Abstract

Let $(R, m)$ be a Noetherian local ring, $a$ an ideal of $R$ and $M$ a finitely generated $R$-module. We investigate some properties of formal local cohomology modules with respect to a Serre subcategory. We provide a common language to indicate some properties of formal local cohomology modules.

1. Introduction

Throughout this paper $(R, m)$ is a commutative Noetherian local ring, $a$ an ideal of $R$ and $M$ is a finitely generated $R$-module. For an integer $i \in \mathbb{N}_0$, $H^i_a(N)$ denotes the $i$-th local cohomology module of $M$ with respect to $a$ as introduced by Grothendieck (cf. [1], [2]).

We shall consider the family of local cohomology modules $\{H^i_m\left(\frac{M}{a^nM}\right)\}_{n \in \mathbb{N}}$ for a non-negative integer $i \in \mathbb{N}_0$. With natural homomorphisms, this family forms an inverse system. Schenzel introduced the $i$-th formal local cohomology of $M$ with respect to $a$ in the form of $f^i_a(M) := \varprojlim_{n \in \mathbb{N}} H^i_m\left(\frac{M}{a^nM}\right)$, which is the $i$-th cohomology module of the $a$-adic completion of the Čech complex $\check{\mathcal{C}}_R M$, where $\mathbf{x}$ denotes a system of elements of $R$ such that $Rad(\mathbf{x}, R) = m$ (see [3, Definition 3.1]). He defines the formal grade as $f grade(a, M) = \inf \{i \in \mathbb{N}_0 | f^i_a(M) \neq 0\}$. For any ideal $a$ of $R$ and finitely generated $R$-module $M$ the following statements hold:

(i) (See [3, Theorem 3.11]). If $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is a short exact sequence of finitely generated $R$-modules, then there is the following long exact sequence:

$$\cdots \rightarrow f^i_a(M') \rightarrow f^i_a(M) \rightarrow f^i_a(M'') \rightarrow \cdots$$

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(ii) (See [3, Theorem 1.3]). \( f. \text{grade}(a, M) \leq \dim(M) - cd(a, M) \); some properties of formal local cohomology have been presented in [3, 4, 5 and 6].

Throughout this paper \( S \) denotes a Serre subcategory of the category of \( R \)-modules and \( R \)-homomorphisms (we recall that a class \( S \) of \( R \)-modules is a Serre subcategory of the category of \( R \)-modules and \( R \)-homomorphisms if \( S \) is closed under taking submodules, quotients and extensions).

Our paper contains three sections. In Section 2, we shall define the formal grade of \( a \) with respect to \( M \) in \( S \) as the infimum of the integers \( i \) such that \( f^i_a(M) \notin S \) and is denoted by \( f. \text{grade}_S(a, M) \). (See definition 2.1). Then we shall obtain some properties of this notion. We show that if \( \Gamma_a(M) \) is a pure submodule of \( M \), then \( \text{Hom}_R(\frac{R}{m}f^i_a(\Gamma_a(M))) \) and \( \text{Hom}_R(\frac{R}{m}f^{i-1}_a(\frac{M}{\Gamma_a(M)})) \) belong to \( S \), where \( t = f. \text{grade}_S(a, M) \).

In Section 3, we shall define the formal cohomological dimension of \( a \) with respect to \( M \) in \( S \) as the supremum of the integers \( i \) such that \( f^i_a(M) \notin S \) and is denoted by \( f. \text{cd}_S(a, M) \). (See definition 3.1). The main result of this section is that if \( f^i_a(M) \in S \) and \( H^i_m(M) \in S \) for all \( i > t \), then \( \frac{R}{a} \otimes_R f^i_a(M) \) belongs to \( S \).

### 2. The formal grade of a module in a Serre subcategory

**Definition 2.1.** The formal grade of \( a \) with respect to \( M \) in \( S \) is the infimum of the integers \( i \) such that \( f^i_a(M) \notin S \) and is denoted by \( f. \text{grade}_S(a, M) \).

**Proposition 2.2.** Let \((R, m)\) be a local ring and \( a \) be an ideal of \( R \). If \( 0 \to L \to M \to N \to 0 \) is an exact sequence of finitely generated \( R \)-modules, then the following statements hold.

(a) \( f. \text{grade}_S(a, M) \geq \min\{f. \text{grade}_S(a, L), f. \text{grade}_S(a, N)\} \).

(b) \( f. \text{grade}_S(a, L) \geq \min\{f. \text{grade}_S(a, M), f. \text{grade}_S(a, N) + 1\} \).

(c) \( f. \text{grade}_S(a, N) \geq \min\{f. \text{grade}_S(a, L) - 1, f. \text{grade}_S(a, M)\} \).

**Proof.** According to [3, Theorem 3.11], the above short exact sequence induces the following long exact sequence.

\[ \cdots \to f^{i-1}_a(N) \to f^i_a(L) \to f^i_a(M) \to f^i_a(N) \to f^{i+1}_a(L) \to \cdots \]

So, the result follows.

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Corollary 2.3. If $x = x_1, ..., x_n$ is a regular $M$-sequence, then $f.\text{grade}_S \left( \frac{a}{\sum M} \right) \geq f.\text{grade}_S (a, M) - n.$

**Proof.** Consider the following exact sequence ($n \in \mathbb{N}$)

$$0 \to \frac{M}{(x_1, ..., x_{n-1})M} \xrightarrow{x_n} \frac{M}{(x_1, ..., x_n)M} \xrightarrow{\text{nat.}} \frac{M}{(x_1, ..., x_n)M} \to 0$$

whenever $n = 1$ by $(x_1, ..., x_{n-1})M$ we means $0$.

Corollary 2.4. Let $a$ and $b$ be ideals of $R$. Then

(a) $f.\text{grade}_S (a \cap b, M) \geq \min\{ f.\text{grade}_S (a, M), f.\text{grade}_S (b, M), f.\text{grade}_S ((a, b), M) + 1 \}.$

(b) $f.\text{grade}_S ((a, b), M) \geq \min\{ f.\text{grade}_S (a \cap b, M) - 1, f.\text{grade}_S (a, M), f.\text{grade}_S (b, M) \}.$

**Proof.** For all $n \in \mathbb{N}$ there is a short exact sequence as follows:

$$0 \to \frac{M}{a^n M} \to \frac{M}{a^n M} \oplus \frac{M}{b^n M} \to \frac{M}{(a^n, b^n)M} \to 0.$$

By using [3, Theorem 5.1], the above exact sequence induces the following long exact sequence.

$$\cdots \to \lim_{\mathbb{N}} H^i_m \left( \frac{M}{(a \cap b)^n M} \right) \to \lim_{\mathbb{N}} H^i_m \left( \frac{M}{a^n M} \right) \oplus \lim_{\mathbb{N}} H^i_m \left( \frac{M}{b^n M} \right) \to \lim_{\mathbb{N}} H^i_m \left( \frac{M}{(a, b)^n M} \right) \to \cdots.$$

So by using an argument similar to that of Proposition 2.2, the result follows.

Corollary 2.5. Assume that $M$ is a finitely generated $R$-module and $N_1$ and $N_2$ are submodules of $M$. Then considering the exact sequence $0 \to \frac{M}{N_1 \cap N_2} \to \frac{M}{N_1} \oplus \frac{M}{N_2} \to \frac{M}{N_1 + N_2} \to 0$ we shall have

(a) $f.\text{grade}_S \left( a, \frac{M}{N_1 \cap N_2} \right) \geq \min\{ f.\text{grade}_S (a, \frac{M}{N_1}), f.\text{grade}_S (a, \frac{M}{N_2}), MN2, f.\text{grade}_S a, MN1+ N2 +1\}.$

(b) $f.\text{grade}_S \left( a, \frac{M}{N_1 \cap N_2} \right) \geq \min\{ f.\text{grade}_S \left( \frac{M}{N_1 \cap N_2} \right) - 1, f.\text{grade}_S (a, \frac{M}{N_1}), f.\text{grade}_S a, MN2 \}.$

**Theorem 2.6.** Let $a$ be an ideal of a local ring $(R, m)$, $M$ be a finitely generated $R$-module and $L$ be a pure submodule of $M$. Then $f.\text{grade}_S (a, L) \geq f.\text{grade}_S (a, M)$ where $S$ is a Serre subcategory of the category of $R$-modules and $R$-homomorphisms. In particular, $\inf \{ i | H^i_m (L) \notin S \} \geq \inf \{ i | H^i_m (M) \notin S \}.$
Proof. Let \( L \) be a pure submodule of \( M \). So \( \frac{L}{a_nL} \rightarrow \frac{M}{a_nM} \) is pure for each \( n \in \mathbb{N} \). Now according to [8, Corollary 3.2 (a)], \( H_m^i \left( \frac{L}{a_nL} \right) \rightarrow H_m^i \left( \frac{M}{a_nM} \right) \) is injective. Since inverse limit is a left exact functor, \( f_a^i(L) \) is isomorphic to a submodule of \( f_a^i(M) \). Consequently, \( f.a.grade_S(a,L) \geq f.a.grade_S(a,M) \). If \( a = 0 \) then, \( f.a.grade_S(0,M) = \inf \{ i | H_m^i(M) \notin S \} \) and the result follows.

Corollary 2.7. If \( 0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 \) is a pure exact sequence of finitely generated \( R \)-modules, then \( \min \{ f.a.grade_S(a,L), f.a.grade_S(a,N) + 1 \} \geq f.a.grade_S(a,M) \).

Proof. Since \( L \) is a pure submodule of \( M \), as a result of the previous theorem, \( f.a.grade_S(a,L) \geq f.a.grade_S(a,M) \). Hence we must prove that \( f.a.grade_S(a,N) + 1 \geq f.a.grade_S(a,M) \). We assume that \( i < f.a.grade_S(a,M) \) and we show that \( i < f.a.grade_S(a,N) + 1 \). Consider the following long exact sequence.

\[ \cdots \rightarrow f_a^{i-1}(M) \rightarrow f_a^i(N) \rightarrow f_a^i(L) \rightarrow f_a^i(M) \rightarrow f_a^i(N) \rightarrow \cdots \quad (***) \]

If \( i < f.a.grade_S(a,M) \), then \( f_a^0(M), f_a^i(M), \ldots, f_a^{i-1}(M), f_a^i(M) \in S \). On the other hand, since \( i < f.a.grade_S(a,M) \leq f.a.grade_S(a,L), f_a^0(L), \ldots, f_a^i(L) \in S \). Hence, it follows from (***), \( f_a^0(N), \ldots, f_a^{i-1}(N) \in S \) and so \( i - 1 < f.a.grade_S(a,N) \).

Theorem 2.8. Let \( (R, m) \) be a local ring, \( a \) be an ideal of \( R \), \( S \) be a Serre subcategory of the category of \( R \)-modules and \( \hom(R, a) \) be a finitely generated \( R \)-module such that \( \Gamma_a(M) \) is a pure submodule of \( M \). Then \( \hom(R, a) = f.a.grade_S(a,M) \).

Proof. Due to the previous theorem, \( f.a.grade_S(a, \Gamma_a(M)) \geq f.a.grade_S(a,M) \). If \( f.a.grade_S(a, \Gamma_a(M)) > f.a.grade_S(a,M) \), then the result is obvious. Accordingly, we assume that \( f.a.grade_S(a, \Gamma_a(M)) = f.a.grade_S(a,M) \). We know that \( Supp(\Gamma_a(M)) \subseteq Var(\ a) \). By using [4, Lemma 2.3], \( \Gamma_a^i(\Gamma_a(M)) \cong H_m^i(\Gamma_a(M)) \) for all \( i \geq 0 \). So, if \( j < f.a.grade_S(a,M) \), then \( \Gamma_a^i(\Gamma_a(M)) \cong H_m^i(\Gamma_a(M)) \in S \) and \( Ext_R^k(\frac{R}{a}, H_m^i(\Gamma_a(M))) \in S \) for all \( k \geq 0 \) and \( j < f.a.grade_S(a,M) \). Moreover \( Ext_R^k(\frac{R}{m}, \Gamma_a(M)) \in S \), because \( \Gamma_a(M) \in S \). Consequently, according to [7, Theorem 2.2],

\( \hom(R, a) \in S \), where \( t = f.a.grade_S(a,M) \).

Corollary 2.9 With the same notations as Theorem 2.8, let \( X \in S \) be a submodule of \( f_a^i(\Gamma_a(M)) \), where \( t = f.a.grade_S(a,M) \). Then \( \hom(R, a) \in S \).

Proof. Consider the long exact sequence:
In accordance with the previous theorem \( \text{Hom}_R \left( \frac{R}{m}, f_{a}^t(\Gamma_a(M)) \right) \to \text{Hom}_R \left( \frac{R}{m}, \frac{f_d(\Gamma_a(M))}{X} \right) \to \text{Ext}_R^1 \left( \frac{R}{m}, X \right) \). (*)

Moreover, \( \text{Ext}_R^1 \left( \frac{R}{m}, X \right) \in \mathcal{S} \). It follows from the exact sequence (*) that \( \text{Hom}_R \left( \frac{R}{m}, \frac{f_d(\Gamma_a(M))}{X} \right) \in \mathcal{S} \).

**Theorem 2.10.** Suppose that \( a \) is an ideal of \( (R, m) \) and \( M \in \mathcal{S} \) is a finitely generated \( R \)-module such that \( \Gamma_a(M) \) is a pure submodule of \( M \). Then \( \text{Hom}_R \left( \frac{R}{m}, f_{a}^{t-1} \left( \frac{M}{\Gamma_a(M)} \right) \right) \in \mathcal{S} \), where \( t = f.\text{grade}_\mathcal{S}(a, M) \).

**Proof.** One has \( f.\text{grade}_\mathcal{S}(a, \Gamma_a(M)) \geq f.\text{grade}_\mathcal{S}(a, M) \), by Theorem 2.6. Now, the exact sequence \( 0 \to \Gamma_a(M) \to M \to \frac{M}{\Gamma_a(M)} \to 0 \) induces the following long exact sequence:

\[
\cdots \to f_{a}^{t-1}(\Gamma_a(M)) \xrightarrow{\alpha} f_{a}^{t-1}(M) \xrightarrow{\beta} f_{a}^{t-1}(\frac{M}{\Gamma_a(M)}) \xrightarrow{\gamma} f_{a}^{t-1}(\Gamma_a(M)) \xrightarrow{\delta} \cdots \text{. (*)}
\]

Using the exact sequence (*), we obtain the short exact sequence \( 0 \to \text{Im}(\beta) \to f_{a}^{t-1}(M) \to \text{Im}(\gamma) \to 0 \). Since \( f_{a}^{t-1}(M) \in \mathcal{S} \), \( \text{Im}(\beta) \in \mathcal{S} \) and \( \text{Im}(\gamma) \in \mathcal{S} \). Furthermore, we have the exact sequence \( 0 \to \text{Im}(\xi) \to H_m^t(\Gamma_a(M)) \to \text{Im}(\varphi) \to 0 \) which induces the following long exact sequence:

\[
0 \to \text{Hom}_R \left( \frac{R}{m}, \text{Im}(\xi) \right) \to \text{Hom}_R \left( \frac{R}{m}, H_m^t(\Gamma_a(M)) \right) \to \cdots.
\]

Thus \( \text{Hom}_R \left( \frac{R}{m}, \text{Im}(\xi) \right) \in \mathcal{S} \). Finally, by considering the short exact sequence \( 0 \to \text{Im}(\gamma) \to f_{a}^{t-1}(\frac{M}{\Gamma_a(M)}) \to \text{Im}(\xi) \to 0 \) we can conclude that \( \text{Hom}_R \left( \frac{R}{m}, f_{a}^{t-1} \left( \frac{M}{\Gamma_a(M)} \right) \right) \in \mathcal{S} \).

**Theorem 2.11.** Suppose that \( R \) is complete with respect to the \( a \)-adic topology and \( M \in \mathcal{S} \) be a finitely generated \( R \)-module and \( t \) a positive integer such that \( f_{a}^{t}(M) \in \mathcal{S} \) for all \( i < t \). Then \( \text{Hom}_R \left( \frac{R}{m}, f_{a}^{t}(M) \right) \in \mathcal{S} \).

**Proof.** We use induction on \( t \). Let \( t = 0 \). Consider the following isomorphisms.

\[
\text{Hom}_R \left( \frac{R}{m}, f_{a}^{0}(M) \right) \cong \lim_{\longrightarrow} \text{Hom}_R \left( \frac{R}{m}, H_m^0 \left( \frac{M}{a^i M} \right) \right) \cong \lim_{\longrightarrow} \text{Hom}_R \left( \frac{R}{m}, \frac{M}{a^i M} \right) \cong \text{Hom}_R \left( \frac{R}{m}, \frac{M}{a^i M} \right) \cong \text{Hom}_R \left( \frac{R}{m}, M \right).
\]
It is clear that $\text{Hom}_R \left( \frac{R}{M}, M \right) \in S$. So by the above isomorphisms, we deduce that

$$\text{Hom}_R \left( \frac{R}{M}, f^i_0(M) \right) \in S.$$ 

Suppose that $t > 0$ and the result is true for all integer $i$ less than $t$. Set $N := f^i_0(M)$. Then $f^i_0(M) \cong f^i_0 \left( \frac{M}{N} \right)$ for all $i > 0$, and so we may assume that $\text{depth}_R(M) > 0$. There is an $M - \text{regular}$ element $x \in m$. The exact sequence $0 \rightarrow M \rightarrow M \rightarrow \frac{M}{xM} \rightarrow 0$ induces the following long exact sequence:

$$\cdots \rightarrow f^{t-2}_a(M) \overset{x}{\rightarrow} f^{t-2}_a(M) \overset{f}{\rightarrow} f^{t-2}_a \left( \frac{M}{xM} \right) \rightarrow f^{t-1}_a(M) \overset{x}{\rightarrow} f^{t-1}_a(M) \overset{g}{\rightarrow} f^{t-1}_a \left( \frac{M}{xM} \right) \rightarrow \cdots \ (\ast)$$

Using the exact sequence $(\ast)$ we obtain the short exact sequence

$$0 \rightarrow f^{t-1}_a(M) \rightarrow f^{t-1}_a \left( \frac{M}{xM} \right) \rightarrow (0 : x) \rightarrow 0.$$

Now, this exact sequence induces the following long exact sequence:

$$0 \rightarrow \text{Hom}_R \left( \frac{R}{m}, f^{t-1}_a(M) \right) \rightarrow \text{Hom}_R \left( \frac{R}{m}, f^{t-1}_a \left( \frac{M}{xM} \right) \right) \rightarrow \text{Hom}_R \left( \frac{R}{m}, (0 : x) \right) \rightarrow \text{Ext}^1_R \left( \frac{R}{m}, f^{t-1}_a(M) \right) \rightarrow \cdots \ (\ast\ast)$$

By using $(\ast)$, $f^i_a \left( \frac{M}{xM} \right) \in S$ for all $i < t - 1$. Therefore by the induction hypothesis $\text{Hom}_R \left( \frac{R}{m}, f^{t-1}_a(M) \right) \in S$. Furthermore $\text{Ext}^1_R \left( \frac{R}{m}, f^{t-1}_a(M) \right) \in S$ because $f^{t-1}_a(M) \in S$. Thus in accordance with $(\ast\ast)$, $\text{Hom}_R \left( \frac{R}{m}, (0 : x) \right) \in S$. Since $x \in m$ according to [9,10.36] we have the following isomorphisms.

$$\text{Hom}_R \left( \frac{R}{m}, (0 : x) \right) \cong \text{Hom}_R \left( \frac{R}{x}, \text{Hom}_R \left( \frac{R}{x}, f^i_a(M) \right) \right) \cong \text{Hom}_R \left( \frac{R}{m} \otimes_R \frac{R}{x}, f^i_a(M) \right) \cong \text{Hom}_R \left( \frac{R}{m}, f^i_a(M) \right).$$

Consequently $\text{Hom}_R \left( \frac{R}{m}, f^i_a(M) \right) \in S$. 

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3. The formal cohomological dimension in a Serre subcategory

We recall from [3,Theorem 1.1] that for a finitely generated \( R \)-module \( M \),
\[
\sup \{ i \in \mathbb{N} \mid f^i_a(M) \neq 0 \} = \dim \left( \frac{M}{aM} \right).
\]

**Definition 3.1.** The formal cohomological dimension of \( M \) with respect to \( a \) in \( S \) is The supremum of the integers \( i \) such that \( f^i_a(M) \notin S \) and is denoted by \( f.c.d_S(a,M) \).

**Theorem 3.2.** Suppose that \( S \) is a Serre subcategory of the category of \( R \)-modules and \( R \rightarrow \homomorphisms \) and \( L \) and \( N \) are two finitely generated \( R \)-modules such that \( \text{Supp}_R(L) \subseteq \text{Supp}_R(N) \). Then \( f.c.d_S(a,L) \leq f.c.d_S(a,N) \).

**Proof.** It is enough to prove that \( f^i_a(L) \in S \) for all \( i > f.c.d_S(a,N) \) and all finitely generated \( R \)-module \( L \) such that \( \text{Supp}_R(L) \subseteq \text{Supp}_R(N) \). We use descending induction on \( i \). For all \( i > \dim \left( \frac{L}{aL} \right) + f.c.d_S(a,N) \), \( f^i_a(L) = 0 \in S \). Let \( i > f.c.d_S(a,N) \) and the result is proved for \( i + 1 \). By Gruson’s theorem, there is a chain \( 0 = L_0 \subset L_1 \subset \ldots \subset L_l = L \) of submodules of \( L \) such that \( \frac{L_i}{L_{i-1}} \) is a homomorphic image of a direct sum of finitely many copies of \( N \). Consider the exact sequence \( 0 \to L_{i-1} \to L_i \to \frac{L_i}{L_{i-1}} \to 0 \) \((i = 0,1,...,l)\). We may assume that \( l = 1 \). The exact sequence \( 0 \to K \to \bigoplus_{i=1}^t N \to L \to 0 \) where \( K \) is a finitely generated \( R \)-module induces the following long exact sequence:

\[
\cdots \to f^i_a\left( \bigoplus_{j=1}^t N \right) \to f^i_a(L) \to f^{i+1}_a(K) \to \cdots \ (\ast)
\]

Based on the induction hypothesis \( f^{i+1}_a(K) \in S \). Moreover \( f^ia_1\left( \bigoplus_{j=1}^t N \right) = \bigoplus_{j=1}^t f^i_a(N) \in S \). Hence it follows from the exact sequence \((\ast)\) that \( f^i_a(L) \in S \).

The next example shows that even if \( \text{Supp}_R(M) = \text{Supp}_R(N) \), then it may not true that \( f.c.d_S(a,M) = f.c.d_S(a,N) \).

**Example 3.3.** (See [4, Example 4.3 (i)]) \( R \) be a 2 dimensional complete regular local ring, \( S = 0 \) and \( a \) be an ideal of \( R \) with \( \dim \left( \frac{R}{a} \right) = 1 \). Then by using [5,Theorem 1.1], \( f.c.d_S(a,R) = 1 \) and \( f.c.d_S\left( a, \frac{R}{m} \right) = 0 \). Set \( M = R \oplus \frac{R}{m} \). Then \( \text{Supp}_R(M) = \text{Supp}_R(R) \).

\[
f.c.d_S(a,M) = \inf \left\{ f.c.d_S(a,R), f.c.d_S\left( a, \frac{R}{m} \right) \right\} = 0.
\]

**Corollary 3.4.** For all \( x \in a \cdot f.c.d_S(a,M) \geq f.c.d_S\left( a, \frac{M}{xM} \right) \).

**Corollary 3.5.** Suppose that \( 0 \to L \to M \to N \to 0 \) is an exact sequence of finitely generated \( R \)-modules. Then \( f.c.d_S(a,M) = \max \{ f.c.d_S(a,L), f.c.d_S(a,N) \} \).
Proof. Since $\text{Supp}_R(M) = \text{Supp}_R(L) \cup \text{Supp}_R(N)$ by referring to Theorem 3.2 we deduce that $f.c.d_\mathcal{S}(a, M) \geq f.c.d_\mathcal{S}(a, L)$ and $f.c.d_\mathcal{S}(a, M) \geq f.c.d_\mathcal{S}(a, N)$. Therefore $f.c.d_\mathcal{S}(a, M) \geq \max \{f.c.d_\mathcal{S}(a, L), f.c.d_\mathcal{S}(a, N)\}$.

Next we prove that $\max \{f.c.d_\mathcal{S}(a, L), f.c.d_\mathcal{S}(a, N)\} \geq f.c.d_\mathcal{S}(a, M)$.

Let $i > \max \{f.c.d_\mathcal{S}(a, L), f.c.d_\mathcal{S}(a, N)\}$. Then $f_i^L(N), f_i^L(L) \in \mathcal{S}$ and from the exact sequence $f_i^L(L) \to f_i^L(M) \to f_i^L(N)$ we conclude that $f_i^L(M) \in \mathcal{S}$. Thus, $\text{max}\{f.c.d_\mathcal{S}(a, L), f.c.d_\mathcal{S}(a, N)\} \geq f.c.d_\mathcal{S}(a, M)$.

We recall that the cohomological dimension of an $R$-module $M$ with respect to an ideal $a$ of $R$ in $\mathcal{S}$ is defined as

$$c.d_\mathcal{S}(a, M) := \sup \{i \in \mathbb{N}_0 | H_i^a(M) \notin \mathcal{S}\}.$$ 

The following lemma shows that when we considering the Artinianness of $f_i^a(M)$, we can assume that $M$ is $a$-torsion-free.

Lemma 3.6. Suppose that $a$ is an ideal of a local ring $(R, m)$ and $t$ be a non-negative integer. If $H_i^m(M) \in \mathcal{S}$ for all $i \geq t$, then the following are equivalent:

(a) $f_i^a(M) \in \mathcal{S}$ for all $i \geq t$.

(b) $f_i^a(M) \in \mathcal{S}$ for all $i \geq t$.

Proof. According to the hypothesis $t \geq c.d_\mathcal{S}(m, M)$. On the other hand $\text{Supp}_R(\Gamma_a(M)) \subseteq \text{Supp}_R(M)$. So by referring to [7,Theorem 3.5], $c.d_\mathcal{S}(m, \Gamma_a(M)) \leq c.d_\mathcal{S}(m, M)$. Thus, $t \geq c.d_\mathcal{S}(m, \Gamma_a(M))$ and $H_i^m(\Gamma_a(M)) \in \mathcal{S}$ for all $i \geq t$. Now, consider the following long exact sequence:

$$\cdots \to f_i^a(\Gamma_a(M)) \to f_i^a(M) \to f_i^a\left(\frac{M}{\Gamma_a(M)}\right) \to f_i^a(\Gamma_a(M)) \to \cdots (\ast)$$

According to [4,Lemma 2.3] $f_i^a(\Gamma_a(M)) \cong H_i^m(\Gamma_a(M))$. By using the hypothesis $f_i^a(\Gamma_a(M)) \in \mathcal{S}$ for all $i \geq t$. So it follows from the exact sequence ($\ast$) that $f_i^a(M) \in \mathcal{S}$ if and only if $f_i^a\left(\frac{M}{\Gamma_a(M)}\right) \in \mathcal{S}$ for all $i \geq t$.

Theorem 3.7. Let $(R, m)$ be a local ring and $M \in \mathcal{S}$ be a finitely generated $R$-module of dimension $d$ such that $c.d_\mathcal{S}(m, M) \leq f.c.d_\mathcal{S}(a, M)$. Then $f_i^a(M) \in \mathcal{S}$ where $t = \frac{f.d_i^a(M)}{a.f_i^a(M)} \in \mathcal{S}$ where $t = f.c.d_\mathcal{S}(a, M)$.

Proof. We use induction on $d = \dim(M)$. If $d = 0$, then $\dim\left(\frac{M}{a.M}\right) = 0$. Accordingly to [3, Theorem 1.1], $f_i^a(M) = 0$ for all $i > 0$. 

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Moreover $f_a^i(M) \cong M \in S$. By definition $H^i_m(M) \in S$ for all $i > t$. Therefore from the above lemma we can assume that $M$ is $a$-torsion-free and there is an $M$-regular element $x \in a$. Consider the long exact sequence:

$$\cdots \to f_a^i(M) \xrightarrow{x} f_a^i(M) \xrightarrow{f_a^i(M)} \frac{M}{xM} \xrightarrow{g} f_a^{i+1}(M) \xrightarrow{h} \cdots. \quad (*)$$

By using the hypothesis $f_a^i(M) \in S$ for all $i > t$ (because $t = f \cdot cd_S(a, M)$). So using the above long exact sequence $f_a^i(M/xM) \in S$ for all $i > t$. By induction hypothesis, $\frac{f_a^i(M/xM)}{a f_a^i(M/xM)} \in S$ because $\dim \left( \frac{M}{xM} \right) = \dim(M) - 1$.

Afterwards from the exact sequence (*) we get the following short exact sequence.

$$0 \to \text{Im}(f) \to f_a^t \left( \frac{M}{xM} \right) \to \text{Im}(g) \to 0$$

So we obtain the following long exact sequence.

$$\cdots \to \text{Tor}_n^R \left( \frac{R}{a}, \text{Im}(g) \right) \to \text{Im}(f) \to \frac{f_a^t(M)}{a f_a^t(M)} \to \text{Im}(g) \to 0,$$

Since $f_a^t(M) \in S$ and $\text{Im}(g)$ is a submodule of $f_a^{t+1}(M)$, we deduce that $\text{Tor}_n^R \left( \frac{R}{a}, \text{Im}(g) \right) \in S$. On the other hand, $\frac{f_a^t(M/xM)}{a f_a^t(M/xM)} \in S$. Therefore, $\frac{\text{Im}(f)}{a \text{Im}(f)} \in S$ by the above long exact sequence.

Now, consider the following long exact sequence.

$$\frac{f_a^t(M)}{a f_a^t(M)} \xrightarrow{x} f_a^t(M) \xrightarrow{f_a^t(M)} \frac{M}{xM} \xrightarrow{g} f_a^{i+1}(M) \xrightarrow{h} \cdots.$$

So, $\frac{f_a^t(M)}{a f_a^t(M)} \cong \frac{\text{Im}(f)}{a \text{Im}(f)}$ because $x \in a$. Consequently, $\frac{f_a^t(M)}{a f_a^t(M)} \in S$.

**Proposition 3.8.** For a finitely generated $R$-module $M$,

$$f \cdot cd_S(a, M) = \max \{ f \cdot cd_S(a, R_P) | P \in \text{Ass}_R(M) \}.$$

**Proof.** Set $N := \bigoplus_{P \in \text{Ass}_R(M)} R_P$. Then $\text{Supp}_R(M) = \text{Supp}_R(N)$. So, by Theorem 3.2 and Corollary 3.3, $f \cdot cd_S(a, M) = f \cdot cd_S(a, N) = \max \{ f \cdot cd_S(a, R_P) | P \in \text{Ass}_R(M) \}.$

**Proposition 3.9.** Assume that $a$ is an ideal of the local ring $(R, m)$. Then $\text{Hom}_R \left( \frac{R}{m}, f_a^0(M) \right) \in S$ if and only if $\text{Hom}_R \left( \frac{R}{m}, \mathcal{M}^a \right) \in S$.

**Proof.** It is enough to consider the following isomorphisms

$$\text{Hom}_R \left( \frac{R}{m}, f_a^0(M) \right) \cong \lim_{\mathcal{N} \in \mathcal{N}} \text{Hom}_R \left( \frac{R}{m}, f_a^0(\left( \frac{M}{a^n M} \right) \right) \cong \lim_{\mathcal{N} \in \mathcal{N}} \text{Hom}_R \left( \frac{R}{m}, \mathcal{M}^a \right).$$
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References