Formal Local Cohomology Modules and Serre Subcategories

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Abstract

Let \( (R, m) \) be a Noetherian local ring, \( a \) an ideal of \( R \) and \( M \) a finitely generated \( R \)-module. We investigate some properties of formal local cohomology modules with respect to a Serre subcategory. We provide a common language to indicate some properties of formal local cohomology modules.

1. Introduction

Throughout this paper \((R, m)\) is a commutative Noetherian local ring, \( a \) an ideal of \( R \) and \( M \) is a finitely generated \( R \)-module. For an integer \( i \in \mathbb{N}_0 \), \( H^i_a(N) \) denotes the \( i \)-th local cohomology module of \( M \) with respect to \( a \) as introduced by Grothendieck (cf. [1], [2]).

We shall consider the family of local cohomology modules \( \{H^i_m\left(\frac{M}{a^iM}\right)\}_{i \in \mathbb{N}} \) for a non-negative integer \( i \in \mathbb{N}_0 \). With natural homomorphisms; this family forms an inverse system. Schenzel introduced the \( i \)-th formal local cohomology of \( M \) with respect to \( a \) in the form of \( f^i_a(M) := \varprojlim_{n \in \mathbb{N}} H^i_m\left(\frac{M}{a^nM}\right) \), which is the \( i \)-th cohomology module of the \( a \)-adic completion of the \( \check{\text{C}}ech \) complex \( \check{\mathcal{C}}_x \otimes_R M \), where \( x \) denotes a system of elements of \( R \) such that \( \text{Rad}(\ x, R ) = m \) (see [3, Definition 3.1]). He defines the formal grade as \( f.\text{grade}(a, M) = \inf \{i \in \mathbb{N}_0 \mid f^i_a(M) \neq 0\} \). For any ideal \( a \) of \( R \) and finitely generated \( R \)-module \( M \) the following statements hold:

(i) (See [3, Theorem 3.11]). If \( 0 \to M' \to M \to M'' \to 0 \) is a short exact sequence of finitely generated \( R \)-modules, then there is the following long exact sequence:

\[ \cdots \to f^i_a(M') \to f^i_a(M) \to f^i_a(M'') \to \cdots \]

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(ii) (See [3, Theorem 1.3]). \( f.\ grade(\ a, M) \leq \dim(M) - cd(\ a, M) \); some properties of formal local cohomology have been presented in [3, 4, 5 and 6].

Throughout this paper \( \mathcal{S} \) denotes a Serre subcategory of the category of \( R \)-modules and \( R \)-homomorphisms (we recall that a class \( \mathcal{S} \) of \( R \)-modules is a Serre subcategory of the category of \( R \)-modules and \( R \)-homomorphisms if \( \mathcal{S} \) is closed under taking submodules, quotients and extensions).

Our paper contains three sections. In Section 2, we shall define the formal grade of \( a \) with respect to \( M \) in \( \mathcal{S} \) as the infimum of the integers \( i \) such that \( f_a^i(M) \notin \mathcal{S} \) and is denoted by \( f.\ grade_\mathcal{S}(\ a, M) \). (See definition 2.1). Then we shall obtain some properties of this notion. We show that if \( \Gamma_a(M) \) is a pure submodule of \( M \), then
\[
\text{Hom}_R\left( \frac{R}{m}, f_a^t\left( \frac{M}{\Gamma_a(M)} \right) \right) \in \mathcal{S},
\]
where \( t = f.\ grade_\mathcal{S}(\ a, M) \).

In Section 3, we shall define the formal cohomological dimension of \( a \) with respect to \( M \) in \( \mathcal{S} \) as the supremum of the integers \( i \) such that \( f_a^i(M) \notin \mathcal{S} \) and is denoted by \( f.\ cd_\mathcal{S}(\ a, M) \). (See definition 3.1). The main result of this section is that if \( f_a^i(M) \in \mathcal{S} \) and \( H_m^i(M) \in \mathcal{S} \) for all \( i > t \), then \( \frac{R}{a} \otimes_R f_a^i(M) \) belongs to \( \mathcal{S} \).

### 2. The formal grade of a module in a Serre subcategory

**Definition 2.1.** The formal grade of \( a \) with respect to \( M \) in \( \mathcal{S} \) is the infimum of the integers \( i \) such that \( f_a^i(M) \notin \mathcal{S} \) and is denoted by \( f.\ grade_\mathcal{S}(\ a, M) \).

**Proposition 2.2.** Let \(( R, m )\) be a local ring and \( a \) be an ideal of \( R \). If \( 0 \to L \to M \to N \to 0 \) is an exact sequence of finitely generated \( R \)-modules, then the following statements hold.

(a) \( f.\ grade_\mathcal{S}(\ a, M) \geq \min\{ f.\ grade_\mathcal{S}(\ a, L), f.\ grade_\mathcal{S}(\ a, N) \} \).

(b) \( f.\ grade_\mathcal{S}(\ a, L) \geq \min\{ f.\ grade_\mathcal{S}(\ a, M), f.\ grade_\mathcal{S}(\ a, N) + 1 \} \).

(c) \( f.\ grade_\mathcal{S}(\ a, N) \geq \min\{ f.\ grade_\mathcal{S}(\ a, L) - 1, f.\ grade_\mathcal{S}(\ a, M) \} \).

**Proof.** According to [3, Theorem 3.11], the above short exact sequence induces the following long exact sequence.

\[
\cdots \to f_a^{i-1}(N) \to f_a^i(L) \to f_a^i(M) \to f_a^i(N) \to f_a^{i+1}(L) \to \cdots.
\]

So, the result follows.

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Corollary 2.3. If \( \underline{x} = x_1, \ldots, x_n \) is a regular \( M \)-sequence, then \( f.\text{grade}_{S} \left( a, \frac{M}{x^{M}} \right) \geq f.\text{grade}_{S} (a,M) - n \).

**Proof.** Consider the following exact sequence \((n \in \mathbb{N})\)
\[
0 \to \frac{M}{(x_1, \ldots, x_{n-1})M} \xrightarrow{x_n} \frac{M}{(x_1, \ldots, x_n)M} \xrightarrow{\text{nat.}} \frac{M}{(x_1, \ldots, x_n)M} \to 0
\]
whenever \( n = 1 \) by \((x_1, \ldots, x_{n-1})M \) we means 0.

Corollary 2.4. Let \( a \) and \( b \) be ideals of \( R \). Then
(a) \( f.\text{grade}_{S} (a \cap b, M) \geq \min\{f.\text{grade}_{S} (a,M), f.\text{grade}_{S} (b,M), f.\text{grade}_{S} ((a,b), M) + 1\} \).
(b) \( f.\text{grade}_{S} ((a,b), M) \geq \min\{f.\text{grade}_{S} (a \cap b,M) - 1, f.\text{grade}_{S} (a,M), f.\text{grade}_{S} (b,M)\} \).

**Proof.** For all \( n \in \mathbb{N} \) there is a short exact sequence as follows:
\[
0 \to \frac{M}{a^nM \cap b^nM} \to \frac{M}{a^nM} \oplus \frac{M}{b^nM} \to \frac{M}{(a^n, b^n)M} \to 0.
\]
By using [3,Theorem 5.1], the above exact sequence induces the following long exact sequence.
\[
\cdots \to \lim_{n \to \infty} H_{m}^{i} \left( \frac{M}{(a^n, b^n)M} \right) \to \lim_{n \to \infty} H_{m}^{i} \left( \frac{M}{a^nM} \right) \oplus \lim_{n \to \infty} H_{m}^{i} \left( \frac{M}{b^nM} \right) \to \lim_{n \to \infty} H_{m}^{i} \left( \frac{M}{(a,b)^nM} \right) \to \cdots
\]
So by using an argument similar to that of Proposition 2.2, the result follows.

Corollary 2.5. Assume that \( M \) is a finitely generated \( R \)-module and \( N_1 \) and \( N_2 \) are submodules of \( M \). Then considering the exact sequence \( 0 \to \frac{M}{N_1} \oplus \frac{M}{N_2} \to \frac{M}{N_1 + N_2} \to 0 \) we shall have
(a) \( f.\text{grade}_{S} \left( a, \frac{M}{N_1 \cap N_2} \right) \geq \min\{f.\text{grade}_{S} (a, M), f.\text{grade}_{S} (a, MN_2), f.\text{grade}_{S} (a, MN_1 + N_2 + 1)\} \).
(b) \( f.\text{grade}_{S} \left( a, \frac{M}{N_1 + N_2} \right) \geq \min\{f.\text{grade}_{S} (a, MN_1), f.\text{grade}_{S} (a, MN_2)\} - 1, f.\text{grade}_{S} (a, MN_1), f.\text{grade}_{S} (a, MN_2) \).

Theorem 2.6. Let \( a \) be an ideal of a local ring \((R, m)\), \( M \) be a finitely generated \( R \)-module and \( L \) be a pure submodule of \( M \). Then \( f.\text{grade}_{S} (a, L) \geq f.\text{grade}_{S} (a, M) \) where \( S \) is a Serre subcategory of the category of \( R \)-modules and \( R \)-homomorphisms. In particular, \( \inf \{i | H_{m}^{i}(L) \notin S \} \geq \inf \{i | H_{m}^{i}(M) \notin S \} \).
Proof. Let $L$ be a pure submodule of $M$. So \( \frac{L}{a^n L} \to \frac{M}{a^n M} \) is pure for each $n \in \mathbb{N}$. Now according to [8, Corollary 3.2 (a)], \( H^i_m \left( \frac{L}{a^n L} \right) \to H^i_m \left( \frac{M}{a^n M} \right) \) is injective. Since inverse limit is a left exact functor, \( f^i_a(L) \) is isomorphic to a submodule of \( f^i_a(M) \). Consequently, \( f.g. \geq \). If $a = 0$ then, \( f.g. = \inf \{ i | H^i_m(M) \notin S \} \) and the result follows.

**Corollary 2.7.** If $0 \to L \to M \to N \to 0$ is a pure exact sequence of finitely generated $R$-modules, then \( \min \{ f.g. + 1 \} \geq f.g. \).

**Proof.** Since $L$ is a pure submodule of $M$, as a result of the previous theorem, \( f.g. \geq f.g. \). Hence we must prove that \( f.g. + 1 \geq f.g. \). We assume that $i < f.g.$ and we show that $i < f.g. + 1$. Consider the following long exact sequence.

$$
\ldots \to f^i_a(N) \to f^i_a(L) \to f^i_a(M) \to f^i_a(N) \to \ldots \quad (**)
$$

If $i < f.g.$, then \( f^0_a(M), f^1_a(M), \ldots, f^{i-1}_a(M), f^i_a(M) \in S \). On the other hand, since $i < f.g.$, \( f^0_a(L), f^1_a(L), \ldots, f^i_a(L) \in S \). Hence, it follows from (**) that \( f^0_a(M), \ldots, f^{i-1}_a(N) \in S \) and so $i - 1 < f.g.$.

**Theorem 2.8.** Let $(R, m)$ be a local ring, $a$ be an ideal of $R$, $S$ be a Serre subcategory of the category of $R$-modules and $R$-homomorphisms and $M \in S$ be a finitely generated $R$-module such that $\Gamma_a(M)$ is a pure submodule of $M$. Then \( Hom_R \left( \frac{R}{a}, f^i_a(\Gamma_a(M)) \right) \in S \), where $t = f.g.$.

**Proof.** Due to the previous theorem, \( f.g. \geq f.g. \). If $f.g. > f.g.$, then the result is obvious. Accordingly, we assume that $f.g. = f.g.$ and $a \in Var$. We know that $Supp(\Gamma_a(M)) \in Var$. By using [4, Lemma 2.3], \( f^i_a(\Gamma_a(M)) \cong H^i_m(\Gamma_a(M)) \) for all $i \geq 0$. So, if $j < f.g.$, then \( f^j_a(\Gamma_a(M)) \cong H^j_m(\Gamma_a(M)) \in S \) and $Ext^k_R \left( \frac{R}{m}, f^j_a(\Gamma_a(M)) \right) \in S$ for all $k \geq 0$ and $j < f.g.$ and $Ext^k_R \left( \frac{R}{m}, f^j_a(\Gamma_a(M)) \right) \in S$.

**Corollary 2.9.** With the same notations as Theorem 2.8, let $X \in S$ be a submodule of $f^i_a(\Gamma_a(M))$, where $t = f.g.$ and $Hom_R \left( \frac{R}{m}, f^i_a(\Gamma_a(M)) \right) \in S$.

**Proof.** Consider the long exact sequence:
In accordance with the previous theorem \( \text{Hom}_R\left( \frac{R}{m}, f^t_a(\Gamma_a(M)) \right) \rightarrow \text{Hom}_R\left( \frac{R}{m}, \frac{f_a(\Gamma_a(M))}{x} \right) \rightarrow \text{Ext}_R^1\left( \frac{R}{m}, X \right). (*) \)

Theorem 2.10. Suppose that \( a \) is an ideal of \( (R, m) \) and \( M \in S \) is a finitely generated \( R \)-module such that \( \Gamma_a(M) \) is a pure submodule of \( M \). Then \( \text{Hom}_R\left( \frac{R}{m}, f^{t-1}_a\left( \frac{M}{\Gamma_a(M)} \right) \right) \in S \), where \( t = f.\text{grade}_S(a, M) \).

**Proof.** One has \( f.\text{grade}_S(a, \Gamma_a(M)) \geq f.\text{grade}_S(a, M) \), by Theorem 2.6. Now, the exact sequence \( 0 \rightarrow \Gamma_a(M) \rightarrow M \rightarrow \frac{M}{\Gamma_a(M)} \rightarrow 0 \) induces the following long exact sequence:

\[
\cdots \rightarrow f^{t-1}_a(\Gamma_a(M)) \xrightarrow{\alpha} f^{t-1}_a(M) \xrightarrow{\beta} f^{t-1}_a\left( \frac{M}{\Gamma_a(M)} \right) \xrightarrow{\gamma} \text{Hom}_R\left( \frac{R}{m}, f^{t-1}_a(\Gamma_a(M)) \right) \xrightarrow{\varphi} \cdots \ (*)
\]

Using the exact sequence (*), we obtain the short exact sequence \( 0 \rightarrow \text{Im}(\beta) \rightarrow f^{t-1}_a(M) \rightarrow \text{Im}(\gamma) \rightarrow 0 \). Since \( f^{t-1}_a(M) \in S \), \( \text{Im}(\beta) \in S \) and \( \text{Im}(\gamma) \in S \). Furthermore, we have the exact sequence \( 0 \rightarrow \text{Im}(\xi) \rightarrow H^t_m(\Gamma_a(M)) \rightarrow \text{Im}(\varphi) \rightarrow 0 \) which induces the following long exact sequence:

\[
0 \rightarrow \text{Hom}_R\left( \frac{R}{m}, \text{Im}(\xi) \right) \rightarrow \text{Hom}_R\left( \frac{R}{m}, H^t_m(\Gamma_a(M)) \right) \rightarrow \cdots.
\]

Thus \( \text{Hom}_R\left( \frac{R}{m}, \text{Im}(\xi) \right) \in S \). Finally, by considering the short exact sequence \( 0 \rightarrow \text{Im}(\gamma) \rightarrow f^{t-1}_a\left( \frac{M}{\Gamma_a(M)} \right) \rightarrow \text{Im}(\xi) \rightarrow 0 \) we can conclude that \( \text{Hom}_R\left( \frac{R}{m}, f^{t-1}_a\left( \frac{M}{\Gamma_a(M)} \right) \right) \in S \).

Theorem 2.11. Suppose that \( R \) is complete with respect to the \( \mathfrak{a} \)-adic topology and \( M \in S \) be a finitely generated \( R \)-module and \( t \) a positive integer such that \( f^t_a(M) \in S \) for all \( i < t \). Then \( \text{Hom}_R\left( \frac{R}{m}, f^t_a(M) \right) \in S \).

**Proof.** We use induction on \( t \). Let \( t = 0 \). Consider the following isomorphisms.

\[
\text{Hom}_R\left( \frac{R}{m}, f^0_a(M) \right) \cong \text{lim}_{\rightarrow \prec} \text{Hom}_R\left( \frac{R}{m}, H^0_{d^i M}(M) \right) \cong \text{lim}_{\rightarrow \prec} \text{Hom}_R\left( \frac{R}{m}, \frac{M}{d^i M} \right)
\]

\[
\cong \text{Hom}_R\left( \frac{R}{m}, \text{lim}_{\rightarrow \prec} \frac{M}{d^i M} \right) \cong \text{Hom}_R\left( \frac{R}{m}, M^2 \right) \cong \text{Hom}_R\left( \frac{R}{m}, M \right)
\]

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It is clear that \( \text{Hom}_R \left( \frac{R}{m}, M \right) \in S \). So by the above isomorphisms, we deduce that

\[
\text{Hom}_R \left( \frac{R}{m}, f_a^0 (M) \right) \in S.
\]

Suppose that \( t > 0 \) and the result is true for all integer \( i \) less than \( t \). Set \( N := f_m (M) \). Then \( f_a^i (M) \cong f_a^i \left( \frac{M}{N} \right) \) for all \( i > 0 \), and so we may assume that \( \text{depth}_R (M) > 0 \). There is an \( M \) - regular element \( x \in m \). The exact sequence \( 0 \to M \xrightarrow{x} M \to \frac{M}{xM} \to 0 \) induces the following long exact sequence:

\[
\ldots \to f_a^{t-2} (M) \xrightarrow{x} f_a^{t-2} (M) \xrightarrow{f} f_a^{t-2} \left( \frac{M}{xM} \right) \\
\to f_a^{t-1} (M) \xrightarrow{x} f_a^{t-1} (M) \xrightarrow{g} f_a^{t-1} \left( \frac{M}{xM} \right) \\
\to f_a^t (M) \xrightarrow{x} f_a^t (M) \xrightarrow{h} \ldots.
\]

Using the exact sequence (\#) we obtain the short exact sequence

\[
0 \to f_a^{t-1} \left( \frac{M}{x f_a^{t-1} (M)} \right) \to f_a^{t-1} \left( \frac{M}{xM} \right) \to \left( 0 : x \right) \to 0.
\]

Now, this exact sequence induces the following long exact sequence:

\[
0 \to \text{Hom}_R \left( \frac{R}{m}, f_a^{t-1} \left( \frac{M}{xM} \right) \right) \to \text{Hom}_R \left( \frac{R}{m}, f_a^{t-1} \left( \frac{M}{x f_a^{t-1} (M)} \right) \right) \to \text{Hom}_R \left( \frac{R}{m}, \left( 0 : x \right) \right) \to
\]

\[
\text{Ext}_R^1 \left( \frac{R}{m}, f_a^{t-1} \left( \frac{M}{x f_a^{t-1} (M)} \right) \right) \to \ldots. (**) \]

By using (\#), \( f_a^i \left( \frac{M}{xM} \right) \in S \) for all \( i < t - 1 \). Therefore by the induction hypothesis

\[
\text{Hom}_R \left( \frac{R}{m}, f_a^{t-1} \left( \frac{M}{xM} \right) \right) \in S.
\]

Furthermore, \( \text{Ext}_R^1 \left( \frac{R}{m}, f_a^{t-1} \left( \frac{M}{x f_a^{t-1} (M)} \right) \right) \in S \) because

\[
f_a^{t-1} (M) \in S.
\]

Thus in accordance with (\#), \( \text{Hom}_R \left( \frac{R}{m}, \left( 0 : x \right) \right) \in S \). Since \( x \in m \) according to [9,10,86] we have the following isomorphisms.

\[
\text{Hom}_R \left( \frac{R}{m}, \left( 0 : x \right) \right) \cong \text{Hom}_R \left( \frac{R}{m}, \text{Hom}_R \left( \frac{R}{x}, f_a^t (M) \right) \right) \cong
\]

\[
\text{Hom}_R \left( \frac{R}{m} \otimes_R \frac{R}{xR}, f_a^t (M) \right) \cong \text{Hom}_R \left( \frac{R}{m}, f_a^t (M) \right).
\]

Consequently \( \text{Hom}_R \left( \frac{R}{m}, f_a^t (M) \right) \in S \).
3. The formal cohomological dimension in a Serre subcategory

We recall from [3, Theorem 1.1] that for a finitely generated $R$-module $M$, \[ \text{sup}\{i \in \mathbb{N}_0 | f_i^a(M) \neq 0\} = \dim \left( \frac{M}{aM} \right). \]

**Definition 3.1.** The formal cohomological dimension of $M$ with respect to $a$ in $S$ is the supremum of the integers $i$ such that $f_i^a(M) \in S$ and is denoted by $f.c.d_S(a, M)$.

**Theorem 3.2.** Suppose that $S$ is a Serre subcategory of the category of $R$-modules and $R$-$\hom$-homomorphisms and $L$ and $N$ are two finitely generated $R$-modules such that $\text{Supp}_R(L) \subseteq \text{Supp}_R(N)$. Then $f.c.d_S(a, L) \leq f.c.d_S(a, N)$.

**Proof.** It is enough to prove that $f_i^a(L) \in S$ for all $i > f.c.d_S(a, N)$ and all finitely generated $R$-module $L$ such that $\text{Supp}_R(L) \subseteq \text{Supp}_R(N)$. We use descending induction on $i$. For all $i > \dim \left( \frac{L}{aL} \right) + f.c.d_S(a, N)$, $f_i^a(L) = 0 \in S$. Let $i > f.c.d_S(a, N)$ and the result is proved for $i + 1$. By Gruson’s theorem, there is a chain $0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_i = L$ of submodules of $L$ such that $\frac{L_i}{L_{i-1}}$ is a homomorphic image of a direct sum of finitely many copies of $N$. Consider the exact sequence $0 \to L_{i-1} \to L_i \to \frac{L_i}{L_{i-1}} \to 0$ ($i = 0, 1, \ldots, l$). We may assume that $l = 1$. The exact sequence $0 \to K \to \bigoplus_{j=1}^l N \to L \to 0$ where $K$ is a finitely generated $R$-module induces the following long exact sequence:

\[ \cdots \to f_i^a \left( \bigoplus_{j=1}^l N \right) \to f_i^a(L) \to f_i^{a+1}(K) \to \cdots. \]

Based on the induction hypothesis $f_i^{a+1}(K) \in S$. Moreover $f_i^a \left( \bigoplus_{j=1}^l N \right) = \bigoplus_{j=1}^l f_i^a(N) \in S$ for all $i > f.c.d_S(a, N)$. Hence it follows from the exact sequence (*) that $f_i^a(L) \in S$.

The next example shows that even if $\text{Supp}_R(M) = \text{Supp}_R(N)$, then it may not true that $f.g.S(a, M) = f.g.S(a, N)$.

**Example 3.3.** (See [4, Example 4.3 (i)]) Let $(R, m)$ be a 2 dimensional complete regular local ring, $S = 0$ and $a$ be an ideal of $R$ with $\dim \left( \frac{R}{a} \right) = 1$. Then by using [5, Theorem 1.1], $f.g.S(a, R) = 1$ and $f.g.S \left( a, \frac{R}{m} \right) = 0$. Set $M := R \bigoplus \frac{R}{m}$. Then $\text{Supp}_R(M) = \text{Supp}_R(R)$. But

\[ f.g.S(a, M) = \inf \{ f.g.S(a, R), f.g.S \left( a, \frac{R}{m} \right) \} = 0. \]

**Corollary 3.4.** For all $x \in a \cdot f.c.d_S(a, M) \geq f.c.d_S \left( a, \frac{M}{xM} \right)$.

**Corollary 3.5.** Suppose that $0 \to L \to M \to N \to 0$ is an exact sequence of finitely generated $R$-modules. Then $f.c.d_S(a, M) = \max \{ f.c.d_S(a, L), f.c.d_S(a, N) \}$. 

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Proof. Since $Supp_R(M) = Supp_R(L) \cup Supp_R(N)$ by referring to Theorem 3.2 we deduce that $f.cds(a, M) \geq f.cds(a, L)$ and $f.cds(a, M) \geq f.cds(a, N)$. Therefore $f.cds(a, M) \geq \max\{f.cds(a, L), f.cds(a, N)\}$.

Next we prove that $\max\{f.cds(a, L), f.cds(a, N)\} \geq f.cds(a, M)$.

Let $i > \max\{f.cds(a, L), f.cds(a, N)\}$. Then $f_i^L(N), f_i^L(L) \in S$ and from the exact sequence $f_i^L(L) \to f_i^L(M) \to f_i^L(N)$ we conclude that $f_i^L(M) \in S$. Thus, $\max\{f.cds(a, L), f.cds(a, N)\} \geq f.cds(a, M)$.

We recall that the cohomological dimension of an $R$-module $M$ with respect to an ideal $a$ of $R$ in $S$ is defined as $cd_s(a, M) := \sup\{i \in \mathbb{N} \mid H_i^a(M) \not\in S\}$.

The following lemma shows that when we considering the Artinianness of $f_i^a(M)$, we can assume that $M$ is $a$-torsion-free.

Lemma 3.6. Suppose that $a$ is an ideal of a local ring $(R, m)$ and $t$ be a non-negative integer. If $H_i^m(M) \in S$ for all $i \geq t$, then the following are equivalent:

(a) $f_i^a(M) \in S$ for all $i \geq t$.

(b) $f_i^a(M) \in S$ for all $i \geq t$.

Proof. According to the hypothesis $t > cd_s(m, M)$. On the other hand $Supp_R(\Gamma_a(M)) \subseteq Supp_R(M)$. So by referring to [7, Theorem 3.5], $cd_s(m, \Gamma_a(M)) \leq cd_s(m, M)$. Thus, $t > cd_s(m, \Gamma_a(M))$ and $H_i^m(\Gamma_a(M)) \in S$ for all $i \geq t$. Now, consider the following long exact sequence:

$$\cdots \to f_i^a(\Gamma_a(M)) \to f_i^a(M) \to f_i^a(\frac{M}{\Gamma_a(M)}) \to f_{i+1}^a(\Gamma_a(M)) \to \cdots \ast$$

According to [4, Lemma 2.3] $f_i^a(\Gamma_a(M)) \cong H_i^m(\Gamma_a(M))$. By using the hypothesis $f_i^a(\Gamma_a(M)) \in S$ for all $i \geq t$. So it follows from the exact sequence $(\ast)$ that $f_i^a(M) \in S$ if and only if $f_i^a(\frac{M}{\Gamma_a(M)}) \in S$ for all $i \geq t$.

Theorem 3.7. Let $(R, m)$ be a local ring and $M \in S$ be a finitely generated $R$-module of dimension $d$ such that $cd_s(m, M) \leq f.cds(a, M)$. Then $\frac{f_i^a(M)}{a_{f_i^a(M)}} \in S$ where $t = f.cds(a, M)$.

Proof. We use induction on $d = \dim(M)$. If $d = 0$, then $\dim(\frac{M}{a_M}) = 0$. Accordingly to [3, Theorem 1.1], $f_i^a(M) = 0$ for all $i > 0$. 

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Moreover \( f_a^0(M) \cong M \in S \). By definition \( H^i_m(M) \in S \) for all \( i > t \). Therefore from the above lemma we can assume that \( M \) is \( a \)-torsion-free and there is an \( M \)-regular element \( x \in a \). Consider the long exact sequence:

\[
\cdots \to f_a^i(M) \xrightarrow{x} f_a^i(M) \to f_a^i\left( \frac{M}{xM} \right) \to f_a^{i+1}(M) \to \cdots. (*)
\]

By using the hypothesis \( f_a^i(M) \in S \) for all \( i > t \) (because \( t = f.cd_S(a, M) \)). So using the above long exact sequence \( f_a^i\left( \frac{M}{xM} \right) \in S \) for all \( i > t \). By induction hypothesis, \( \frac{f_a^i\left( \frac{M}{xM} \right)}{a f_a^i\left( \frac{M}{xM} \right)} \in S \) because \( \dim \left( \frac{M}{xM} \right) = \dim(M) - 1 \).

Afterwards from the exact sequence \( (*) \) we get the following short exact sequence:

\[
0 \to \operatorname{Im}(f) \to f_a^t\left( \frac{M}{xM} \right) \to \operatorname{Im}(g) \to 0
\]

So we obtain the following long exact sequence.

\[
\cdots \to \operatorname{Tor}_R^R\left( \frac{R}{a}, \operatorname{Im}(g) \right) \to \operatorname{Im}(f) \to \frac{f_a^t\left( \frac{M}{xM} \right)}{a f_a^t\left( \frac{M}{xM} \right)} \to \operatorname{Im}(g) \to 0.
\]

Since \( f_a^t(M) \in S \) and \( \operatorname{Im}(g) \) is a submodule of \( f_a^{t+1}(M) \), we deduce that \( \operatorname{Tor}_R^R\left( \frac{R}{a}, \operatorname{Im}(g) \right) \in S \). On the other hand, \( \frac{f_a^t\left( \frac{M}{xM} \right)}{a f_a^t\left( \frac{M}{xM} \right)} \in S \). Therefore, \( \frac{\operatorname{Im}(f)}{a \operatorname{Im}(f)} \in S \) by the above long exact sequence.

Now, consider the following long exact sequence.

\[
\frac{f_a^t(M)}{a f_a^t(M)} \xrightarrow{x} \frac{f_a^{t+1}(M)}{a f_a^{t+1}(M)} \to \frac{\operatorname{Im}(f)}{a \operatorname{Im}(f)} \to 0.
\]

So, \( \frac{f_a^t(M)}{a f_a^t(M)} \cong \frac{\operatorname{Im}(f)}{a \operatorname{Im}(f)} \) because \( x \in a \). Consequently, \( \frac{f_a^t(M)}{a f_a^t(M)} \in S \).

**Proposition 3.8.** For a finitely generated \( R \)-module \( M \),

\[
f.cd_S(a, M) = \max \{ f.cd_S(a, \frac{R}{P}) | P \in \text{Ass}_R(M) \}.
\]

**Proof.** Set \( N := \bigoplus_{P \in \text{Ass}_R(M)} \frac{R}{P} \). Then \( \text{Supp}_R(M) = \text{Supp}_R(N) \). So, by Theorem 3.2 and Corollary 3.5, \( f.cd_S(a, M) = f.cd_S(a, N) = \max \{ f.cd_S(a, \frac{R}{P}) | P \in \text{Ass}_R(M) \} \).

**Proposition 3.9.** Assume that \( a \) is an ideal of the local ring \((R, m)\). Then \( \text{Hom}_R\left( \frac{R}{m}, f_a^0(M) \right) \in S \) if and only if \( \text{Hom}_R\left( \frac{R}{m}, \tilde{M}^a \right) \in S \).

**Proof.** It is enough to consider the following isomorphisms

\[
\text{Hom}_R\left( \frac{R}{m}, f_a^0(M) \right) \cong \lim_{\mathbb{N}} \text{Hom}_R\left( \frac{R}{m}, H^0_{\mathbb{N}}(M) \right) \cong \lim_{\mathbb{N}} \text{Hom}_R\left( \frac{R}{m}, a^n M \right) \cong \text{Hom}_R\left( \frac{R}{m}, \tilde{M}^a \right).
\]
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