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Numerical Solution of Fractional Volterra Integro-differential Equations via the Rationalized Haar Functions

Y. Ordokhani*, N. Rahimi;

Department of Mathematics, Alzahra University

Abstract

In this paper rationalized Haar (RH) functions method is applied to approximate the numerical solution of the fractional Volterra integro-differential equations (FVIDEs). The fractional derivatives are described in Caputo sense. The properties of RH functions are presented, and the operational matrix of the fractional integration together with the product operational matrix is used to reduce the computation of FVIDEs into a system of algebraic equations. By using this technique for solving FVIDEs computation time is low. Numerical examples are given to demonstrate application of the presented method with RH functions base.

Introduction

In resent years, many important problems in fluid mechanics, viscoelasticity, electromagne-tics, chemistry, biology, physics, engineering and other areas of science can be modeled by fractional derivatives and integrals, see [1], [2]. In this work, we study numerical solution of FVIDEs of the type

$$ {}_{*}D^{\alpha}\, y(x) = f(x) + g(x)y(x) + \lambda \int_{0}^{x} k(x,t)G\big(t, y(t)\big)dt, \tag{1}$$

$$0 \le x \le 1, \qquad n-1 < \alpha \le n, \qquad n \in N,$$

with n initial condition

$$y^{i}(0) = \delta_i, \quad i = 0,1,2,3,\dots,n-1, \tag{2}$$

where ${}_{*}D^{\alpha}$ is Caputo's fractional derivative and α is a parameter describing the order of fractional derivative. Also, λ is a real known constant, $f,g \in L^2([0,1])$ and $k \in L^2([0,1]^2)$

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* Correspondence Author ordokhani@alzahra.ac.ir

are given functions, y(x) is the solution to be determined and G(t,y(t)) is an analytic function of the unknown function y(x). Such kind of equations arise in the mathematical modeling of various physical phenomena, such as heat encountered in combined conductions, convection and radiation problems [3], [4], [5]. Since, FVIDEs are usually difficult to be solved analytically, several methods have been used for the solution of FVIDES. Examples of such methods are, Adomian decomposition method (ADM) [6], [7], Spline collocation method [8], Fractional differential transform method [9], Homotopy pertubation method [10], Operational Tau method (OTM) [11] and other methods [12], [13], [14]. Ordokhani [15] has described the orthogonal set of Haar functions and transformed it to RH functions. In this method, we want to expand the ${}_{*}D^{\alpha}y(x)$ by RH functions with unknown coefficients and by using Newton-Cotes nodes [16] we can evaluate the unknown coefficients and find an approximate solution to Eq. (1).

The article is organized as follows:

In section 2, we will introduce some necessary definitions and preliminaries of the fractional calculus theory. We shall present the properties of RH functions required for our subsequent development in section 3. Section 4 is devoted to the solution of Eq. (1) by using RH functions, and in section 5 we will report our numerical findings and demonstrate the accuracy of the proposed method by numerical examples.

Definitions and preliminaries

In this section, we give some definitions and mathematics preliminaries of the fractional calculus theory.

Definition1. The Riemann-Liouville fractional integral operator of order α is defined as [2], [17]

$$I^{\alpha}y(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\,y(t)dt, \quad \alpha > 0, \quad x > 0,$$

$$I^0 y(x) = y(x),$$

whrere Γ(.) is Gamma function. It has the following properties:

$$I^{\alpha}x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}, \qquad \gamma > -1.$$

Definition2. The Caputo definition of fractional derivative operator is given by [18], [19]

$$_{*}D^{\alpha}\, y(x) = I^{n-\alpha}\, {}_{*}D^{n}\, y(x) = \frac{1}{\Gamma(n-\alpha)}\int_0^x (x-t)^{n-\alpha-1}\, y^{(n)}(t)dt,$$

where n-1 < $\alpha \leq$ n, $n \in N$, x > 0.

It has the following properties

$$_{*}D^{\alpha}I^{\alpha}\, y(x) = y(x),$$

$$I^{\alpha}\, {}_{*}D^{\alpha}\, y(x) = y(x) - \sum_{k=0}^{n-1} y^{(k)}\,(0^{+})\frac{x^k}{k!}, \qquad x > 0.$$

Properties of rationalized Haar functions

1. Rationalized Haar functions

The RH functions $h_r(x)$, r=1,2,..., are composed of three values +1, -1 and 0 can be defined on the interval [0,1) as [15]

$$h_r(x) = \begin{cases} 1, & J_1 \leq x < J_{1/2} \\ -1, & J_{1/2} \leq x < J_0 \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

where $J_u = \frac{j-u}{2^i}$, $u = 0, \frac{1}{2}, 1$.

The value of r is defined by two parameters i and j as

$$r = 2^i + j - 1, \quad i = 0,1,2,3,\dots, \quad j = 1,2,3,\dots,2^i,$$

$h_0(x)$ is defined for i=j=0 and given by

$$h_0(x) = 1, \qquad 0 \leq x < 1. \tag{4}$$

The orthogonality property of RH functions is given by

$$\int_0^1 h_r(x)\, h_v(x)dx = \begin{cases} 2^{-i}, & r = v, \\ 0, & r \neq v, \end{cases}$$

where

$$v = 2^n + m - 1, \qquad n = 0,1,2,3,\dots, \qquad m = 1,2,3,\dots,2^n.$$

2. Function approximation

A function $f(x) \in L^2([0,1])$ may be expanded into RH functions as [15]

$$f(x) = \sum_{r=0}^{\infty} c_r\, h_r(x), \tag{5}$$

where c_r given by

$$c_r = 2^i \int_0^1 f(x)\, h_r(x)\, dx, \qquad r = 0,1,2,\dots,$$

with $r=2^i+j-1$, $i=0,1,2,3,\dots$, $j=1,2,3,\dots,2^i$ and $r=0$ for $i=j=0$.

The series in Eq. (5) contains an infinite number of terms. If, we let $i=0,1,2,\dots,\beta$ then the infinite series in Eq. (5) is truncated up to its first m terms as

$$f(x) = \sum_{r=0}^{m-1} c_r \, h_r(x) = C_m^T H_m(x), \tag{6}$$

where, $m = 2^{\beta+1}$, $\beta = 0,1,2,3,\dots$.

The RH function coefficient vector C_m and RH function vector $H_m(x)$ are defined as

$$C_m = [c_0, c_1, \dots, c_{m-1}]^T, \tag{7}$$

$$H_m(x) = [h_0(x), h_1(x), \dots, h_{m-1}(x)]^T, \tag{8}$$

Also, we can expand the function $k(x,t) \in L^2([0,1]^2)$ into RH function as

$$k(x,t) = \sum_{v=0}^{m-1} \sum_{r=0}^{m-1} k_{vr}\, h_r(x)\, h_v(t)$$

where

$$k_{vr} = 2^{i+n} \int_0^1 \int_0^1 k(x,t)\, h_r(x)\, h_v(t) dx\, dt$$

Hence we have

$$k(x,t) = H_m^T(x)\, K\, H_m(t), \tag{9}$$

$$K = (k_{vr})_{m\times m}^T. \tag{10}$$

Taking the Newton-Cotes nodes as following [16]

$$x_i = \frac{2i-1}{2m}, \qquad i = 1,2,\dots,m. \tag{11}$$

The m-square Haar matrix $\Phi_{m\times m}$ can be expressed as

$$\Phi_{m\times m} = \left[H_m\left(\frac{1}{2m}\right), H_m\left(\frac{3}{2m}\right), \dots, H_m\left(\frac{2m-1}{2m}\right)\right], \tag{12}$$

for example if $m=8$ we have

$$\Phi_{8\times 8} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

Using Eq. (6) we get

$$\left[f\left(\frac{1}{2m}\right), f\left(\frac{3}{2m}\right), \dots, f\left(\frac{2m-1}{2m}\right)\right] = C_m^T \, \Phi_{m\times m}\,. \tag{13}$$

From Eqs. (10) and (13) we have

$$K = (\Phi_{m\times m}^{-1})^T \; \widetilde{K}\Phi_{m\times m}^{-1}\,, \tag{14}$$

where

$$\widehat{K} = (\hat{k}_{lp})m\times m\,, \qquad \hat{k}_{lp= k(\frac{2l-1}{2m},\frac{2p-1}{2m})}, \qquad p, l = 1,2,\dots, m.$$

3. Operational matrix of the fractional integration

The integration of the vector $H_m(x)$ defined in Eq. (8) can be expanded into Haar series with Haar coefficient matrix $P_{m\times m}$ as follows [15]

$$\int_0^x H_m\,(t)dt = P_{m\times m}\, H_m(x), \tag{15}$$

where $P_{m\times m}$ is called the RH functions operational matrix of integration. In this section our purpose is to derive the RH functions operational matrix of the fractional integration [12]. For this purpose, we consider an m-set of block-pulse functions as

$$b_i(x) = \begin{cases} 1, & \frac{i}{m} \le x < \frac{i+1}{m}, \\ 0, & \text{other wise,} \end{cases}$$

where i=0,1,2,…,m-1.

The function $b_i(x)$, are disjoint and orthogonal. That is

$$b_i(x)b_j(x) = \begin{cases} 0, & i \ne j, \\ b_i(x), & i = j, \end{cases}$$

$$\int_0^1 b_i(x)b_j(x)dx = \begin{cases} 0, & i \ne j, \\ \frac{1}{m}, & i = j, \end{cases}$$

The RH functions may be expanded into an m-set of block-pulse functions as

$$H_m(x) = \Phi_{m\times m}B_m(x), \tag{16}$$

where $B_m(x) = [b_0(x), b_1(x), ..., b_{m-1}(x)]^T$ and $\Phi_{m\times m}$ is an m×m matrix defined in Eq. (12).

In [20], Kilicman and Alzhour have given the block-pulse operational matrix of the fractional integration F^α as follows:

$$I^\alpha B_m(x) = F^\alpha B_m(x), \tag{17}$$

where

$$F^\alpha = \frac{1}{m^\alpha}\frac{1}{\Gamma(\alpha+1)}\begin{bmatrix} 1 & \xi_1 & \xi_2 & \xi_3 & \cdots & \xi_{m-1} \\ 0 & 1 & \xi_1 & \xi_2 & \ldots & \xi_{m-2} \\ 0 & 0 & 1 & \xi_1 & \ldots & \xi_{m-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & \xi_1 \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \tag{18}$$

with $\xi_k = (k+1)^\alpha - 2\,k^{\alpha+1} + (k-1)^{\alpha-1}$.

Next, we derive the RH function operational matrix of the fractional integration.

Let

$$I^\alpha H_m(x) = P^\alpha_{m\times m} H_m(x), \tag{19}$$

where $P^\alpha_{m\times m}$ is called the RH functions operational matrix of the fractional integration.

Using Eqs. (16) and (17), we have

$$I^\alpha H_m(x) = \Phi_{m\times m}B_m(x) = \Phi_{m\times m}I^\alpha B_m(x) = \Phi_{m\times m}F^\alpha B_m(x),$$

from (16) and (19), we get

$$P^\alpha_{m\times m}H_m(x) = P^\alpha_{m\times m}\Phi_{m\times m}B_m(x) = \Phi_{m\times m}F^\alpha B_m(x).$$

Then, $P^\alpha_{m\times m}$ is given by

$$P^\alpha_{m\times m} = \Phi_{m\times m}F^\alpha \Phi^{-1}_{m\times m}, \tag{20}$$

where, $\Phi^{-1}_{m\times m}$ is inverse of matrix $\Phi_{m\times m}$.

Therefore, we have found the operational matrix of fractional integration for RH functions.

For example, let m=4, then we have

$$P^\alpha_{4\times 4} = \frac{1}{4^\alpha}\frac{1}{\Gamma(\alpha+1)}\begin{bmatrix} \frac{3\xi_1}{4} + \frac{\xi_2}{2} + \frac{\xi_3}{2} + 1 & -\frac{\xi_1}{4} - \frac{\xi_2}{2} - \frac{\xi_3}{4} & -\frac{\xi_2}{2} & -\frac{\xi_3}{2} \\ \frac{\xi_1}{4} + \frac{\xi_2}{2} + \frac{\xi_3}{4} & \frac{\xi_1}{4} - \frac{\xi_2}{2} - \frac{\xi_3}{4} + 1 & -\frac{\xi_1}{2} & \xi_1 - \frac{\xi_3}{2} \\ \frac{\xi_3}{4} & \frac{\xi_1}{2} - \frac{\xi_3}{4} & 1 - \frac{\xi_1}{2} & \xi_2 - \frac{\xi_1}{2} - \frac{\xi_3}{4} \\ \frac{\xi_1}{4} & -\frac{\xi_1}{4} & 0 & 1 - \frac{\xi_1}{2} \end{bmatrix},$$

and for α=0.25, the operational matrix $P_{4\times4}^{0.25}$ can be expressed as following

$$P_{4\times4}^{0.25} = \begin{bmatrix} 0.8826 & -0.1404 & -0.1181 & -0.0433 \\ 0.1404 & 0.6018 & -0.1181 & 0.1928 \\ 0.0217 & 0.0964 & 0.5060 & -0.0420 \\ 0.0590 & -0.0590 & 0 & 0.5060 \end{bmatrix}.$$

3.4. The product operational matrix

In this work, it is necessary to evaluate the product of $H_m(x)$ and $H_m^T(x)$, that is called the product matrix of RH functions.

For this purpose, let

$$H_m(x)H_m^T(x)C_m = \tilde{C}_{m\times m}H_m(x), \tag{21}$$

where vector C_m is what defined in (7) and $\tilde{C}_{m\times m}$ is m×m-dimensional coefficient matrix.

Using (16), we have

$$H_m(x)H_m^T(x)C_m = \Phi_{m\times m}B_m(x)B_m^T(x)\Phi_{m\times m}^T C_m, \tag{22}$$

Let

$$C_m^* = \Phi_{m\times m}^T C_m = [c_0^*, c_1^*, \dots, c_{m-1}^*]^T. \tag{23}$$

From Eqs. (16), (21), (22) and (23), we have

$$H_m(x)H_m^T(x)C_m = \tilde{C}_{m\times m}H_m(x) = \Phi_{m\times m}\tilde{C}_{m\times m}^*\Phi_{m\times m}^{-1}H_m(x), \tag{24}$$

where $\tilde{C}_{m\times m}^* = \mathrm{diag}(c_0^*, c_1^*, \dots, c_{m-1}^*)$, is the product operational matrix of block-pulse functions.

Therefore, we have the coefficient matrix as $\tilde{C}_{m\times m} = \Phi_{m\times m}\tilde{C}_{m\times m}^*\Phi_{m\times m}^{-1}$.

For m=4 we have

$$\tilde{C}_{4\times4} = \begin{bmatrix} c_0 & c_1 & c_2 & c_3 \\ c_1 & c_0 & c_2 & -c_3 \\ \frac{c_2}{2} & \frac{c_2}{2} & c_0 + c_1 & 0 \\ \frac{c_3}{2} & -\frac{c_3}{2} & 0 & c_0 - c_1 \end{bmatrix}.$$

Solution of FVIDEs

In this section we consider the FVIDEs given in Eq. (1). To solve for y(x), we first approxmate ${}_*D^{\alpha}\, y(x)$ as

$${}_*D^{\alpha}\, y(x) \cong C_m^T H_m(x), \tag{25}$$

where C_m is the RH functions coefficient vector and $H_m(x)$ is RH functions vector.By using the initial conditions in Eq. (2) and Eqs. (19), (25) and properties of Caputo derivative we have

$$
\begin{gathered}
{}_*D^{n-1}\, y(x) = I^{n-\alpha+1}\, {}_*D^{\alpha}\, y(x) + \square^{(\square-1)}(0) \cong C_m^T P_{m\times m}^{\alpha-n+1} H_m(x) + \delta_{n-1}, \\
{}_*D^{n-2}\, y(x) \cong \left(C_m^T P_{m\times m}^{\alpha-n+2} + \delta_{n-1} e^T \Phi_{m\times m}^{-1} P_{m\times m}^{1}\right) H_m(x) + \delta_{n-2}, \\
\vdots \\
y(x) \cong (C_m^T P_{m\times m}^{\alpha} + e^T \Phi_{m\times m}^{-1} \textstyle\sum_{i=0}^{n-1} \delta_i\, (P_{m\times m}^{1})^{i}) H_m(x) \qquad (26)
\end{gathered}
$$

where

$e=(1,1,...,1)^T$, $P_{m\times m}^{1}$ is operational matrix of RH functions defined in Eq. (15) and $P^0=I$ is $m\times m$-dimensional identity matrix.

Also, we let

$$z(x) = G(x, y(x)). \qquad (27)$$

Suppose z(x), f(x), g(x) and k(x,t) can be expressed approximately as

$$z(x)=Z_m^T H_m(x), \quad f(x)= F_m^T H_m(x), \quad g(x)= G_m^T H_m(x), \quad k(x,t)=H_m^T(x)\, K\, H_m(t), \qquad (28)$$

where Z_m, F_m, G_m and K are given in Eqs. (6) and (14) respectively.

Using Eqs. (9), (15), (24) and (28), we have

$$\int_0^x k(x,t) G(t,y(t)) dt = \int_0^x H_m^T(x)\, K\, H_m(t) H_m^T(t)\, Z_m dt = H_m^T(x)\, K\, \tilde{Z}_m P_{m\times m} H_m(x), \qquad (29)$$

Let

$$A_m = (C_m^T\, P_{m\times m}^{\alpha} + e^T \Phi_{m\times m}^{-1} \sum_{i=0}^{n-1} \delta_i\, (P_{m\times m}^{1})^{i})^{T}, \qquad (30)$$

with substituting Eqs. (25), (26), (28), (29) and (30) in Eq. (1), we have

$$C_m H_m(x) = F_m^T H_m(x) + G_m^T H_m(x) H_m^T(x) A_m + \lambda H_m^T K Z_m P_{m\times m} H_m(x), \qquad (31)$$

by using (24), Eq. (31) can be written as

$$C_m H_m(x) = F_m^T H_m(x) + G_m^T \tilde{A}_m H_m(x) + \lambda H_m^T(x) K \tilde{Z}_m P_{m\times m} H_m(x), \qquad (32)$$

from Eqs. (26), (27), (28) and (30) we get

$$Z_m^T\, H_m(x) = G\left(x, A_m^T\, Hm\,(x)\right). \qquad (33)$$

In order to construct the approximation for y(x) we collocate Eqs. (32) and (33) in m points. For a suitable collocation points we choose Newton-Cotes nodes defined in Eq. (11). By using Eqs. (8), (11) and (12) we have

$$H_m(x_i) = \Phi_{m\times m} e_i, \qquad i = 1,2,\dots,m,$$

where

$$e_{i}=(\underbrace{0,0,\dots,0}_{i-1},1,\underbrace{0,\dots,0}_{m-i})^{T}.$$

Then, Eqs. (32) and (11) can be expressed as

$$\begin{cases} C_m^T\Phi_{m\times m}e_i = F_m^T\Phi_{m\times m}e_i + G_m^T\,\widetilde{A}_m\Phi_{m\times m}e_i + \lambda\, e_i^T\Phi_{m\times m}^T K\,\widetilde{Z}_m P_{m\times m}\Phi_{m\times m}e_i \\ Z_m^T\,\Phi_{m\times m}e_i = G\left(x_i, A_m^T\,\Phi_{m\times m}e_i\right), \\ \qquad\qquad i = 0,1,2,3,\dots,m. \end{cases} \tag{34}$$

Therefore, we convert Eq. (1) to the systems of algebraic equations. Eq. (34) can be solved for the unknowns C_m and Z_m, then the required approximation to the solution y(x) in Eq. (1) is obtained.

Numerical examples

In this section, we apply the present method and solve some examples that were given in different papers. All calculations were performed using MATLAB software.

Example 1. Consider the following nonlinear FVIDE ([11])

$$_{*}D^{\alpha}\, y(x) = x^{3}(-1+e^{\sin x})-\sin x-\int_0^x x^3\cos t\; e^{y(t)}dt, \qquad x > 0, \quad 1 < \alpha \le 2,$$

with the initial conditions:

$$y(0) = 0, \qquad y'(0) = 1.$$

The only case which we know the exact solution for $\alpha=2$ is $y(x)=\sin x$.

We have solved this example for m=128 for different α and have compared it with OTM method [11]. The comparison is shown in Table 1 and Table 2.

Note that in the theory of fractional calculus, it is obvious that as α ($n-1< \alpha \le n$) approaches to positive integer number n, then the numerical solution continuously converges to the exact solution of the problem with derivation n. i.e. in the limit, the solution of fractional equations approaches to integer-order equations [11], [19].

Table1. Comparison of the solution of OTM and RH for different α of example 1

x	$\alpha = 1.25$		$\alpha = 1.5$		$\alpha = 1.75$		Exact
	y_{OTM}	y_{RH}	y_{OTM}	y_{RH}	y_{OTM}	y_{RH}	for α=2
0.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.097355	0.097793	0.098677	0.099047	0.099419	0.099597	0.099833
0.2	0.185847	0.189532	0.193716	0.194627	0.196812	0.197299	0.198669
0.3	0.272375	0.274037	0.283807	0.285247	0.290975	0.291790	0.295520
0.4	0.347526	0.350668	0.367799	0.369858	0.380815	0.381966	0.389418
0.5	0.412246	0.419073	0.444601	0.447660	0.465333	0.466862	0.479425
0.6	0.464013	0.479133	0.513081	0.518045	0.543605	0.545631	0.564642
0.7	0.499207	0.531011	0.571968	0.580614	0.614770	0.617556	0.644217
0.8	0.512786	0.5751171	0.619749	0.635167	0.678009	0.682040	0.717356
0.9	0.497956	0.612450	0.654568	0.681736	0.732528	0.738624	0.783326
1.0	0.445845	0.644121	0.674130	0.720628	0.777544	0.787000	0.841470
CPU	-	7.7788 s	-	6.2642 s	-	6.0147 s	-

Table2. Comparison of the solution of OTM and RH of example 1

x	$\alpha = 2$		$\alpha = 2$	
	y_{OTM}	error	y_{RH}	error
0.0	0.000000	0	0.000000	0
0.2	0.198669	0	0.198669	0
0.4	0.389418	0	0.389418	0
0.6	0.564648	6.0000×10^{-6}	0.564640	2.9187×10^{-6}
0.8	0.717397	5.1000×10^{-5}	0.717350	5.5916×10^{-6}
1.0	0.841666	1.9600×10^{-4}	0.841471	4.7900×10^{-7}
CPU	-	-	5.6256S	-

The results in Table 1 show as $\alpha \to 2$, numerical results tend to exact solution of $\alpha = 2$. From Table 2 we conclude that approximate results with present method is in good agreement with the exact solution when $\alpha = 2$. So, for cases $\alpha = 1.25$, $\alpha = 1.5$ and $\alpha = 1.75$ that the exact solution is unknown present method is reliable tool.

Example 2. Consider the nonlinear FVIDE ([7,21,22])

$$ {}_{*}D^{\alpha}\, y(x) = 1 + \int_0^x e^{-t}\, y(t)^2\, dt, \qquad 0 \le x \le 1 \quad 3 < \alpha \le 4, $$

with the boundary condition:

$$ y(0) = y'(0) = 1, $$

$$ y(1) = y'(1) = e, $$

The only case which we know the exact solution is for α=4 and given by $y(x) = e^x$. We have solved this example for m=128 for different α and have compared it with methods of [22]. The comparison is shown in Table 3 and Table 4.

Table3. Approximate and exact solutions for different α of example 3

x	Method of [22]			Present Method			Exact for α = 4
	α = 3.25	α = 3.5	α = 3.75	α = 3.25	α=3.5	α=3.75	
0.0	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
0.1	1.106551	1.106750	1.106151	1.105236	1.105194	1.105178	1.105170
0.2	1.223931	1.224324	1.223227	1.222011	1.221655	1.221484	1.221402
0.3	1.353200	1.353756	1.352308	1.352093	1.350861	1.350204	1.349858
0.4	1.495600	1.496270	1.494636	1.497436	1.494480	1.492783	1.491824
0.5	1.652553	1.653273	1.651615	1.660166	1.651950	1.650830	1.648721
0.6	1.825654	1.826354	1.824824	1.842597	1.832533	1.826128	1.822118
0.7	2.016687	2.017294	2.0160234	2..047219	2.031227	2.020643	2.013752
0.8	2.227634	2.228084	2.227176	2.276726	2.252871	2.236539	2.225540
0.9	2.460690	2.460931	2.460458	2.534034	2.500118	2.476198	2.459603
1.0	2.718281	2.718281	2.718281	2.822302	2.775866	2.742238	2.718282
CPU	-	-	-	7.7788s	-	6.0147 s	-

Table4. Comparison of present method and method of [22] in case *a*=4

x	Method of [22]		Present Method	
	Numerical solution	Absolute error	Numerical solution	Absolute error
0.0	1.000000	0	1.000000	0
0.1	1.105160	1.0000×10^{-5}	1.105171	9.2860×10^{-10}
0.2	1.221382	2.0000×10^{-5}	1.221403	5.4492×10^{-8}
0.3	1.349829	2.9000×10^{-5}	1.349859	1.3930×10^{-7}
0.4	1.491788	3.6000×10^{-5}	1.491825	5.0993×10^{-8}
0.5	1.648681	4.0000×10^{-5}	1.648721	6.2063×10^{-7}
0.6	1.822078	4.0000×10^{-5}	1.822119	1.9676×10^{-7}
0.7	2.013716	3.6000×10^{-5}	2.013754	1.1158×10^{-6}
0.8	2.225513	2.7000×10^{-5}	2.225543	1.6183×10^{-6}
0.9	2.459587	1.6000×10^{-5}	2.459604	8.9789×10^{-7}
1.0	2.718282	0	2.718280	2.1930×10^{-6}
CPU	-	-	5. 6734s	-

Numerical results in Table 4 show our numerical solutions using the RH functions is more accurate than the numerical solutions obtained using the method of [22]. Therefore, we conclude that the solutions for α = 3.25, α = 3.5 and α = 3.75 that show in Table 3 are also credible. In Table 3 as α→ *4* numerical results tend to be the exact solution of α = 4 but numerical results of method of [22] do not have this property. Although, for obtaining a good accuracy with our method number of values must be very large but in this method computation time is very low..

Example3. Consider the following FVIDE

$$_{*}D^{\frac{3}{2}}\, y(x) = f(x) - \int_0^x \cos(x-t)\, y(t)^3 dt,$$

$$y(0) = y'(0) = 0,$$

where $f(x) = 720\,(x - \sin x) - 120\,x^3 + 6\,x^5 + \frac{2}{\Gamma(1.5)}\,x^{\frac{1}{2}}$ and the exact solution is $y(x) = x^2$.

We have solved this example for different m . The absolute error in Table 5 shows that the accuracy improves with increasing the m.

Table5. Absolute error for different m of example 3

x	m=8	m=16	m=32	m=64
0.0	0	0	0	0
0.1	2.0042×10^{-3}	1.3790×10^{-3}	2.7975×10^{-4}	1.3648×10^{-4}
0.2	55160×10^{-3}	1.1190×10^{-3}	5.4594×10^{-4}	1.3911×10^{-4}
0.3	6.2680×10^{-3}	1.3709×10^{-3}	6.3356×10^{-4}	1.6980×10^{-4}
0.4	4.4735×10^{-3}	2.8321×10^{-3}	5.5630×10^{-4}	2.3325×10^{-4}
0.5	4.8269×10^{-3}	1.7195×10^{-3}	6.0926×10^{-4}	2.1538×10^{-4}
0.6	5.4539×10^{-3}	2.5265×10^{-3}	6.7714×10^{-4}	2.7592×10^{-4}
0.7	8.2439×10^{-3}	2.0628×10^{-3}	8.7750×10^{-4}	2.5602×10^{-4}
0.8	8.5314×10^{-3}	2.1743×10^{-3}	9.1878×10^{-4}	2.7110×10^{-4}
0.9	6.2813×10^{-3}	2.8480×10^{-3}	7.9774×10^{-4}	3.2023×10^{-4}
1	6.0884×10^{-3}	2.2171×10^{-3}	7.9890×10^{-4}	2.8596×10^{-4}
CPU	0.1533s	1.4405 s	1.8215 s	2.4966s

Conclusion

In the present work RH functions are used to solve the FVIDEs. We reduce the FVIDEs to a system of algebraic equations via the RH functions and collocation points. In this method time computations is short, because the matrix $\Phi_{m\times m}$ introduces in Eq. (12) contain many zeros, and these zeros make the RH functions fast and easy to use. Numerical examples with satisfactory results are given to demonstrate it is a useful tool for solving the FVIDEs.

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