Formal Local Cohomology Modules and Serre Subcategories

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Abstract

Let \(( R, m )\) be a Noetherian local ring, \( a \) an ideal of \( R \) and \( M \) a finitely generated \( R\)-module. We investigate some properties of formal local cohomology modules with respect to a Serre subcategory. We provide a common language to indicate some properties of formal local cohomology modules.

1. Introduction

Throughout this paper \(( R, m )\) is a commutative Noetherian local ring, \( a \) an ideal of \( R \) and \( M \) is a finitely generated \( R\)-module. For an integer \( i \in \mathbb{N}_0 \), \( H^i_a(N) \) denotes the \( i \)-th local cohomology module of \( M \) with respect to \( a \) as introduced by Grothendieck (cf. [1], [2]).

We shall consider the family of local cohomology modules \( \{ H^i_m(\frac{M}{a^nM}) \}_{n \in \mathbb{N}} \) for a non-negative integer \( i \in \mathbb{N}_0 \). With natural homomorphisms; this family forms an inverse system. Schenzel introduced the \( i \)-th formal local cohomology of \( M \) with respect to \( a \) in the form of \( f^i_a(M) : = \lim_{\mathbb{N}} H^i_m(\frac{M}{a^nM}) \), which is the \( i \)-th cohomology module of the \( a \)-adic completion of the \( \check{\text{C}}ech \) complex \( \check{\Gamma}_x \otimes_R M \), where \( x \) denotes a system of elements of \( R \) such that \( \text{Rad} \left( \frac{a}{x} \right) = m \) (see [3, Definition 3.1]). He defines the formal grade as \( f.grade( a, M ) = \inf \{ i \in \mathbb{N}_0 \mid f^i_a(M) \neq 0 \} \). For any ideal \( a \) of \( R \) and finitely generated \( R \)-module \( M \) the following statements hold:

(i) (See [3, Theorem 3.11]). If \( 0 \to M' \to M \to M'' \to 0 \) is a short exact sequence of finitely generated \( R \)-modules, then there is the following long exact sequence:

\[
\cdots \to f^i_a(M') \to f^i_a(M) \to f^i_a(M'') \to \cdots.
\]

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(ii) (See [3, Theorem 1.3]). \( f.\ grade(\mathfrak{a}, M) \leq \dim(M) - cd(\mathfrak{a}, M) \); some properties of formal local cohomology have been presented in [3, 4, 5 and 6].

Throughout this paper \( \mathcal{S} \) denotes a Serre subcategory of the category of \( R \)-modules and \( R \)-homomorphisms (we recall that a class \( \mathcal{S} \) of \( R \)-modules is a Serre subcategory of the category of \( R \)-modules and \( R \)-homomorphisms if \( \mathcal{S} \) is closed under taking submodules, quotients and extensions).

Our paper contains three sections. In Section 2, we shall define the formal grade of \( \mathfrak{a} \) with respect to \( M \) in \( \mathcal{S} \) as the infimum of the integers \( i \) such that \( f^i_a(M) \notin \mathcal{S} \) and is denoted by \( f.\ grade_\mathcal{S}(\mathfrak{a}, M) \). (See definition 2.1). Then we shall obtain some properties of this notion. We show that if \( \Gamma_a(M) \) is a pure submodule of \( M \), then \( Hom_R(\frac{R}{m}, \Gamma_a(M)) \) and \( Hom_R(\frac{R}{m}, f_a^{i-1}(\frac{M}{\Gamma_a(M)})) \) belong to \( \mathcal{S} \), where \( t = f.\ grade_\mathcal{S}(\mathfrak{a}, M) \).

In Section 3, we shall define the formal cohomological dimension of \( \mathfrak{a} \) with respect to \( M \) in \( \mathcal{S} \) as the supremum of the integers \( i \) such that \( f^i_a(M) \notin \mathcal{S} \) and is denoted by \( f.\ cd_\mathcal{S}(\mathfrak{a}, M) \). (See definition 3.1). The main result of this section is that if \( f^i_a(M) \in \mathcal{S} \) and \( H^i_m(M) \in \mathcal{S} \) for all \( i > t \), then \( \frac{R}{a} \otimes_R f_a^i(M) \) belongs to \( \mathcal{S} \).

### 2. The formal grade of a module in a Serre subcategory

**Definition 2.1.** The formal grade of \( \mathfrak{a} \) with respect to \( M \) in \( \mathcal{S} \) is the infimum of the integers \( i \) such that \( f^i_a(M) \notin \mathcal{S} \) and is denoted by \( f.\ grade_\mathcal{S}(\mathfrak{a}, M) \).

**Proposition 2.2.** Let \( (R, m) \) be a local ring and \( \mathfrak{a} \) be an ideal of \( R \). If \( 0 \to L \to M \to N \to 0 \) is an exact sequence of finitely generated \( R \)-modules, then the following statements hold.

(a) \( f.\ grade_\mathcal{S}(\mathfrak{a}, M) \geq \min\{f.\ grade_\mathcal{S}(\mathfrak{a}, L), f.\ grade_\mathcal{S}(\mathfrak{a}, N)\} \).

(b) \( f.\ grade_\mathcal{S}(\mathfrak{a}, L) \geq \min\{f.\ grade_\mathcal{S}(\mathfrak{a}, M), f.\ grade_\mathcal{S}(\mathfrak{a}, N) + 1\} \).

(c) \( f.\ grade_\mathcal{S}(\mathfrak{a}, N) \geq \min\{f.\ grade_\mathcal{S}(\mathfrak{a}, L) - 1, f.\ grade_\mathcal{S}(\mathfrak{a}, M)\} \).

**Proof.** According to [3, Theorem 3.11], the above short exact sequence induces the following long exact sequence.

\[ \cdots \to f_a^{i-1}(N) \to f_a^i(L) \to f_a^i(M) \to f_a^i(N) \to f_a^{i+1}(L) \to \cdots \]

So, the result follows.
Corollary 2.3. If $x = x_1, \ldots, x_n$ is a regular $M$-sequence, then $f.\text{grade}_S \left( a, \frac{M}{\sum M} \right) \geq f.\text{grade}_S (a, M) - n$.

Proof. Consider the following exact sequence ($n \in \mathbb{N}$)

\[ 0 \rightarrow \frac{M}{(x_1, \ldots, x_{n-1})M} \rightarrow \frac{M}{(x_1, \ldots, x_n - 1)M} \rightarrow \frac{M}{(x_1, \ldots, x_n)M} \rightarrow 0 \]

whenever $n = 1$ by $(x_1, \ldots, x_{n-1})M$ we means 0.

Corollary 2.4. Let $a$ and $b$ be ideals of $R$. Then

(a) $f.\text{grade}_S (a \cap b, M) \geq \min\{f.\text{grade}_S (a, M), f.\text{grade}_S (b, M), f.\text{grade}_S ((a, b), M) + 1\}$.

(b) $f.\text{grade}_S ((a, b), M) \geq \min\{f.\text{grade}_S (a \cap b, M) - 1, f.\text{grade}_S (a, M), f.\text{grade}_S (b, M)\}$.

Proof. For all $n \in \mathbb{N}$ there is a short exact sequence as follows:

\[ 0 \rightarrow \frac{M}{a^n M \cap b^n M} \rightarrow \frac{M}{a^n M} \oplus \frac{M}{b^n M} \rightarrow \frac{M}{a^n b^n M} \rightarrow 0. \]

By using [3, Theorem 5.1], the above exact sequence induces the following long exact sequence.

\[ \ldots \rightarrow \lim_{n \in \mathbb{N}} H^i_M \left( \frac{M}{(a \cap b)^n M} \right) \rightarrow \lim_{n \in \mathbb{N}} H^i_M \left( \frac{M}{a^n M} \right) \oplus \lim_{n \in \mathbb{N}} H^i_M \left( \frac{M}{b^n M} \right) \rightarrow \lim_{n \in \mathbb{N}} H^i_M \left( \frac{M}{(a \cap b)^n M} \right) \rightarrow \ldots \]

So by using an argument similar to that of Proposition 2.2, the result follows.

Corollary 2.5. Assume that $M$ is a finitely generated $R$-module and $N_1$ and $N_2$ are submodules of $M$. Then considering the exact sequence $0 \rightarrow \frac{M}{N_1 \cap N_2} \rightarrow \frac{M}{N_1} \oplus \frac{M}{N_2} \rightarrow \frac{M}{N_1 + N_2} \rightarrow 0$ we shall have

(a) $f.\text{grade}_S \left( a, \frac{M}{N_1 \cap N_2} \right) \geq \min\{f.\text{grade}_S (a, \frac{M}{N_1}), f.\text{grade}_S (a, \frac{M}{N_2}), MN2, f.\text{grade}_S a, MN1 + N2 + 1\}$.

(b) $f.\text{grade}_S \left( a, \frac{M}{N_1 + N_2} \right) \geq \min\{f.\text{grade}_S \left( \frac{M}{N_1 \cap N_2} \right) - 1, f.\text{grade}_S (a, \frac{M}{N_1}), MN1, f.\text{grade}_S a, MN2\}$.

Theorem 2.6. Let $a$ be an ideal of a local ring $(R, m)$, $M$ be a finitely generated $R$-module and $L$ be a pure submodule of $M$. Then $f.\text{grade}_S (a, L) \geq f.\text{grade}_S (a, M)$ where $S$ is a Serre subcategory of the category of $R$-modules and $R$-homomorphisms. In particular, if $\inf \{i | H^i_M (L) \notin S\} \geq \inf \{i | H^i_M (M) \notin S\}$.
Proof. Let $L$ be a pure submodule of $M$. So $\frac{L}{a^nL} \rightarrow \frac{M}{a^nM}$ is pure for each $n \in \mathbb{N}$. Now according to [8, Corollary 3.2 (a)], $H^i_m\left(\frac{L}{a^nL}\right) \rightarrow H^i_m\left(\frac{M}{a^nM}\right)$ is injective. Since inverse limit is a left exact functor, $f_a^i(L)$ is isomorphic to a submodule of $f_a^i(M)$. Consequently, $f.grade_S(a, L) \geq f.grade_S(a, M)$. If $a = 0$ then, $f.grade_S(0, M) = \inf \{ |H^i_m(M) \notin S \}$ and the result follows.

Corollary 2.7. If $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is a pure exact sequence of finitely generated $R$-modules, then $\min \{ f.grade_S(a, L), f.grade_S(a, N) + 1 \} \geq f.grade_S(a, M)$.

Proof. Since $L$ is a pure submodule of $M$, as a result of the previous theorem, $f.grade_S(a, L) \geq f.grade_S(a, M)$. Hence we must prove that $f.grade_S(a, N) + 1 \geq f.grade_S(a, M)$. We assume that $i < f.grade_S(a, M)$ and we show that $i < f.grade_S(a, N) + 1$. Consider the following long exact sequence.

$$\cdots \rightarrow f_a^{i-1}(M) \rightarrow f_a^i(N) \rightarrow f_a^i(L) \rightarrow f_a^i(M) \rightarrow f_a^i(N) \rightarrow \cdots.$$ (**) 

If $i < f.grade_S(a, M)$, then $f_a^0(M), f_a^1(M), \ldots, f_a^{i-1}(M), f_a^i(M) \in S$. On the other hand, since $i < f.grade_S(a, M) \leq f.grade_S(a, L), f_a^0(L), \ldots, f_a^i(L) \in S$. Hence, it follows from (**) that $f_a^0(N), \ldots, f_a^{i-1}(N) \in S$ and so $i - 1 < f.grade_S(a, N)$.

Theorem 2.8. Let $(R, m)$ be a local ring, $a$ be an ideal of $R$, $S$ be a Serre subcategory of the category of $R$-modules and $R$-homomorphisms and $M \in S$ be a finitely generated $R$-module such that $\Gamma_a(M)$ is a pure submodule of $M$. Then $Hom_R\left(\frac{R}{a}, f_a^i(\Gamma_a(M))\right) \in S$, where $t = f.grade_S(a, M)$.

Proof. Due to the previous theorem, $f.grade_S(a, \Gamma_a(M)) \geq f.grade_S(a, M)$. If $f.grade_S(a, \Gamma_a(M)) > f.grade_S(a, M)$, then the result is obvious. Accordingly, we assume that $f.grade_S(a, \Gamma_a(M)) = f.grade_S(a, M)$. We know that $\text{Supp}(\Gamma_a(M)) \subseteq \text{Var}(a)$. By using [4, Lemma 2.3], $f_a^i(\Gamma_a(M)) \cong H^i_m(\Gamma_a(M))$ for all $i \geq 0$. So, if $j < f.grade_S(a, M)$, then $f_a^j(\Gamma_a(M)) \cong H^j_m(\Gamma_a(M)) \in S$ and $\text{Ext}^k_R(\frac{R}{m}, f_a^i(\Gamma_a(M))) \in S$ for all $k \geq 0$ and $j < f.grade_S(a, M)$. Moreover $\text{Ext}^i_R(\frac{R}{m}, \Gamma_a(M)) \in S$, because $\Gamma_a(M) \in S$. Consequently, according to [7, Theorem 2.2],

$$Hom_R(\frac{R}{m}, H^i_m(\Gamma_a(M)) \in S, where t = f.grade_S(a, M).$$

Corollary 2.9 With the same notations as Theorem 2.8, let $X \in S$ be a submodule of $f_a^i(\Gamma_a(M))$, where $t = f.grade_S(a, M)$. Then $Hom_R(\frac{R}{m}, \frac{f_a^i(\Gamma_a(M))}{X}) \in S$.

Proof. Consider the long exact sequence:
\[ \text{Hom}_R \left( \frac{R}{m}, f^t_a(\Gamma_a(M)) \right) \to \text{Hom}_R \left( \frac{R}{m}, f^t_a(\Gamma_a(M)) \right) \to \text{Ext}^1_R \left( \frac{R}{m}, X \right). \]

In accordance with the previous theorem \( \text{Hom}_R \left( \frac{R}{m}, f^t_a(\Gamma_a(M)) \right) \in S. \) Moreover \( \text{Ext}^1_R \left( \frac{R}{m}, X \right) \in S. \) It follows from the exact sequence \( (*) \) that \( \text{Hom}_R \left( \frac{R}{m}, f^t_a(\Gamma_a(M)) \right) \in S. \)

**Theorem 2.10.** Suppose that \( a \) is an ideal of \( (R, m) \) and \( M \in S \) is a finitely generated \( R \)-module such that \( \Gamma_a(M) \) is a pure submodule of \( M. \) Then \( \text{Hom}_R \left( \frac{R}{m}, f^{t-1}_a \left( \frac{M}{\Gamma_a(M)} \right) \right) \in S, \) where \( t = f. \text{grade}_S(a, M). \)

**Proof.** One has \( f. \text{grade}_S(a, \Gamma_a(M)) \geq f. \text{grade}_S(a, M), \) by Theorem 2.6. Now, the exact sequence \( 0 \to \Gamma_a(M) \to M \to \frac{M}{\Gamma_a(M)} \to 0 \) induces the following long exact sequence:

\[ \cdots \to f^{t-1}_a(\Gamma_a(M)) \to f^{t-1}_a(M) \to f^{t-1}_a \left( \frac{M}{\Gamma_a(M)} \right) \to f^{t-1}_a(\Gamma_a(M)) \to \cdots. \]

Using the exact sequence \( (*) \), we obtain the short exact sequence \( 0 \to \text{Im}(\beta) \to f^{t-1}_a(M) \to \text{Im}(\gamma) \to 0. \) Since \( f^{t-1}_a(M) \in S, \) \( \text{Im}(\beta) \in S \) and \( \text{Im}(\gamma) \in S. \) Furthermore, we have the exact sequence \( 0 \to \text{Im}(\xi) \to \text{H}^t_a(\Gamma_a(M)) \to \text{Im}(\varphi) \to 0 \) which induces the following long exact sequence:

\[ 0 \to \text{Hom}_R(\frac{R}{m}, \text{Im}(\xi)) \to \text{Hom}_R(\frac{R}{m}, \text{H}^t_a(\Gamma_a(M))) \to \cdots. \]

Thus \( \text{Hom}_R \left( \frac{R}{m}, \text{Im}(\xi) \right) \in S. \) Finally, by considering the short exact sequence \( 0 \to \text{Im}(\gamma) \to f^{t-1}_a \left( \frac{M}{\Gamma_a(M)} \right) \to \text{Im}(\xi) \to 0 \) we can conclude that \( \text{Hom}_R \left( \frac{R}{m}, f^{t-1}_a \left( \frac{M}{\Gamma_a(M)} \right) \right) \in S. \)

**Theorem 2.11.** Suppose that \( R \) is complete with respect to the \( a \)-adic topology and \( M \in S \) be a finitely generated \( R \)-module and \( t \) a positive integer such that \( f^t_a(M) \in S \) for all \( i < t. \) Then \( \text{Hom}_R \left( \frac{R}{m}, f^t_a(M) \right) \in S. \)

**Proof.** We use induction on \( t. \) Let \( t=0. \) Consider the following isomorphisms.

\[ \text{Hom}_k \left( \frac{R}{m}, f^0_a(M) \right) \cong \lim_{\rightarrow a} \text{Hom}_k \left( \frac{R}{m}, \text{H}^0_a\left( \frac{M}{a^i M} \right) \right) \cong \lim_{\rightarrow a} \text{Hom}_k \left( \frac{R}{m}, \frac{M}{a^i M} \right) \]

\[ \cong \text{Hom}_k \left( \frac{R}{m}, \lim_{\rightarrow a} \left( \frac{M}{a^i M} \right) \right) \cong \text{Hom}_k \left( \frac{R}{m}, \text{H}^0_a \left( \frac{M}{a^t M} \right) \right) \cong \text{Hom}_k \left( \frac{R}{m}, M \right) \]

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It is clear that $\text{Hom}_R\left(\frac{R}{m}, M\right) \in S$. So by the above isomorphisms, we deduce that

$$\text{Hom}_R\left(\frac{R}{m}, f^t_a(M)\right) \in S.$$ 

Suppose that $t > 0$ and the result is true for all integer $i$ less than $t$. Set $N := f_t^a(M)$. Then $f^i_a(M) \cong f^i_a\left(\frac{M}{N}\right)$ for all $i > 0$, and so we may assume that $\text{depth}_R(M) > 0$. There is an $M$-regular element $x \in m$. The exact sequence $0 \rightarrow M \rightarrow M \rightarrow M/\langle M \rangle \rightarrow 0$ induces the following long exact sequence:

$$\cdots \rightarrow f_a^{t-2}(M) \xrightarrow{x} f_a^{t-2}(M) \xrightarrow{f} f_a^{t-2}\left(\frac{M}{xM}\right)$$

$$\rightarrow f_a^{t-1}(M) \xrightarrow{x} f_a^{t-1}(M) \xrightarrow{g} f_a^{t-1}\left(\frac{M}{xM}\right)$$

$$\rightarrow f_a^t(M) \xrightarrow{x} f_a^t(M) \xrightarrow{h} \cdots.$$ 

Using the exact sequence $(*)$ we obtain the short exact sequence

$$0 \rightarrow f_a^{t-1}\left(\frac{M}{x f_a^{t-1}(M)}\right) \rightarrow f_a^{t-1}\left(\frac{M}{xM}\right) \rightarrow (0 : x)_{\langle y(M) \rangle} \rightarrow 0.$$ 

Now, this exact sequence induces the following long exact sequence:

$$0 \rightarrow \text{Hom}_R\left(\frac{R}{m}, f_a^{t-1}(M)\right) \rightarrow \text{Hom}_R\left(\frac{R}{m}, f_a^{t-1}\left(\frac{M}{xM}\right)\right) \rightarrow \text{Hom}_R\left(\frac{R}{m}, (0 : x)_{\langle y(M) \rangle}\right) \rightarrow$$

$$\text{Ext}_R^1\left(\frac{R}{m}, f_a^{t-1}\left(\frac{M}{xM}\right)\right) \rightarrow \cdots.$$ 

By using $(*)$, $f^i_a\left(\frac{M}{xM}\right) \in S$ for all $i < t - 1$. Therefore by the induction hypothesis

$$\text{Hom}_R\left(\frac{R}{m}, f_a^{t-1}\left(\frac{M}{xM}\right)\right) \in S.$$ 

Furthermore $\text{Ext}_R^1\left(\frac{R}{m}, f_a^{t-1}\left(\frac{M}{xM}\right)\right) \in S$ because $f_a^{t-1}(M) \in S$. Thus in accordance with $(**)$, $\text{Hom}_R\left(\frac{R}{m}, (0 : x)_{\langle y(M) \rangle}\right) \in S$. Since $x \in m$ according to [9,10.86] we have the following isomorphisms.

$$\text{Hom}_R\left(\frac{R}{m}, (0 : x)\right) \cong \text{Hom}_R\left(\frac{R}{m}, \text{Hom}_R\left(\frac{R}{xR}, f_a^t(M)\right)\right) \cong$$

$$\text{Hom}_R\left(\frac{R}{m} \otimes_R \frac{R}{xR}, f_a^t(M)\right) \cong \text{Hom}_R\left(\frac{R}{m}, f_a^t(M)\right).$$

Consequently $\text{Hom}_R\left(\frac{R}{m}, f_a^t(M)\right) \in S$. 


3. The formal cohomological dimension in a Serre subcategory

We recall from [3,Theorem 1.1] that for a finitely generated $R$-module $M$, $\sup\{i \in \mathbb{N}_0 | f_i^a(M) \neq 0\} = \dim \left( \frac{M}{a^i M} \right)$.

**Definition 3.1.** The formal cohomological dimension of $M$ with respect to $a$ in $S$ is the supremum of the integers $i$ such that $f_i^a(M) \not\in S$ and is denoted by $f.c.d_S(a, M)$.

**Theorem 3.2.** Suppose that $S$ is a Serre subcategory of the category of $R$-modules and $R$-homomorphisms and $L$ and $N$ are two finitely generated $R$-modules such that $\text{Supp}_R(L) \subseteq \text{Supp}_R(N)$. Then $f.c.d_S(a, L) \leq f.c.d_S(a, N)$.

**Proof.** It is enough to prove that $f_i^a(L) \in S$ for all $i > f.c.d_S(a, N)$ and all finitely generated $R$-module $L$ such that $\text{Supp}_R(L) \subseteq \text{Supp}_R(N)$. We use descending induction on $i$. For all $i > \dim \left( \frac{L}{a^i L} \right) + f.c.d_S(a, N)$, $f_i^a(L) = 0 \in S$. Let $i > f.c.d_S(a, N)$ and the result is proved for $i + 1$. By Gruson’s theorem, there is a chain $0 = L_0 \subset L_1 \subset \cdots \subset L_I = L$ of submodules of $L$ such that $\frac{L_i}{L_{i-1}}$ is a homomorphic image of a direct sum of finitely many copies of $N$. Consider the exact sequence $0 \to L_{i-1} \to L_i \to \frac{L_i}{L_{i-1}} \to 0$ ($i = 0, 1, \ldots, I$). We may assume that $I = I$. The exact sequence $0 \to K \to \bigoplus_{j=1}^r N \to L \to 0$ where $K$ is a finitely generated $R$-module induces the following long exact sequence:

$$\cdots \to f_i^a\left(\bigoplus_{j=1}^r N\right) \to f_i^a(L) \to f_i^{a+1}(K) \to \cdots \ (*)$$

Based on the induction hypothesis $f_i^{a+1}(K) \in S$. Moreover $f_i^a(\bigoplus_{j=1}^r N) = \bigoplus_{j=1}^r f_i^a(N) \in S$ for all $i > f.c.d_S(a, N)$. Hence it follows from the exact sequence $(*)$ that $f_i^a(L) \in S$.

The next example shows that even if $\text{Supp}_R(M) = \text{Supp}_R(N)$, then it may not true that $f.g.d_S(a, M) = f.g.d_S(a, N)$.

**Example 3.3.** (See [4, Example 4.3 (i)]) Let $(R, m)$ be a 2 dimensional complete regular local ring, $S = 0$ and $a$ be an ideal of $R$ with $\dim \left( \frac{R}{a} \right) = 1$. Then by using [5,Theorem 1.1], $f.g.d_S(a, R) = 1$ and $f.g.d_S\left(a, \frac{R}{m}\right) = 0$. Set $M := R \bigoplus \frac{R}{m}$.

Then $\text{Supp}_R(M) = \text{Supp}_R(R)$. But

$$f.g.d_S(a, M) = \inf \{ f.g.d_S(a, R), f.g.d_S\left(a, \frac{R}{m}\right) \} = 0.$$

**Corollary 3.4.** For all $x \in a$, $f.c.d_S(a, M) \geq f.c.d_S\left(a, \frac{M}{xM}\right)$.

**Corollary 3.5.** Suppose that $0 \to L \to M \to N \to 0$ is an exact sequence of finitely generated $R$-modules. Then $f.c.d_S(a, M) = \max\{ f.c.d_S(a, L), f.c.d_S(a, N)\}$. 

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Proof. Since $\text{Supp}_R(M) = \text{Supp}_R(L) \cup \text{Supp}_R(N)$ by referring to Theorem 3.2 we deduce that $f.\text{cd}_S(a, M) \geq f.\text{cd}_S(a, L)$ and $f.\text{cd}_S(a, M) \geq f.\text{cd}_S(a, N)$. Therefore $f.\text{cd}_S(a, M) \geq \max \{f.\text{cd}_S(a, L), f.\text{cd}_S(a, N)\}$.

Next we prove that $\max \{f.\text{cd}_S(a, L), f.\text{cd}_S(a, N)\} \geq f.\text{cd}_S(a, M)$.

Let $i > \max \{f.\text{cd}_S(a, L), f.\text{cd}_S(a, N)\}$. Then $f^i_a(N), f^i_a(L) \in S$ and from the exact sequence $f^i_a(L) \to f^i_a(M) \to f^i_a(N)$ we conclude that $f^i_a(M) \in S$. Thus, $\max \{f.\text{cd}_S(a, L), f.\text{cd}_S(a, N)\} \geq f.\text{cd}_S(a, M)$.

We recall that the cohomological dimension of an $R$-module $M$ with respect to an ideal $a$ of $R$ in $S$ is defined as

$$\text{cd}_S(a, M) := \sup \{i \in \mathbb{N} | H^i_a(M) \not\in S\}.$$ 

The following lemma shows that when we considering the Artinianness of $f^i_a(M)$, we can assume that $M$ is $a$-torsion-free.

Lemma 3.6. Suppose that $a$ is an ideal of a local ring $(R, m)$ and $t$ be a non-negative integer. If $H^i_m(M) \in S$ for all $i \geq t$, then the following are equivalent:

(a) $f^i_a(M) \in S$ for all $i \geq t$.

(b) $f^i_a(M) \in S$ for all $i \geq t$.

Proof. According to the hypothesis $t > \text{cd}_S(m, M)$. On the other hand $\text{Supp}_R(\Gamma a(M)) \subseteq \text{Supp}_R(M)$. So by referring to [7, Theorem 3.5], $\text{cd}_S(m, \Gamma a(M)) \leq \text{cd}_S(m, M)$. Thus, $t > \text{cd}_S(m, \Gamma a(M))$ and $H^i_m(\Gamma a(M)) \in S$ for all $i \geq t$. Now, consider the following long exact sequence:

$$\cdots \to f^i_a(\Gamma a(M)) \to f^i_a(M) \to f^i_a\left(\frac{M}{\Gamma a(M)}\right) \to f^{i+1}_a(\Gamma a(M)) \to \cdots (*)$$

According to [4, Lemma 2.3] $f^i_a(\Gamma a(M)) \cong H^i_m(\Gamma a(M))$. By using the hypothesis $f^i_a(\Gamma a(M)) \in S$ for all $i \geq t$. So it follows from the exact sequence $(*)$ that $f^i_a(M) \in S$ if and only if $f^i_a\left(\frac{M}{\Gamma a(M)}\right) \in S$ for all $i \geq t$.

Theorem 3.7. Let $(R, m)$ be a local ring and $M \in S$ be a finitely generated $R$-module of dimension $d$ such that $\text{cd}_S(m, M) \leq f.\text{cd}_S(a, M)$. Then $f^i_a(m) \in S$ where $t = f.\text{cd}_S(a, M)$.

Proof. We use induction on $d = \dim (M)$. If $d = 0$, then $\dim \left(\frac{M}{aM}\right) = 0$. Accordingly to [3, Theorem 1.1], $f^i_a(M) = 0$ for all $i > 0$.
Moreover $f_a^0(M) \cong M \in \mathcal{S}$. By definition $H^i_a(M) \in \mathcal{S}$ for all $i > t$. Therefore from the above lemma we can assume that $M$ is $a$-torsion-free and there is an $M$-regular element $x \in a$. Consider the long exact sequence:

$$\cdots \rightarrow f_a^i(M) \xrightarrow{x} f_a^i(M) \xrightarrow{f} f_a^i\left(\frac{M}{xM}\right) \xrightarrow{g} f_a^{i+1}(M) \xrightarrow{h} \cdots \ (\ast)$$

By using the hypothesis $f_a^i(M) \in S$ for all $i > t$ (because $t = f \cdot cd_S(a, M)$). So using the above long exact sequence $f_a^i\left(\frac{M}{xM}\right) \in \mathcal{S}$ for all $i > t$. By induction hypothesis, $\frac{f_a^i\left(\frac{M}{xM}\right)}{a f_a^i\left(\frac{M}{xM}\right)} \in \mathcal{S}$ because $\dim\left(\frac{M}{xM}\right) = \dim(M) - 1$.

Afterwards from the exact sequence $(\ast)$ we get the following short exact sequence:

$$0 \rightarrow \text{Im}(f) \rightarrow f_a^i\left(\frac{M}{xM}\right) \rightarrow \text{Im}(g) \rightarrow 0$$

So we obtain the following long exact sequence.

$$\cdots \rightarrow \text{Tor}_1^R\left(\frac{R}{a}, \text{Im}(g)\right) \rightarrow \text{Im}(f) \rightarrow \frac{f_a^i\left(\frac{M}{xM}\right)}{a f_a^i\left(\frac{M}{xM}\right)} \rightarrow \text{Im}(g) \rightarrow 0.$$

Since $f_a^i(M) \in \mathcal{S}$ and $\text{Im}(g)$ is a submodule of $f_a^{i+1}(M)$, we deduce that $\text{Tor}_1^R\left(\frac{R}{a}, \text{Im}(g)\right) \in \mathcal{S}$. On the other hand, $\frac{f_a^i\left(\frac{M}{xM}\right)}{a f_a^i\left(\frac{M}{xM}\right)} \in \mathcal{S}$. Therefore, $\frac{\text{Im}(f)}{a \text{Im}(f)} \in \mathcal{S}$ by the above long exact sequence.

Now, consider the following long exact sequence.

$$\frac{f_a^i(M)}{a f_a^i(M)} \xrightarrow{x} \frac{f_a^i(M)}{a f_a^i(M)} \xrightarrow{\text{Im}(f)} \frac{\text{Im}(f)}{a \text{Im}(f)} \rightarrow 0.$$
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