Formal Local Cohomology Modules and Serre Subcategories

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Abstract

Let $(R, m)$ be a Noetherian local ring, $a$ an ideal of $R$ and $M$ a finitely generated $R$-module. We investigate some properties of formal local cohomology modules with respect to a Serre subcategory. We provide a common language to indicate some properties of formal local cohomology modules.

1. Introduction

Throughout this paper $(R, m)$ is a commutative Noetherian local ring, $a$ an ideal of $R$ and $M$ is a finitely generated $R$-module. For an integer $i \in \mathbb{N}_0$, $H^i_a(N)$ denotes the $i$-th local cohomology module of $M$ with respect to $a$ as introduced by Grothendieck (cf. [1], [2]).

We shall consider the family of local cohomology modules $\{H^i_m\left(\frac{M}{a^nM}\right)\}_{n \in \mathbb{N}}$ for a non-negative integer $i \in \mathbb{N}_0$. With natural homomorphisms, this family forms an inverse system. Schenzel introduced the $i$-th formal local cohomology of $M$ with respect to $a$ in the form of $f^i_a(M) := \varprojlim_{n \in \mathbb{N}} H^i_m\left(\frac{M}{a^nM}\right)$, which is the $i$-th cohomology module of the $a$-adic completion of the Čech complex $\check{\mathcal{C}}_x \otimes_R M$, where $x$ denotes a system of elements of $R$ such that $\text{Rad}(x, R) = m$ (see [3, Definition 3.1]). He defines the formal grade as $f.\text{grade}(a, M) = \inf \{i \in \mathbb{N}_0 \mid f^i_a(M) \neq 0\}$. For any ideal $a$ of $R$ and finitely generated $R$-module $M$ the following statements hold:

(i) (See [3, Theorem 3.11]). If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of finitely generated $R$-modules, then there is the following long exact sequence:

$$\cdots \to f^i_a(M') \to f^i_a(M) \to f^i_a(M'') \to \cdots.$$ 

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(ii) (See [3, Theorem 1.3]). $\text{f. grade}(a, M) \leq \dim(M) - \text{cd}(a, M)$; some properties of formal local cohomology have been presented in [3, 4, 5 and 6].

Throughout this paper $S$ denotes a Serre subcategory of the category of $R$-modules and $R$-homomorphisms (we recall that a class $S$ of $R$-modules is a Serre subcategory of the category of $R$-modules and $R$-homomorphisms if $S$ is closed under taking submodules, quotients and extensions).

Our paper contains three sections. In Section 2, we shall define the formal grade of $a$ with respect to $M$ in $S$ as the infimum of the integers $i$ such that $f_a^i(M) \not\in S$ and is denoted by $f_{\text{grade}_S}(a, M)$. (See definition 2.1). Then we shall obtain some properties of this notion. We show that if $\Gamma_a(M)$ is a pure submodule of $M$, then $\text{Hom}_R \left( \frac{R}{m}, f_a^i(\Gamma_a(M)) \right)$ and $\text{Hom}_R \left( \frac{R}{m}, f_a^{i-1}(\Gamma_a(M)) \right)$ belong to $S$, where $t = f_{\text{grade}_S}(a, M)$.

In Section 3, we shall define the formal cohomological dimension of $a$ with respect to $M$ in $S$ as the supremum of the integers $i$ such that $f_a^i(M) \not\in S$ and is denoted by $f_{\text{cd}_S}(a, M)$. (See definition 3.1). The main result of this section is that if $f_a^i(M) \in S$ and $H^i_m(M) \in S$ for all $i > t$, then $\frac{R}{a} \otimes_R f_a^i(M)$ belongs to $S$.

### 2. The formal grade of a module in a Serre subcategory

**Definition 2.1.** The formal grade of $a$ with respect to $M$ in $S$ is the infimum of the integers $i$ such that $f_a^i(M) \not\in S$ and is denoted by $f_{\text{grade}_S}(a, M)$.

**Proposition 2.2.** Let $(R, m)$ be a local ring and $a$ be an ideal of $R$. If $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is an exact sequence of finitely generated $R$-modules, then the following statements hold.

(a) $f_{\text{grade}_S}(a, M) \geq \min\{f_{\text{grade}_S}(a, L), f_{\text{grade}_S}(a, N)\}$.

(b) $f_{\text{grade}_S}(a, L) \geq \min\{f_{\text{grade}_S}(a, M), f_{\text{grade}_S}(a, N) + 1\}$.

(c) $f_{\text{grade}_S}(a, N) \geq \min\{f_{\text{grade}_S}(a, L) - 1, f_{\text{grade}_S}(a, M)\}$.

**Proof.** According to [3, Theorem 3.11], the above short exact sequence induces the following long exact sequence.

$$\cdots \rightarrow f_a^{i-1}(N) \rightarrow f_a^i(L) \rightarrow f_a^i(M) \rightarrow f_a^i(N) \rightarrow f_a^{i+1}(L) \rightarrow \cdots$$

So, the result follows. 

338
Corollary 2.3. If $x = x_1, ..., x_n$ is a regular $M$-sequence, then $f.\text{grade}_S \left( \frac{a}{\text{rad}(a)} \right) \geq f.\text{grade}_S (a, M) - n$.

**Proof.** Consider the following exact sequence $(n \in \mathbb{N})$

$$0 \to \frac{M}{(x_1, ..., x_{n-1})M} \to \frac{M}{(x_1, ..., x_n)M} \to \frac{M}{(x_1, ..., x_n)M} \to 0$$

whenever $n = 1$ by $(x_1, ..., x_{n-1})M$ we mean 0.

Corollary 2.4. Let $a$ and $b$ be ideals of $R$. Then

(a) $f.\text{grade}_S (a \cap b, M) \geq \min \{f.\text{grade}_S (a, M), f.\text{grade}_S (b, M), f.\text{grade}_S ((a, b), M) + 1 \}$.

(b) $f.\text{grade}_S ((a, b), M) \geq \min \{f.\text{grade}_S (a \cap b, M) - 1, f.\text{grade}_S (a, M), f.\text{grade}_S (b, M) \}$.

**Proof.** For all $n \in \mathbb{N}$ there is a short exact sequence as follows:

$$0 \to \frac{M}{a^nM \cap b^nM} \to \frac{M}{a^nM} \oplus \frac{M}{b^nM} \to \frac{M}{(a^n, b^n)M} \to 0.$$  

By using [3, Theorem 5.1], the above exact sequence induces the following long exact sequence.

$$\cdots \to \lim_{n \in \mathbb{N}} H_m^1 \left( \frac{M}{(a \cap b)^nM} \right) \to \lim_{n \in \mathbb{N}} H_m^1 \left( \frac{M}{a^nM} \right) \oplus \lim_{n \in \mathbb{N}} H_m^1 \left( \frac{M}{b^nM} \right) \to \lim_{n \in \mathbb{N}} H_m^1 \left( \frac{M}{(a, b)^nM} \right) \to \cdots.$$  

So by using an argument similar to that of Proposition 2.2, the result follows.

Corollary 2.5. Assume that $M$ is a finitely generated $R$-module and $N_1$ and $N_2$ are submodules of $M$. Then considering the exact sequence $0 \to \frac{M}{N_1 \cap N_2} \to \frac{M}{N_1} \oplus \frac{M}{N_2} \to \frac{M}{N_1 + N_2} \to 0$ we shall have

(a) $f.\text{grade}_S \left( a, \frac{M}{N_1 \cap N_2} \right) \geq \min \{f.\text{grade}_S (a, \frac{M}{N_1}), f.\text{grade}_S (a, \frac{M}{N_2}), f.\text{grade}_S (a, \frac{M}{N_1 + N_2}) + 1 \}$.

(b) $f.\text{grade}_S \left( a, \frac{M}{N_1 + N_2} \right) \geq \min \{f.\text{grade}_S (a, \frac{M}{N_1}), f.\text{grade}_S (a, \frac{M}{N_2}), f.\text{grade}_S (a, \frac{M}{N_1 + N_2}) - 1, f.\text{grade}_S (a, \frac{M}{N_1 + N_2}) + 1 \}$.

Theorem 2.6. Let $a$ be an ideal of a local ring $(R, m)$, $M$ be a finitely generated $R$-module and $L$ be a pure submodule of $M$. Then $f.\text{grade}_S (a, L) \geq f.\text{grade}_S (a, M)$ where $S$ is a Serre subcategory of the category of $R$-modules and $R$-homomorphisms. In particular, $\inf \{i | H^i_m(L) \notin S \} \geq \inf \{i | H^i_m(M) \notin S \}$.  

339
Proof. Let \( L \) be a pure submodule of \( M \). So \( \frac{L}{a^nL} \rightarrow \frac{M}{a^nM} \) is pure for each \( n \in \mathbb{N} \). Now according to [8, Corollary 3.2 (a)], \( H_i^m\left(\frac{L}{a^nL}\right) \rightarrow H_i^m\left(\frac{M}{a^nM}\right) \) is injective. Since inverse limit is a left exact functor, \( f^i_a(L) \) is isomorphic to a submodule of \( f^i_a(M) \). Consequently, \( f_{grade_S(a,L)} \geq f_{grade_S(a,M)} \). If \( a = 0 \) then, \( f_{grade_S(0,M)} = \inf \{i|H_i^m(M) \not\in S\} \) and the result follows.

Corollary 2.7. If \( 0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 \) is a pure exact sequence of finitely generated \( R \)-modules, then \( \min \{f_{grade_S(a,L)}, f_{grade_S(a,N)} + 1\} \geq f_{grade_S(a,M)} \).

Proof. Since \( L \) is a pure submodule of \( M \), as a result of the previous theorem, \( f_{grade_S(a,L)} \geq f_{grade_S(a,M)} \). Hence we must prove that \( f_{grade_S(a,N)} + 1 \geq f_{grade_S(a,M)} \). We assume that \( i < f_{grade_S(a,M)} \) and we show that \( i < f_{grade_S(a,N)} + 1 \). Consider the following long exact sequence.

\[ \cdots \rightarrow f^i_a(M) \rightarrow f^{i-1}_a(N) \rightarrow f^i_a(L) \rightarrow f^i_a(M) \rightarrow f^i_a(N) \rightarrow \cdots \] (**)

If \( i < f_{grade_S(a,M)} \), then \( f^0_a(M), f^1_a(M), \ldots, f^{i-1}_a(M), f^i_a(M) \in S \). On the other hand, since \( i < f_{grade_S(a,M)} \leq f_{grade_S(a,L)}, f^0_a(L), \ldots, f^i_a(L) \in S \). Hence, it follows from (**) that \( f^0_a(N), \ldots, f^{i-1}_a(N) \in S \) and so \( i-1 < f_{grade_S(a,N)} \).

Theorem 2.8. Let \((R,m)\) be a local ring, \( a \) be an ideal of \( R \), \( S \) be a Serre subcategory of the category of \( R \)-modules and \( homomorphisms \) and \( M \in S \) be a finitely generated \( R \)-module such that \( \Gamma_a(M) \) is a pure submodule of \( M \). Then \( Hom_R\left(\frac{R}{a}, f^i_a(\Gamma_a(M))\right) \in S \), where \( t = f_{grade_S(a,M)} \).

Proof. Due to the previous theorem, \( f_{grade_S(a,\Gamma_a(M))} \geq f_{grade_S(a,M)} \). If \( f_{grade_S(a,\Gamma_a(M))} > f_{grade_S(a,M)} \), then the result is obvious. Accordingly, we assume that \( f_{grade_S(a,\Gamma_a(M))} = f_{grade_S(a,M)} \). We know that \( Supp(\Gamma_a(M)) \subseteq Var(a) \). By using [4, Lemma 2.3], \( f^i_a(\Gamma_a(M)) \cong H_i^m(\Gamma_a(M)) \) for all \( i \geq 0 \). So, if \( j < f_{grade_S(a,M)} \), then \( f^i_a(\Gamma_a(M)) \cong H_i^m(\Gamma_a(M)) \in S \) and \( Ext^k_R\left(\frac{R}{m}, H_j^m(\Gamma_a(M))\right) \in S \) for all \( k \geq 0 \) and \( j < f_{grade_S(a,M)} \). Moreover \( Ext^k_R\left(\frac{R}{m}, \Gamma_a(M)\right) \in S \), because \( \Gamma_a(M) \in S \). Consequently, according to [7, Theorem 2.2],

\[ Hom_R\left(\frac{R}{m}, H_j^m(\Gamma_a(M))\right) \in S \], where \( t = f_{grade_S(a,M)} \).

Corollary 2.9 With the same notations as Theorem 2.8, let \( X \in S \) be a submodule of \( f^i_a(\Gamma_a(M)) \), where \( t = f_{grade_S(a,M)} \). Then \( Hom_R\left(\frac{R}{m}, \frac{f_a^i(\Gamma_a(M))}{X}\right) \in S \).

Proof. Consider the long exact sequence:
In accordance with the previous theorem $\text{Hom}_R \left( \frac{R}{m}, f_a^t(\Gamma_a(M)) \right) \rightarrow \text{Hom}_R \left( \frac{R}{m}, \frac{f_a^t(\Gamma_a(M))}{x} \right)$ \rightarrow \text{Ext}_R^1 \left( \frac{R}{m}, X \right) \cdot (*)$

Theorem 2.10. Suppose that $a$ is an ideal of $(R, m)$ and $M \in \mathcal{S}$ is a finitely generated $R$-module such that $\Gamma_a(M)$ is a pure submodule of $M$. Then

$\text{Hom}_R \left( \frac{R}{m}, f_a^{t-1}\left( \frac{M}{\Gamma_a(M)} \right) \right) \in \mathcal{S},$ where $t = f. \text{grade}_a(R, M)$.

Proof. One has $f. \text{grade}_a(\alpha, \Gamma_a(M)) \geq f. \text{grade}_a(\alpha, M)$, by Theorem 2.6. Now, the exact sequence $0 \rightarrow \Gamma_a(M) \rightarrow M \rightarrow \frac{M}{\Gamma_a(M)} \rightarrow 0$ induces the following long exact sequence:

$\ldots \rightarrow f_a^{t-1}(\Gamma_a(M)) \rightarrow \beta \rightarrow f_a^{t-1}(M) \rightarrow \gamma \rightarrow f_a^{t-1}\left( \frac{M}{\Gamma_a(M)} \right) \rightarrow \xi \rightarrow f_a^{t-1}(\Gamma_a(M)) \rightarrow \varphi \rightarrow \ldots \cdot (*)$

Using the exact sequence $(*)$, we obtain the short exact sequence $0 \rightarrow \text{Im}(\beta) \rightarrow f_a^{t-1}(M) \rightarrow \text{Im}(\gamma) \rightarrow 0$. Since $f_a^{t-1}(M) \in \mathcal{S}$, $\text{Im}(\beta) \in \mathcal{S}$, and $\text{Im}(\gamma) \in \mathcal{S}$. Furthermore, we have the exact sequence $0 \rightarrow \text{Im}(\xi) \rightarrow H_a^t(\Gamma_a(M)) \rightarrow \text{Im}(\varphi) \rightarrow 0$ which induces the following long exact sequence:

$0 \rightarrow \text{Hom}_R \left( \frac{R}{m}, \text{Im}(\xi) \right) \rightarrow \text{Hom}_R \left( \frac{R}{m}, H_a^t(\Gamma_a(M)) \right) \rightarrow \ldots$.

Thus $\text{Hom}_R \left( \frac{R}{m}, \text{Im}(\xi) \right) \in \mathcal{S}$. Finally, by considering the short exact sequence $0 \rightarrow \text{Im}(\gamma) \rightarrow f_a^{t-1}\left( \frac{M}{\Gamma_a(M)} \right) \rightarrow \text{Im}(\xi) \rightarrow 0$ we can conclude that $\text{Hom}_R \left( \frac{R}{m}, f_a^{t-1}\left( \frac{M}{\Gamma_a(M)} \right) \right) \in \mathcal{S}$.

Theorem 2.11. Suppose that $R$ is complete with respect to the $a$-adic topolog and $M \in \mathcal{S}$ be a finitely generated $R$-module and $t$ a positive integer such that $f_a^t(M) \in \mathcal{S}$ for all $i < t$. Then $\text{Hom}_R \left( \frac{R}{m}, f_a^t(M) \right) \in \mathcal{S}$.

Proof. We use induction on $t$. Let $t=0$. Consider the following isomorphisms:

$\text{Hom}_R \left( \frac{R}{m}, f_a^0(M) \right) \cong \text{lim} \rightarrow \text{Hom}_R \left( \frac{R}{m}, H_a^2 \left( \frac{M}{a^i \cdot M} \right) \right) \cong \text{lim} \rightarrow \text{Hom}_R \left( \frac{R}{m}, \frac{M}{a^i \cdot M} \right)$

$\cong \text{Hom}_R \left( \frac{R}{m}, \lim \rightarrow \text{Hom}_R \left( \frac{R}{m}, \frac{M}{a^i \cdot M} \right) \right) \cong \text{Hom}_R \left( \frac{R}{m}, M^2 \right) \cong \text{Hom}_R \left( \frac{R}{m}, M \right)$
It is clear that $\text{Hom}_R \left( \frac{R}{m}, M \right) \in S$. So by the above isomorphisms, we deduce that $\text{Hom}_R \left( \frac{R}{m}, f_a^t(M) \right) \in S$.

Suppose that $t > 0$ and the result is true for all integer $i$ less than $t$. Set $N := f_a^t(M)$. Then $f_a^i(M) \cong f_a^i \left( \frac{M}{N} \right)$ for all $i > 0$, and so we may assume that $\text{depth}_R(M) > 0$. There is an $M$-regular element $x \in m$. The exact sequence $0 \to M \xrightarrow{x} M \to \frac{M}{xM} \to 0$ induces the following long exact sequence:

$$\cdots \to f_a^{t-2}(M) \xrightarrow{x} f_a^{t-2}(M) \xrightarrow{f} f_a^{t-2} \left( \frac{M}{xM} \right) \to f_a^{t-1}(M) \xrightarrow{\partial} f_a^{t-1} \left( \frac{M}{xM} \right) \to f_a^{t}(M) \xrightarrow{\partial} f_a^{t} \left( M \right) \to \cdots \quad \text{(**)}$$

Using the exact sequence (**) we obtain the short exact sequence

$$0 \to f_a^{t-1} \left( \frac{M}{x f_a^{t-1}(M)} \right) \to f_a^{t-1} \left( \frac{M}{xM} \right) \to (0 : x) \to 0.$$

Now, this exact sequence induces the following long exact sequence:

$$0 \to \text{Hom}_R \left( \frac{R}{m}, f_a^{t-1} \left( \frac{M}{xM} \right) \right) \to \text{Hom}_R \left( \frac{R}{m}, f_a^{t-1} \left( \frac{M}{x f_a^{t-1}(M)} \right) \right) \to \text{Hom}_R \left( \frac{R}{m}, (0 : x) \right) \to \text{Ext}_R^1 \left( \frac{R}{m}, f_a^{t-1} \left( \frac{M}{x f_a^{t-1}(M)} \right) \right) \to \cdots \quad \text{(***)}$$

By using (**), $f_a^i \left( \frac{M}{xM} \right) \in S$ for all $i < t - 1$. Therefore by the induction hypothesis $\text{Hom}_R \left( \frac{R}{m}, f_a^{t-1} \left( \frac{M}{xM} \right) \right) \in S$. Furthermore $\text{Ext}_R^1 \left( \frac{R}{m}, f_a^{t-1} \left( \frac{M}{x f_a^{t-1}(M)} \right) \right) \in S$ because $f_a^{t-1}(M) \in S$. Thus in accordance with (***) $\text{Hom}_R \left( \frac{R}{m}, (0 : x) \right) \in S$. Since $x \in m$ according to [9,10.86] we have the following isomorphisms.

$$\text{Hom}_R \left( \frac{R}{m}, (0 : x) \right) \cong \text{Hom}_R \left( \frac{R}{xR}, f_a^t \left( M \right) \right) \cong \text{Hom}_R \left( \frac{R}{xR}, f_a^t \left( M \right) \right) \cong \text{Hom}_R \left( \frac{R}{m}, f_a^t \left( M \right) \right) \cong \text{Hom}_R \left( \frac{R}{m}, f_a^t \left( M \right) \right) \in S.$$
3. The formal cohomological dimension in a Serre subcategory

We recall from [3, Theorem 1.1] that for a finitely generated \( R \)-module \( M \),
\[
\sup \{ i \in \mathbb{N}_0 | f^i_a(M) \neq 0 \} = \dim \left( \frac{M}{aM} \right).
\]

**Definition 3.1.** The formal cohomological dimension of \( M \) with respect to \( a \) in \( S \) is the supremum of the integers \( i \) such that \( f^i_a(M) \notin S \) and is denoted by \( f.c.d_S(a, M) \).

**Theorem 3.2.** Suppose that \( S \) is a Serre subcategory of the category of \( R \)-modules and \( R \)-homomorphisms and \( L \) and \( N \) are two finitely generated \( R \)-modules such that \( \text{Supp}_R(L) \subseteq \text{Supp}_R(N) \). Then \( f.c.d_S(a, L) \leq f.c.d_S(a, N) \).

**Proof.** It is enough to prove that \( f^i_a(L) \in S \) for all \( i > f.c.d_S(a, N) \) and all finitely generated \( R \)-module \( L \) such that \( \text{Supp}_R(L) \subseteq \text{Supp}_R(N) \). We use descending induction on \( i \). For all \( i > \dim \left( \frac{L}{aL} \right) + f.c.d_S(a, N) \), \( f^i_a(L) = 0 \in S \). Let \( i > f.c.d_S(a, N) \) and the result is proved for \( i + 1 \). By Gruson’s theorem, there is a chain \( 0 = L_0 \subset L_1 \subset \cdots \subset L_l = L \) of submodules of \( L \) such that \( \frac{L_i}{L_{i-1}} \) is a homomorphic image of a direct sum of finitely many copies of \( N \). Consider the exact sequence \( 0 \to L_{i-1} \to L_i \to \frac{L_i}{L_{i-1}} \to 0 \) \((i = 0, l, \ldots, l)\). We may assume that \( l = 1 \). The exact sequence \( 0 \to K \to \bigoplus_{j=i}^t N \to L \to 0 \) where \( K \) is a finitely generated \( R \)-module induces the following long exact sequence:
\[
\cdots \to f^i_a\left( \bigoplus_{j=i}^t N \right) \to f^i_a(L) \overset{f^i_a(K)}{\to} f^{i+1}_a(K) \to \cdots \tag{*}
\]
Based on the induction hypothesis \( f^{i+1}_a(K) \in S \). Moreover \( f^i_a\left( \bigoplus_{j=i}^t N \right) = \bigoplus_{j=i}^t f^i_a(N) \in S \) for all \( i > f.c.d_S(a, N) \). Hence it follows from the exact sequence \((*)\) that \( f^i_a(L) \in S \).

The next example shows that even if \( \text{Supp}_R(M) = \text{Supp}_R(N) \), then it may not true that \( f.gra_S(a, M) = f.gra_S(a, N) \).

**Example 3.3.** (See [4, Example 4.3 (i)]) Let \( (R, m) \) be a 2 dimensional complete regular local ring, \( S = 0 \) and \( a \) be an ideal of \( R \) with \( \dim \left( \frac{R}{a} \right) = 1 \). Then by using [5, Theorem 1.1], \( f.gra_S(a, R) = 1 \) and \( f.gra_S\left( a, \frac{R}{m} \right) = 0 \). Set \( M := R \oplus \frac{R}{m} \).

Then \( \text{Supp}_R(M) = \text{Supp}_R(R) \). But
\[
f.gra_S(a, M) = \inf \left\{ f.gra_S(a, R), f.gra_S\left( a, \frac{R}{m} \right) \right\} = 0.
\]

**Corollary 3.4.** For all \( x \in a \), \( f.c.d_S(a, M) \geq f.c.d_S\left( a, \frac{M}{xM} \right) \).

**Corollary 3.5.** Suppose that \( 0 \to L \to M \to N \to 0 \) is an exact sequence of finitely generated \( R \)-modules. Then \( f.c.d_S(a, M) = \max \{ f.c.d_S(a, L), f.c.d_S(a, N) \} \).
Proof. Since $\text{Supp}_R(M) = \text{Supp}_R(L) \cup \text{Supp}_R(N)$ by referring to Theorem 3.2 we deduce that $f. cd_S(a, M) \geq f. cd_S(a, L)$ and $f. cd_S(a, M) \geq f. cd_S(a, N)$. Therefore $f. cd_S(a, M) \geq \max \{f. cd_S(a, L), f. cd_S(a, N)\}$.

Next we prove that $\max \{f. cd_S(a, L), f. cd_S(a, N)\} \geq f. cd_S(a, M)$.

Let $i > \max \{f. cd_S(a, L), f. cd_S(a, N)\}$. Then $f^i_a(N), f^i_a(L) \in S$ and from the exact sequence $f^i_a(L) \rightarrow f^i_a(M) \rightarrow f^i_a(N)$ we conclude that $f^i_a(M) \in S$. Thus, $\max \{f. cd_S(a, L), f. cd_S(a, N)\} \geq f. cd_S(a, M)$.

We recall that the cohomological dimension of an $R$-module $M$ with respect to an ideal $a$ of $R$ in $S$ is defined as

$$cd_S(a, M) := \sup \{i \in \mathbb{N} | H^i_a(M) \notin S\}.$$  

The following lemma shows that when we considering the Artinianness of $f^i_a(M)$, we can assume that $M$ is $a$-torsion-free.

Lemma 3.6. Suppose that $a$ is an ideal of a local ring $(R, m)$ and $t$ be a non-negative integer. If $H^i_m(M) \in S$ for all $i \geq t$, then the following are equivalent:

(a) $f^i_a(M) \in S$ for all $i \geq t$.

(b) $f^i_a(M) \in S$ for all $i \geq t$.

Proof. According to the hypothesis $t > cd_S(m, M)$. On the other hand $\text{Supp}_R(\Gamma_a(M)) \subseteq \text{Supp}_R(M)$. So by referring to [7, Theorem 3.5], $cd_S(m, \Gamma_a(M)) \leq cd_S(m, M)$. Thus, $t > cd_S(m, \Gamma_a(M))$ and $H^i_m(\Gamma_a(M)) \in S$ for all $i \geq t$. Now, consider the following long exact sequence:

$$\cdots \rightarrow f^i_a(\Gamma_a(M)) \rightarrow f^i_a(M) \rightarrow f^i_a(M) \rightarrow f^{i+1}_a(M) \rightarrow \cdots.$$ (*)

According to [4, Lemma 2.3] $f^i_a(\Gamma_a(M)) \cong H^i_m(\Gamma_a(M))$. By using the hypothesis $f^i_a(\Gamma_a(M)) \in S$ for all $i \geq t$. So it follows from the exact sequence (*) that $f^i_a(M) \in S$ if and only if $f^i_a(M) \in S$ for all $i \geq t$.

Theorem 3.7. Let $(R, m)$ be a local ring and $M \in S$ be a finitely generated $R$-module of dimension $d$ such that $cd_S(m, M) \leq f. cd_S(a, M)$. Then $f^i_a(M) \in S$ where $t = f. cd_S(a, M)$.

Proof. We use induction on $d = \dim (M)$. If $d = 0$, then $\dim \left(\frac{M}{aM}\right) = 0$. Accordingly to [3, Theorem 1.1], $f^i_a(M) = 0$ for all $i > 0$. 

344
Moreover $f_a^0(M) \cong M \in \mathcal{S}$. By definition $H^i_M(N) \in \mathcal{S}$ for all $i > t$. Therefore from the above lemma we can assume that $M$ is $a$-torsion-free and there is an $M$-regular element $x \in a$. Consider the long exact sequence:

$$\cdots \to f_a^i(M) \xrightarrow{x} f_a^i(M) \xrightarrow{f} f_a^{i+1}(M) \xrightarrow{g} f_a^{i+1}(M) \xrightarrow{h} \cdots.$$ 

By using the hypothesis $f_a^i(M) \in \mathcal{S}$ for all $i > t$ (because $t = f_{cd}(a, M)$). So using the above long exact sequence $f_a^i\left(\frac{M}{xM}\right) \in \mathcal{S}$ for all $i > t$. By induction hypothesis, $\frac{f_a^i(M)}{a f_a^i(M)} \in \mathcal{S}$ because $\dim\left(\frac{M}{xM}\right) = \dim(M) - 1$.

Afterwards from the exact sequence (*) we get the following short exact sequence.

$$0 \to \text{Im}(f) \to f_a^i\left(\frac{M}{xM}\right) \xrightarrow{g} \text{Im}(g) \to 0.$$ 

So we obtain the following long exact sequence.

$$\cdots \to Tor^R_1\left(\frac{R}{a}, \text{Im}(g)\right) \xrightarrow{a f_a^i(M)} f_a^i\left(\frac{M}{xM}\right) \xrightarrow{a f_a^i(M)} \text{Im}(g) \to 0.$$ 

Since $f_a^i(M) \in \mathcal{S}$ and $\text{Im}(g)$ is a submodule of $f_a^{i+1}(M)$, we deduce that $Tor^R_1\left(\frac{R}{a}, \text{Im}(g)\right) \in \mathcal{S}$. On the other hand, $\frac{f_a^i(M)}{a f_a^i(M)} \in \mathcal{S}$. Therefore, $\frac{\text{Im}(f)}{a f_a^i(M)} \in \mathcal{S}$ by the above long exact sequence.

Now, consider the following long exact sequence.

$$\frac{f_a^i(M)}{a f_a^i(M)} \xrightarrow{a f_a^i(M)} f_a^i\left(\frac{M}{xM}\right) \xrightarrow{a f_a^i(M)} \text{Im}(f) \to 0.$$ 

So, $\frac{f_a^i(M)}{a f_a^i(M)} \cong \frac{\text{Im}(f)}{a f_a^i(M)}$ because $x \in a$. Consequently, $\frac{f_a^i(M)}{a f_a^i(M)} \in \mathcal{S}$.

**Proposition 3.8.** For a finitely generated $R$-module $M$,

$$f_{cd}(a, M) = \max \{f_{cd}(a, \frac{R}{P}) | P \in \text{Ass}_R(M)\}.$$ 

**Proof.** Set $N := \bigoplus_{P \in \text{Ass}_R(M)} \frac{R}{P}$. Then $\text{Supp}_R(M) = \text{Supp}_R(N)$. So, by Theorem 3.2 and Corollary 3.5, $f_{cd}(a, M) = f_{cd}(a, N) = \max \{f_{cd}(a, \frac{R}{P}) | P \in \text{Ass}_R(M)\}$.

**Proposition 3.9.** Assume that $a$ is an ideal of the local ring $(R, m)$. Then $Hom^R_R(\frac{R}{m}, f_a^0(M)) \in \mathcal{S}$ if and only if $Hom^R_R(\frac{R}{m}, \tilde{M}) \in \mathcal{S}$.

**Proof.** It is enough to consider the following isomorphisms

$$Hom^R_R\left(\frac{R}{m}, f_a^0(M)\right) \cong \bigoplus_{n \in \mathbb{N}} Hom^R_R\left(\frac{R}{m}, f_a^0(M)\right) \cong \bigoplus_{n \in \mathbb{N}} Hom^R_R\left(\frac{R}{m}, a^n M\right) \cong Hom_R\left(\frac{R}{m}, \tilde{M}\right).$$

345
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References