Formal Local Cohomology Modules and Serre Subcategories

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Abstract

Let \((R, m)\) be a Noetherian local ring, \(a\) an ideal of \(R\) and \(M\) a finitely generated \(R\)-module. We investigate some properties of formal local cohomology modules with respect to a Serre subcategory. We provide a common language to indicate some properties of formal local cohomology modules.

1. Introduction

Throughout this paper \((R, m)\) is a commutative Noetherian local ring, \(a\) an ideal of \(R\) and \(M\) is a finitely generated \(R\)-module. For an integer \(i \in \mathbb{N}_0\), \(H^i_a(N)\) denotes the \(i\)-th local cohomology module of \(M\) with respect to \(a\) as introduced by Grothendieck (cf. [1], [2]).

We shall consider the family of local cohomology modules \(\{H^i_m\left(\frac{M}{a^nM}\right)\}_{n \in \mathbb{N}}\) for a non-negative integer \(i \in \mathbb{N}_0\). With natural homomorphisms; this family forms an inverse system. Schenzel introduced the \(i\)-th formal local cohomology of \(M\) with respect to \(a\) in the form of \(f^i_a(M) := \lim_{n \to \infty} H^i_m\left(\frac{M}{a^nM}\right)\), which is the \(i\)-th cohomology module of the \(a\)-adic completion of the \(\check{\text{C}}ech\) complex \(\check{\cdot} \otimes_R M\), where \(\check{\cdot}\) denotes a system of elements of \(R\) such that \(\text{Rad}(\check{\cdot}, R) = m\) (see [3, Definition 3.1]). He defines the formal grade as \(f.\text{grade}(a, M) = \inf \{i \in \mathbb{N}_0 | f^i_a(M) \neq 0\}\). For any ideal \(a\) of \(R\) and finitely generated \(R\)-module \(M\) the following statements hold:

(i) (See [3, Theorem 3.11]). If \(0 \to M' \to M \to M'' \to 0\) is a short exact sequence of finitely generated \(R\)-modules, then there is the following long exact sequence:
\[
\cdots \to f^i_a(M') \to f^i_a(M) \to f^i_a(M'') \to \cdots.
\]

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(ii) (See [3, Theorem 1.3]). $f.\ grade (a, M) \leq \dim(M) - cd(a, M)$; some properties of formal local cohomology have been presented in [3, 4, 5 and 6].

Throughout this paper $S$ denotes a Serre subcategory of the category of $R$-modules and $R$-homomorphisms (we recall that a class $S$ of $R$-modules is a Serre subcategory of the category of $R$-modules and $R$-homomorphisms if $S$ is closed under taking submodules, quotients and extensions).

Our paper contains three sections. In Section 2, we shall define the formal grade of $a$ with respect to $M$ in $S$ as the infimum of the integers $i$ such that $f^i_a(M) \notin S$ and is denoted by $f.\ grade_S(a, M)$. (See definition 2.1). Then we shall obtain some properties of this notion. We show that if $\Gamma_a(M)$ is a pure submodule of $M$, then $\text{Hom}_R(\frac{R}{m}, f^t_a(\Gamma_a(M)))$ and $\text{Hom}_R(\frac{R}{m}, f^{t-1}_a(\frac{M}{\Gamma_a(M)}))$ belong to $S$, where $t = f.\ grade_S(a, M)$.

In Section 3, we shall define the formal cohomological dimension of $a$ with respect to $M$ in $S$ as the supremum of the integers $i$ such that $f^i_a(M) \notin S$ and is denoted by $f.\ cd_S(a, M)$. (See definition 3.1). The main result of this section is that if $f^i_a(M) \in S$ and $H^t_m(M) \in S$ for all $i > t$, then $\frac{R}{a} \otimes_R f^i_a(M)$ belongs to $S$.

2. The formal grade of a module in a Serre subcategory

**Definition 2.1.** The formal grade of $a$ with respect to $M$ in $S$ is the infimum of the integers $i$ such that $f^i_a(M) \notin S$ and is denoted by $f.\ grade_S(a, M)$.

**Proposition 2.2.** Let $(R, m)$ be a local ring and $a$ be an ideal of $R$. If $0 \to L \to M \to N \to 0$ is an exact sequence of finitely generated $R$-modules, then the following statements hold.

(a) $f.\ grade_S(a, M) \geq \min\{f.\ grade_S(a, L), f.\ grade_S(a, N)\}$.

(b) $f.\ grade_S(a, L) \geq \min\{f.\ grade_S(a, M), f.\ grade_S(a, N) + 1\}$.

(c) $f.\ grade_S(a, N) \geq \min\{f.\ grade_S(a, L) - 1, f.\ grade_S(a, M)\}$.

**Proof.** According to [3, Theorem 3.11], the above short exact sequence induces the following long exact sequence.

$$\cdots \to f^{i-1}_a(N) \to f^i_a(L) \to f^i_a(M) \to f^i_a(N) \to f^{i+1}_a(L) \to \cdots.$$ 

So, the result follows.
Corollary 2.3. If $\underline{x} = x_1, ..., x_n$ is a regular $M$-sequence, then $f.\text{grade}_S\left(\frac{a}{\sum M}\right) \geq f.\text{grade}_S(\alpha, M) - n$.

Proof. Consider the following exact sequence $(n \in \mathbb{N})$

$$0 \rightarrow \frac{M}{(x_1, \ldots, x_{n-1})M} \xrightarrow{x_n} \frac{M}{(x_1, \ldots, x_{n})M} \xrightarrow{\text{nat.}} \frac{M}{(x_1, \ldots, x_n)M} \rightarrow 0$$

whenever $n = 1$ by $(x_1, ..., x_{n-1})M$ we means 0.

Corollary 2.4. Let $a$ and $b$ be ideals of $R$. Then

(a) $f.\text{grade}_S(a \cap b, M) \geq \min\{f.\text{grade}_S(a, M), f.\text{grade}_S(b, M), f.\text{grade}_S((a, b), M) + 1\}$.

(b) $f.\text{grade}_S((a, b), M) \geq \min\{f.\text{grade}_S(a \cap b, M) - 1, f.\text{grade}_S(a, M), f.\text{grade}_S(b, M)\}$.

Proof. For all $n \in \mathbb{N}$ there is a short exact sequence as follows:

$$0 \rightarrow \frac{M}{a^nM \cap b^nM} \rightarrow \frac{M}{a^nM} \oplus \frac{M}{b^nM} \rightarrow \frac{M}{(a^n, b^n)M} \rightarrow 0.$$

By using [3, Theorem 5.1], the above exact sequence induces the following long exact sequence.

$$\ldots \rightarrow \lim_{n \in \mathbb{N}} H^1_m\left(\frac{M}{(a \cap b)^nM}\right) \rightarrow \lim_{n \in \mathbb{N}} H^1_m\left(\frac{M}{a^nM}\right) \oplus \lim_{n \in \mathbb{N}} H^1_m\left(\frac{M}{b^nM}\right) \rightarrow \lim_{n \in \mathbb{N}} H^1_m\left(\frac{M}{(a, b)^nM}\right) \rightarrow \ldots.$$

So by using an argument similar to that of Proposition 2.2, the result follows.

Corollary 2.5. Assume that $M$ is a finitely generated $R$-module and $N_1$ and $N_2$ are submodules of $M$. Then considering the exact sequence $0 \rightarrow \frac{M}{N_1 \cap N_2} \rightarrow \frac{M}{N_1} \oplus \frac{M}{N_2} \rightarrow \frac{M}{N_1 + N_2} \rightarrow 0$ we shall have

(a) $f.\text{grade}_S\left(a, \frac{M}{N_1 \cap N_2}\right) \geq \min\{f.\text{grade}_S(a, \frac{M}{N_1}), f.\text{grade}_S(a, \frac{M}{N_2}), f.\text{grade}_S(a, MN2), f.\text{grade}_S(a, MN1 + N2 + 1)\}$.

(b) $f.\text{grade}_S\left(a, \frac{M}{N_1 + N_2}\right) \geq \min\{f.\text{grade}_S\left(a, \frac{M}{N_1}\right) - 1, f.\text{grade}_S\left(a, \frac{M}{N_2}\right), f.\text{grade}_S(a, MN1), f.\text{grade}_S(a, MN2)\}$.

Theorem 2.6. Let $a$ be an ideal of a local ring $(R, \mathfrak{m})$, $M$ be a finitely generated $R$-module and $L$ be a pure submodule of $M$. Then $f.\text{grade}_S(a, L) \geq f.\text{grade}_S(a, M)$ where $S$ is a Serre subcategory of the category of $R$-modules and $R$-homomorphisms. In particular, $\inf \{i | H^i_m(L) \notin S\} \geq \inf \{i | H^i_m(M) \notin S\}$.
Proof. Let \( L \) be a pure submodule of \( M \). So \( \frac{L}{a^nL} \to \frac{M}{a^nM} \) is pure for each \( n \in \mathbb{N} \). Now according to [8, Corollary 3.2 (a)], \( H^i_m \left( \frac{L}{a^nL} \right) \to H^i_m \left( \frac{M}{a^nM} \right) \) is injective. Since inverse limit is a left exact functor, \( f^i_m(L) \) is isomorphic to a submodule of \( f^i_m(M) \). Consequently \( f.g. grade_S(a,L) \geq f.g. grade_S(a,M) \). If \( a = 0 \) then \( f.g. grade_S(0,M) = \inf \{ |H^i_m(M) \notin S \} \) and the result follows.

Corollary 2.7. If \( 0 \to L \to M \to N \to 0 \) is a pure exact sequence of finitely generated \( R \)-modules, then \( \min \{ f.g. grade_S(a,L), f.g. grade_S(a,N) + 1 \} \geq f.g. grade_S(a,M) \).

Proof. Since \( L \) is a pure submodules of \( M \), as a result of the previous theorem, \( f.g. grade_S(a,L) \geq f.g. grade_S(a,M) \). Hence we must prove that \( f.g. grade_S(a,N) + 1 \geq f.g. grade_S(a,M) \). We assume that \( i < f.g. grade_S(a,M) \) and we show that \( i < f.g. grade_S(a,N) + 1 \). Consider the following long exact sequence.

\[
... \to f^i_a^{-1}(M) \to f^i_a^{-1}(N) \to f^i_a(L) \to f^i_a(M) \to f^i_a(N) \to ... \, (**)
\]

If \( i < f.g. grade_S(a,M) \), then \( f^i_a(M), f^i_a(M), ..., f^1_a(M) \in S \). On the other hand, since \( i < f.g. grade_S(a,N) \leq f.g. grade_S(a,L), f^i_a(L), ..., f^2_a(L) \in S \). Hence, it follows from (**) that \( f^i_a(N), ..., f^i_a(N) \in S \) and so \( i - 1 < f.g. grade_S(a,N) \).

Theorem 2.8. Let \( (R, m) \) be a local ring, \( a \) be an ideal of \( R \), \( S \) be a Serre subcategory of the category of \( R \)-modules and \( R \)-homomorphisms and \( M \in S \) be a finitely generated \( R \)-module such that \( \Gamma_a(M) \) is a pure submodule of \( M \). Then \( \text{Hom}_R \left( \frac{R}{a}, f^i_a(\Gamma_a(M)) \right) \in S \), where \( t = f.g. grade_S(a,M) \).

Proof. Due to the previous theorem, \( f.g. grade_S(a, \Gamma_a(M)) \geq f.g. grade_S(a,M) \). If \( f.g. grade_S(a, \Gamma_a(M)) > f.g. grade_S(a,M) \), then the result is obvious. Accordingly, we assume that \( f.g. grade_S(a, \Gamma_a(M)) = f.g. grade_S(a,M) \). We know that \( \text{Supp}(\Gamma_a(M)) \subseteq \text{Var}(a) \). By using [4, Lemma 2.3], \( f^i_a(\Gamma_a(M)) \cong H^i_m(\Gamma_a(M)) \) for all \( i \geq 0 \). So, if \( j < f.g. grade_S(a,M) \), then \( f^j_a(\Gamma_a(M)) \cong H^j_m(\Gamma_a(M)) \in S \) and \( \text{Ext}^k_R \left( \frac{R}{m}, H^j_m(\Gamma_a(M)) \right) \in S \) for all \( k \geq 0 \) and \( j < f.g. grade_S(a,M) \). Moreover \( \text{Ext}^i_R \left( \frac{R}{m}, \Gamma_a(M) \right) \in S \), because \( \Gamma_a(M) \in S \). Consequently, according to [7, Theorem 2.2],

\[
\text{Hom}_R \left( \frac{R}{m}, H^i_m(\Gamma_a(M)) \right) \in S \), where \( t = f.g. grade_S(a,M) \).
\]

Corollary 2.9 With the same notations as Theorem 2.8, let \( X \in S \) be a submodule of \( f^i_a(\Gamma_a(M)) \), where \( t = f.g. grade_S(a,M) \). Then \( \text{Hom}_R \left( \frac{R}{m}, \frac{f^i_a(\Gamma_a(M))}{X} \right) \in S \).

Proof. Consider the long exact sequence:
In accordance with the previous theorem $\text{Hom}_R \left( \frac{R}{m}, f_{a}^{t}(\Gamma_{a}(M)) \right) \rightarrow \text{Ext}_R^{1} \left( \frac{R}{m}, X \right) \rightarrow \text{Ext}_R^{1} \left( \frac{R}{m}, X \right)$. Moreover $\text{Ext}_R^{1} \left( \frac{R}{m}, X \right) \in S$. It follows from the exact sequence (*) that $\text{Hom}_R \left( \frac{R}{m}, \frac{f_{a}^{t}(\Gamma_{a}(M))}{x} \right) \in S$.

**Theorem 2.10.** Suppose that $a$ is an ideal of $(R, m)$ and $M \in S$ is a finitely generated $R$-module such that $\Gamma_{a}(M)$ is a pure submodule of $M$. Then $\text{Hom}_R \left( \frac{R}{m}, f_{a}^{t-1} \left( \frac{M}{\Gamma_{a}(M)} \right) \right) \in S$, where $t = f. \text{grade}_{S}(a, M)$.

**Proof.** One has $f. \text{grade}_{S}(a, \Gamma_{a}(M)) \geq f. \text{grade}_{S}(a, M)$, by Theorem 2.6. Now, the exact sequence $0 \rightarrow \Gamma_{a}(M) \rightarrow M \rightarrow \frac{M}{\Gamma_{a}(M)} \rightarrow 0$ induces the following long exact sequence:

$$\cdots \rightarrow f_{a}^{t-1} \left( \Gamma_{a}(M) \right) \rightarrow \frac{\beta}{f_{a}^{t-1} \left( M \right)} \rightarrow \frac{\gamma}{f_{a}^{t-1} \left( \frac{M}{\Gamma_{a}(M)} \right)} \rightarrow f_{a}^{t} \left( \Gamma_{a}(M) \right) \rightarrow \cdots \rightarrow \frac{\psi}{\ldots} \rightarrow \frac{\xi}{\ldots} \cdots \rightarrow \ldots \rightarrow \frac{\alpha}{\ldots} \rightarrow \frac{\beta}{\ldots} \rightarrow \frac{\gamma}{\ldots} \rightarrow \frac{\xi}{\ldots} \rightarrow \frac{\psi}{\ldots} \rightarrow \cdots \rightarrow \ldots \rightarrow \ldots \rightarrow \ldots \rightarrow \ldots$$

Using the exact sequence (*), we obtain the short exact sequence $0 \rightarrow \text{Im}(\beta) \rightarrow f_{a}^{t-1} \left( M \right) \rightarrow \text{Im}(\gamma) \rightarrow 0$. Since $f_{a}^{t-1} \left( M \right) \in S$, $\text{Im}(\beta) \in S$ and $\text{Im}(\gamma) \in S$. Furthermore, we have the exact sequence $0 \rightarrow \text{Im}(\xi) \rightarrow H_{m}^{t} \left( \Gamma_{a}(M) \right) \rightarrow \text{Im}(\varphi) \rightarrow 0$ which induces the following long exact sequence:

$$0 \rightarrow \text{Hom}_R \left( \frac{R}{m}, \text{Im}(\xi) \right) \rightarrow \text{Hom}_R \left( \frac{R}{m}, H_{m}^{t} \left( \Gamma_{a}(M) \right) \right) \rightarrow \cdots.$$ 

Thus $\text{Hom}_R \left( \frac{R}{m}, \text{Im}(\xi) \right) \in S$. Finally, by considering the short exact sequence $0 \rightarrow \text{Im}(\gamma) \rightarrow f_{a}^{t-1} \left( \frac{M}{\Gamma_{a}(M)} \right) \rightarrow \text{Im}(\xi) \rightarrow 0$ we can conclude that $\text{Hom}_R \left( \frac{R}{m}, f_{a}^{t-1} \left( \frac{M}{\Gamma_{a}(M)} \right) \right) \in S$.

**Theorem 2.11.** Suppose that $R$ is complete with respect to the $a$-adic topology and $M \in S$ be a finitely generated $R$-module and $t$ a positive integer such that $f_{a}^{t} \left( M \right) \in S$ for all $i < t$. Then $\text{Hom}_R \left( \frac{R}{m}, f_{a}^{t} \left( M \right) \right) \in S$.

**Proof.** We use induction on $t$. Let $t=0$. Consider the following isomorphisms.

$$\text{Hom}_R \left( \frac{R}{m}, f_{a}^{0}(M) \right) \cong \lim_{\rightarrow} \text{Hom}_R \left( \frac{R}{m}, H_{a}^{0} \left( \frac{M}{a^{i}M} \right) \right) \cong \lim_{\rightarrow} \text{Hom}_R \left( \frac{R}{m}, a^{i}M \right) \cong \text{Hom}_R \left( \frac{R}{m}, M \right).$$

$$\text{Hom}_R \left( \frac{R}{m}, f_{a}^{0}(M) \right) \cong \lim_{\rightarrow} \text{Hom}_R \left( \frac{R}{m}, H_{a}^{0} \left( \frac{M}{a^{i}M} \right) \right) \cong \lim_{\rightarrow} \text{Hom}_R \left( \frac{R}{m}, M \right)$$

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It is clear that $\text{Hom}_R \left( \frac{R}{m}, M \right) \in S$. So by the above isomorphisms, we deduce that 

$$\text{Hom}_R \left( \frac{R}{m}, f^i_a(M) \right) \in S.$$ 

Suppose that $t > 0$ and the result is true for all integer $i$ less than $t$. Set $N := f^i_m(M)$. Then $f^i_a(M) \cong \left( \frac{M}{N} \right)$ for all $i > 0$, and so we may assume that $\text{depth}_R(M) > 0$. There is an $M$-regular element $x \in m$. The exact sequence $0 \rightarrow M \rightarrow M \xrightarrow{x} \frac{M}{mM} \rightarrow 0$ induces the following long exact sequence:

$$\cdots \rightarrow f^{t-2}_a(M) \xrightarrow{x} f^{t-2}_a(M) \xrightarrow{f} f^{t-2}_a \left( \frac{M}{M} \right) \rightarrow f^{t-1}_a(M) \xrightarrow{x} f^{t-1}_a(M) \xrightarrow{g} f^{t-1}_a \left( \frac{M}{M} \right) \rightarrow f^t_a(M) \xrightarrow{x} f^t_a(M) \xrightarrow{h} \cdots. \quad (*)$$

Using the exact sequence $(*)$ we obtain the short exact sequence

$$0 \rightarrow f^{t-1}_a \left( \frac{M}{M} \right) \rightarrow f^t_a \left( \frac{M}{M} \right) \rightarrow (0 : x) \rightarrow 0.$$

Now, this exact sequence induces the following long exact sequence:

$$0 \rightarrow \text{Hom}_R \left( \frac{R}{m}, f^{t-2}_a(M) \right) \rightarrow \text{Hom}_R \left( \frac{R}{m}, f^{t-1}_a \left( \frac{M}{M} \right) \right) \rightarrow \text{Hom}_R \left( \frac{R}{m}, (0 : x) \right) \rightarrow$$

$$\text{Ext}^1_R \left( \frac{R}{m}, f^{t-1}_a \left( \frac{M}{M} \right) \right) \rightarrow \cdots. \quad (**)$$

By using $(*)$, $f^i_a \left( \frac{M}{M} \right) \in S$ for all $i < t - 1$. Therefore by the induction hypothesis $\text{Hom}_R \left( \frac{R}{m}, f^{t-1}_a \left( \frac{M}{M} \right) \right) \in S$. Furthermore $\text{Ext}^1_R \left( \frac{R}{m}, f^{t-1}_a \left( \frac{M}{M} \right) \right) \in S$ because $f^{t-1}_a(M) \in S$. Thus in accordance with $(**)$, $\text{Hom}_R \left( \frac{R}{m}, (0 : x) \right) \in S$. Since $x \in m$ according to [9,10.86] we have the following isomorphisms.

$$\text{Hom}_R \left( \frac{R}{m}, (0 : x) \right) \cong \text{Hom}_R \left( \frac{R}{m}, \text{Hom}_R \left( \frac{R}{xM}, f^t_a(M) \right) \right) \cong$$

$$\text{Hom}_R \left( \frac{R}{m} \otimes_R \frac{R}{xM}, f^t_a(M) \right) \cong \text{Hom}_R \left( \frac{R}{m}, f^t_a(M) \right).$$

Consequently $\text{Hom}_R \left( \frac{R}{m}, f^t_a(M) \right) \in S$. 

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3. The formal cohomological dimension in a Serre subcategory

We recall from [3,Theorem 1.1] that for a finitely generated $R$-module $M$,\[
\sup\{i \in \mathbb{N}_0 | f^i_a(M) \neq 0\} = \dim \left( \frac{M}{aM} \right).
\]

**Definition 3.1.** The formal cohomological dimension of $M$ with respect to $a$ in $S$ is the supremum of the integers $i$ such that $f^i_a(M) \not\in S$ and is denoted by $f.cd_S(a,M)$.

**Theorem 3.2.** Suppose that $S$ is a Serre subcategory of the category of $R$-modules and $R$-homomorphisms and $L$ and $N$ are two finitely generated $R$-modules such that $\text{Supp}_R(L) \subseteq \text{Supp}_R(N)$. Then $f.cd_S(a,L) \leq f.cd_S(a,N)$.

**Proof.** It is enough to prove that $f^i_a(L) \in S$ for all $i > f.cd_S(a,N)$ and all finitely generated $R$-module $L$ such that $\text{Supp}_R(L) \subseteq \text{Supp}_R(N)$. We use descending induction on $i$. For all $i > \dim \left( \frac{L}{aL} \right) + f.cd_S(a,N)$, $f^i_a(L) = 0 \in S$. Let $i > f.cd_S(a,N)$ and the result is proved for $i + l$. By Gruson’s theorem, there is a chain $0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_l = L$ of submodules of $L$ such that $\frac{L_i}{L_{i-l}}$ is a homomorphic image of a direct sum of finitely many copies of $N$. Consider the exact sequence $0 \to L_{i-l} \to L_i \to \frac{L_i}{L_{i-l}} \to 0$ ($i = 0, l, \ldots, l$). We may assume that $l = l$. The exact sequence $0 \to K \to \bigoplus_{j=i}^l N \to L \to 0$ where $K$ is a finitely generated $R$-module induces the following long exact sequence:

\[
\cdots \to f^i_a\left( \bigoplus_{j=i}^l N \right) \to f^i_a(L) \to f^{i+1}_a(K) \to \cdots \tag{*}
\]

Based on the induction hypothesis $f^{i+1}_a(K) \in S$. Moreover $f^i_a\left( \bigoplus_{j=i}^l N \right) = \bigoplus_{j=i}^l f^i_a(N) \in S$ for all $i > f.cd_S(a,N)$. Hence it follows from the exact sequence (*) that $f^i_a(L) \in S$.

The next example shows that even if $\text{Supp}_R(M) = \text{Supp}_R(N)$, then it may not true that $f.grade_S(a,M) = f.grade_S(a,N)$.

**Example 3.3.** (See [4, Example 4.3 (i)]) Let $(R, m)$ be a 2 dimensional complete regular local ring, $S = 0$ and $\mathbf{a}$ be an ideal of $R$ with $\dim \left( \frac{R}{a} \right) = 1$. Then by using [5, Theorem 1.1], $f.grade_S(a,R) = 1$ and $f.grade_S\left( a, \frac{R}{m} \right) = 0$. Set $M := R \oplus \frac{R}{m}$. Then $\text{Supp}_R(M) = \text{Supp}_R(R)$. But $f.grade_S(a,M) = \inf \left\{ f.grade_S(a,R), f.grade_S\left( a, \frac{R}{m} \right) \right\} = 0$.

**Corollary 3.4.** For all $x \in a \cdot f.cd_S(a,M) \geq f.cd_S\left( a, \frac{M}{xM} \right)$.

**Corollary 3.5.** Suppose that $0 \to L \to M \to N \to 0$ is an exact sequence of finitely generated $R$-modules. Then $f.cd_S(a,M) = \max \{ f.cd_S(a,L), f.cd_S(a,N) \}$. 

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Proof. Since $\text{Supp}_R(M) = \text{Supp}_R(L) \cup \text{Supp}_R(N)$ by referring to Theorem 3.2 we deduce that $f. cd_S(a, M) \geq f. cd_S(a, L)$ and $f. cd_S(a, M) \geq f. cd_S(a, N)$. Therefore $f. cd_S(a, M) \geq \max \{f. cd_S(a, L), f. cd_S(a, N)\}$. Next we prove that $\max \{f. cd_S(a, L), f. cd_S(a, N)\} \geq f. cd_S(a, M)$.

Let $i > \max \{f. cd_S(a, L), f. cd_S(a, N)\}$. Then $f^i_a(N), f^i_a(L) \in S$ and from the exact sequence $f^i_a(L) \to f^i_a(M) \to f^i_a(N)$ we conclude that $f^i_a(M) \in S$. Thus, $\max \{f. cd_S(a, L), f. cd_S(a, N)\} \geq f. cd_S(a, M)$.

We recall that the cohomological dimension of an $R$-module $M$ with respect to an ideal $a$ of $R$ in $S$ is defined as

$$cd_S(a, M) := \sup \{i \in \mathbb{N} \mid H^i(M) \not\in S\}.$$

The following lemma shows that when we considering the Artinianess of $f^i_a(M)$, we can assume that $M$ is $a$-torsion-free.

Lemma 3.6. Suppose that $a$ is an ideal of a local ring $(R, m)$ and $t$ be a non-negative integer. If $H^i_m(M) \in S$ for all $i \geq t$, then the following are equivalent:

(a) $f^i_a(M) \in S$ for all $i \geq t$.

(b) $f^i_a(M/M_{\Gamma_a(M)}) \in S$ for all $i \geq t$.

Proof. According to the hypothesis $t > cd_S(m, M)$. On the other hand $\text{Supp}_R(\Gamma_a(M)) \subseteq \text{Supp}_R(M)$. So by referring to [7, Theorem 3.5], $cd_S(m, \Gamma_a(M)) \leq cd_S(m, M)$. Thus, $t > cd_S(m, \Gamma_a(M))$ and $H^i_m(M) \in S$ for all $i \geq t$. Now, consider the following long exact sequence:

$$\cdots \to f^i_a(\Gamma_a(M)) \to f^i_a(M) \to f^i_a(M/M_{\Gamma_a(M)}) \to f^{i+1}_a(\Gamma_a(M)) \to \cdots \tag{*}$$

According to [4, Lemma 2.3] $f^i_a(\Gamma_a(M)) \cong H^i_m(\Gamma_a(M))$. By using the hypothesis $f^i_a(\Gamma_a(M)) \in S$ for all $i \geq t$. So it follows from the exact sequence (2) that $f^i_a(M) \in S$ if and only if $f^i_a(M/M_{\Gamma_a(M)}) \in S$ for all $i \geq t$.

Theorem 3.7. Let $(R, m)$ be a local ring and $M \in S$ be a finitely generated $R$-module of dimension $d$ such that $cd_S(m, M) \leq f. cd_S(a, M)$. Then $f^i_a(M/aM) \in S$ where $t = f. cd_S(a, M)$.

Proof. We use induction on $d = \dim(M)$. If $d = 0$, then $\dim(M/aM) = 0$. Accordingly to [3, Theorem 1.1], $f^i_a(M) = 0$ for all $i > 0$. 

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Moreover $f_a^t(M) \cong M \in S$. By definition $H^i_m(M) \in S$ for all $i > t$. Therefore from the above lemma we can assume that $M$ is $\mathfrak{a}$-torsion-free and there is an $M$-regular element $x \in \mathfrak{a}$. Consider the long exact sequence:

$$\cdots \rightarrow f_a^t(M) \rightarrow f_a^t(M) \rightarrow f_a^t(M) \rightarrow f_a^{t+1}(M) \rightarrow \cdots. \tag{*}$$

By using the hypothesis $f_a^i(M) \in S$ for all $i > t$ (because $t = f.\cdots(\mathfrak{a},M)$). So using the above long exact sequence $f_a^i(M) \in S$ for all $i > t$. By induction hypothesis, $\frac{f_a^i(M)}{a f_a^i(M)} \in S$ because $\dim(M) = \dim(M) - 1$.

Afterwards from the exact sequence (*) we get the following short exact sequence:

$$0 \rightarrow \text{Im}(f) \rightarrow f_a^t(M) \rightarrow \text{Im}(g) \rightarrow 0.$$

So we obtain the following long exact sequence.

$$\cdots \rightarrow \text{Tor}_p^R\left(\frac{R}{\mathfrak{a}}, \text{Im}(g)\right) \rightarrow \text{Im}(f) \rightarrow \frac{f_a^t(M)}{a f_a^t(M)} \rightarrow \text{Im}(g) \rightarrow 0.$$

Since $f_a^t(M) \in S$ and Im$(g)$ is a submodule of $f_a^{t+1}(M)$, we deduce that $\text{Tor}_p^R\left(\frac{R}{\mathfrak{a}}, \text{Im}(g)\right) \in S$. On the other hand, $\frac{f_a^t(M)}{a f_a^t(M)} \in S$. Therefore, $\frac{\text{Im}(f)}{\text{Im}(g)} \in S$ by the above long exact sequence.

Now, consider the following long exact sequence.

$$f_a^t(M) \rightarrow f_a^t(M) \rightarrow \text{Im}(f) \rightarrow 0.$$

So, $\frac{f_a^t(M)}{a f_a^t(M)} \cong \frac{\text{Im}(f)}{\text{Im}(g)}$ because $x \in \mathfrak{a}$. Consequently, $\frac{f_a^t(M)}{a f_a^t(M)} \in S$.

**Proposition 3.8.** For a finitely generated $R$-module $M$,

$$f.\cdots(\mathfrak{a},M) = \max \{f.\cdots(\mathfrak{a},\frac{R}{P}) | P \in \text{Ass}_R(M)\}.$$

**Proof.** Set $N := \bigoplus_{P \in \text{Ass}_R(M)}$. Then $\text{Supp}_R(M) = \text{Supp}_R(N)$. So, by Theorem 3.2 and Corollary 3.5, $f.\cdots(\mathfrak{a},M) = f.\cdots(\mathfrak{a},N) = \max \{f.\cdots(\mathfrak{a},\frac{R}{P}) | P \in \text{Ass}_R(M)\}$.

**Proposition 3.9.** Assume that $\mathfrak{a}$ is an ideal of the local ring $(R, \mathfrak{m})$. Then $\text{Hom}_R(\mathfrak{m}, f_a^0(M)) \in S$ if and only if $\text{Hom}_R(\mathfrak{m}, \mathfrak{M}^\mathfrak{a}) \in S$.

**Proof.** It is enough to consider the following isomorphisms

$$\text{Hom}_R\left(\frac{R}{\mathfrak{m}}, f_a^0(M)\right) \cong \lim_{\mathfrak{n} \in \mathfrak{N}} \text{Hom}_R\left(\frac{R}{\mathfrak{m}}, H^0_m(M)\right) \cong \lim_{\mathfrak{n} \in \mathfrak{N}} \text{Hom}_R\left(\frac{R}{\mathfrak{m}}, M\right) \cong \text{Hom}_R\left(\frac{R}{\mathfrak{m}}, \mathfrak{M}^\mathfrak{a}\right) \cong \text{Hom}_R\left(\frac{R}{\mathfrak{m}}, M\right).$$

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