty} \hat{x}_n(f) = lim_{n\to\infty} f(x_n) = f(e_1) = f(x) = \hat{x}(f) = \| x \|_r^2 = (\| f \|_r^*)^2.$$

But

$$lim_{n\to\infty} \| \hat{x}_n - \hat{x} \|_r^{**} = lim_{n\to\infty} \| x_n - x \|_r = lim_{n\to\infty} \| e_n \|_r = lim_{n\to\infty} \left(1 + 2^{-\frac{n}{2}}\right) = 1$$

and hence, by the definition $\|.\|_r^*$ is not Frechet differentiable at $f \epsilon D_X$. By Theorem 13, $\|.\|_r$ is not LUR.

Corollary8. We cannot set WLUR instead of LUR in Theorem13.

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