Formal Local Cohomology Modules and Serre Subcategories

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Abstract

Let \((R, \mathfrak{m})\) be a Noetherian local ring, \(\mathfrak{a}\) an ideal of \(R\) and \(M\) a finitely generated \(R\)-module. We investigate some properties of formal local cohomology modules with respect to a Serre subcategory. We provide a common language to indicate some properties of formal local cohomology modules.

1. Introduction

Throughout this paper \((R, \mathfrak{m})\) is a commutative Noetherian local ring, \(\mathfrak{a}\) an ideal of \(R\) and \(M\) is a finitely generated \(R\)-module. For an integer \(i \in \mathbb{N}_0\), \(H^i_\mathfrak{a}(N)\) denotes the \(i\)-th local cohomology module of \(M\) with respect to \(\mathfrak{a}\) as introduced by Grothendieck (cf. [1], [2]).

We shall consider the family of local cohomology modules \(\{H^i_\mathfrak{m}(\frac{M}{\mathfrak{a}^nM})\}_{n \in \mathbb{N}}\) for a non-negative integer \(i \in \mathbb{N}_0\). With natural homomorphisms; this family forms an inverse system. Schenzel introduced the \(i\)-th formal local cohomology of \(M\) with respect to \(\mathfrak{a}\) in the form of \(f^i_\mathfrak{a}(M) := \lim_{\to} H^i_\mathfrak{m}(\frac{M}{\mathfrak{a}^nM})\), which is the \(i\)-th cohomology module of the \(\mathfrak{a}\)-adic completion of the \(\check{\mathcal{C}}ech\) complex \(\check{\mathcal{C}}_x \otimes_R M\), where \(x\) denotes a system of elements of \(R\) such that \(\text{Rad}(x, R) = \mathfrak{m}\) (see [3, Definition 3.1]). He defines the formal grade as \(f.grade(\mathfrak{a}, M) = \inf \{i \in \mathbb{N}_0 \mid f^i_\mathfrak{a}(M) \neq 0\}\). For any ideal \(\mathfrak{a}\) of \(R\) and finitely generated \(R\)-module \(M\) the following statements hold:

(i) (See [3, Theorem 3.11]). If \(0 \to M' \to M \to M'' \to 0\) is a short exact sequence of finitely generated \(R\)-modules, then there is the following long exact sequence:

\[
\cdots \to f^i_\mathfrak{a}(M') \to f^i_\mathfrak{a}(M) \to f^i_\mathfrak{a}(M'') \to \cdots.
\]

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(ii) (See [3, Theorem 1.3]). \( f. \text{grade}(a, M) \leq \dim(M) - \text{cd}(a, M) \); some properties of formal local cohomology have been presented in [3, 4, 5 and 6].

Throughout this paper \( S \) denotes a Serre subcategory of the category of \( R \)-modules and \( R \)-homomorphisms (we recall that a class \( S \) of \( R \)-modules is a Serre subcategory of the category of \( R \)-modules and \( R \)-homomorphisms if \( S \) is closed under taking submodules, quotients and extensions).

Our paper contains three sections. In Section 2, we shall define the formal grade of \( a \) with respect to \( M \) in \( S \) as the infimum of the integers \( i \) such that \( f_a^i(M) \notin S \) and is denoted by \( f. \text{grade}_S(a, M) \). (See definition 2.1). Then we shall obtain some properties of this notion. We show that if \( \Gamma_a(M) \) is a pure submodule of \( M \), then \( \text{Hom}_R(R_m, f_a^i(\Gamma_a(M))) \) and \( \text{Hom}_R(R_m, f_a^{i-1}(M/\Gamma_a(M))) \) belong to \( S \), where \( t = f. \text{grade}_S(a, M) \).

In Section 3, we shall define the formal cohomological dimension of \( a \) with respect to \( M \) in \( S \) as the supremum of the integers \( i \) such that \( f_a^i(M) \notin S \) and is denoted by \( f. \text{cd}_S(a, M) \). (See definition 3.1). The main result of this section is that if \( f_a^i(M) \in S \) and \( H_m^i(M) \in S \) for all \( i > t \), then \( \frac{R}{a} \otimes_R f_a^i(M) \) belongs to \( S \).

2. The formal grade of a module in a Serre subcategory

Definition 2.1. The formal grade of \( a \) with respect to \( M \) in \( S \) is the infimum of the integers \( i \) such that \( f_a^i(M) \notin S \) and is denoted by \( f. \text{grade}_S(a, M) \).

Proposition 2.2. Let \((R, m)\) be a local ring and \( a \) be an ideal of \( R \). If \( 0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 \) is an exact sequence of finitely generated \( R \)-modules, then the following statements hold.

(a) \( f. \text{grade}_S(a, M) \geq \min\{f. \text{grade}_S(a, L), f. \text{grade}_S(a, N)\} \).

(b) \( f. \text{grade}_S(a, L) \geq \min\{f. \text{grade}_S(a, M), f. \text{grade}_S(a, N) + 1\} \).

(c) \( f. \text{grade}_S(a, N) \geq \min\{f. \text{grade}_S(a, L) - 1, f. \text{grade}_S(a, M)\} \).

Proof. According to [3, Theorem 3.11], the above short exact sequence induces the following long exact sequence.

\[ \cdots \rightarrow f_a^{i-1}(N) \rightarrow f_a^i(L) \rightarrow f_a^i(M) \rightarrow f_a^i(N) \rightarrow f_a^{i+1}(L) \rightarrow \cdots \]

So, the result follows.
Corollary 2.3. If $\underline{x} = x_1, ..., x_n$ is a regular $M$-sequence, then $f.\text{grade}_S \left( \frac{a}{\sum M} \right) \geq f.\text{grade}_S (a, M) - n.$

Proof. Consider the following exact sequence ($n \in \mathbb{N}$)
\[ 0 \to \frac{M}{(x_1, ..., x_{n-1})M} \xrightarrow{\beta} \frac{M}{(x_1, ..., x_n)M} \xrightarrow{\text{nat.}} \frac{M}{(x_1, ..., x_n)M} \to 0 \]
whenever $n = 1$ by $(x_1, ..., x_{n-1})M$ we means 0.

Corollary 2.4. Let $a$ and $b$ be ideals of $R.$ Then
(a) $f.\text{grade}_S (a \cap b, M) \geq\min\{ f.\text{grade}_S (a, M), f.\text{grade}_S (b, M), f.\text{grade}_S ((a, b), M) + 1\}. $

(b) $f.\text{grade}_S ((a, b), M) \geq\min\{ f.\text{grade}_S (a \cap b, M) - 1, f.\text{grade}_S (a, M), f.\text{grade}_S (b, M) \}.$

Proof. For all $n \in \mathbb{N}$ there is a short exact sequence as follows:
\[ 0 \to \frac{M}{a^nM \cap b^nM} \to \frac{M}{a^nM} \oplus \frac{M}{b^nM} \to \frac{M}{(a^n, b^n)M} \to 0. \]
By using [3,Theorem 5.1], the above exact sequence induces the following long exact sequence.
\[ \cdots \to \lim_{n \in \mathbb{N}} H^i_m \left( \frac{M}{(a \cap b)^nM} \right) \to \lim_{n \in \mathbb{N}} H^i_m \left( \frac{M}{a^nM} \right) \oplus \lim_{n \in \mathbb{N}} H^i_m \left( \frac{M}{b^nM} \right) \to \lim_{n \in \mathbb{N}} H^i_m \left( \frac{M}{(a \cap b)^nM} \right) \to \cdots. \]
So by using an argument similar to that of Proposition 2.2, the result follows.

Corollary 2.5. Assume that $M$ is a finitely generated $R$-module and $N_1$ and $N_2$ are submodules of $M.$ Then considering the exact sequence $0 \to \frac{M}{N_1 \cap N_2} \to \frac{M}{N_1} \oplus \frac{M}{N_2} \to \frac{M}{N_1 + N_2} \to 0$ we shall have
(a) $f.\text{grade}_S \left( a, \frac{M}{N_1 \cap N_2} \right) \geq\min\{ f.\text{grade}_S (a, \frac{M}{N_1}), f.\text{grade}_S (a, \frac{M}{N_2}), f.\text{grade}_S (a, \frac{M}{N_1 + N_2}) \}.$

(b) $f.\text{grade}_S \left( a, \frac{M}{N_1 \cap N_2} \right) \geq\min\{ f.\text{grade}_S (a, \frac{M}{N_1}), f.\text{grade}_S (a, \frac{M}{N_2}), f.\text{grade}_S (a, \frac{M}{N_1 \cap N_2}) \}.$

Theorem 2.6. Let $a$ be an ideal of a local ring $(R, m),$ $M$ be a finitely generated $R$-module and $L$ be a pure submodule of $M.$ Then $f.\text{grade}_S (a, L) \geq f.\text{grade}_S (a, M)$ where $S$ is a Serre subcategory of the category of $R$-modules and $R$-homomorphisms.

In particular, $\inf \{ i|H^i_m(L) \notin S \} \geq \inf \{ i|H^i_m(M) \notin S \}.$
Proof. Let $L$ be a pure submodule of $M$. So \( \frac{L}{a^n L} \to \frac{M}{a^n M} \) is pure for each $n \in \mathbb{N}$. Now according to [8, Corollary 3.2 (a)], $H^i_m \left( \frac{L}{a^n L} \right) \to H^i_m \left( \frac{M}{a^n M} \right)$ is injective. Since inverse limit is a left exact functor, $f^i_a(L)$ is isomorphic to a submodule of $f^i_a(M)$. Consequently, $f$.grade$_S(a, L) \geq f$.grade$_S(a, M)$. If $a = 0$ then, $f$.grade$_S(0, M) = \inf \{ l | H^i_m(0) \notin S \}$ and the result follows.

Corollary 2.7. If $0 \to L \to M \to N \to 0$ is a pure exact sequence of finitely generated $R$-modules, then $\min \{ f$.grade$_S(a, L), f$.grade$_S(a, N) + 1 \} \geq f$.grade$_S(a, M)$.

Proof. Since $L$ is a pure submodule of $M$, as a result of the previous theorem, $f$.grade$_S(a, L) \geq f$.grade$_S(a, M)$. Hence we must prove that $f$.grade$_S(a, N) + 1 \geq f$.grade$_S(a, M)$. We assume that $i < f$.grade$_S(a, M)$ and we show that $i < f$.grade$_S(a, N) + 1$. Consider the following long exact sequence.

\[
\cdots \to f^{i-1}_a(M) \to f^i_a(L) \to f^i_a(M) \to f^i_a(N) \to \cdots . (**)
\]

If $i < f$.grade$_S(a, M)$, then $f^0_a(M), f^1_a(M), ..., f^{i-1}_a(M), f^i_a(M) \in S$. On the other hand, since $i < f$.grade$_S(a, M) \leq f$.grade$_S(a, L), f^0_a(L), ..., f^i_a(L) \in S$. Hence, it follows from $(**)$ that $f^0_a(N), ..., f^{i-1}_a(N) \in S$ and so $i - 1 < f$.grade$_S(a, N)$.

Theorem 2.8. Let $(R, m)$ be a local ring, $a$ be an ideal of $R$, $S$ be a Serre subcategory of the category of $R$-modules and $R$-homomorphisms and $M \in S$ be a finitely generated $R$-module such that $\Gamma_a(M)$ is a pure submodule of $M$. Then $\text{Hom}_R \left( \frac{R}{a}, f^i_a(\Gamma_a(M)) \right) \in S$, where $t = f$.grade$_S(a, M)$.

Proof. Due to the previous theorem, $f$.grade$_S(a, \Gamma_a(M)) \geq f$.grade$_S(a, M)$. If $f$.grade$_S(a, \Gamma_a(M)) > f$.grade$_S(a, M)$, then the result is obvious. Accordingly, we assume that $f$.grade$_S(a, \Gamma_a(M)) = f$.grade$_S(a, M)$. We know that $\text{Supp}(\Gamma_a(M)) \subseteq \text{Var}(a)$. By using [4, Lemma 2.3], $f^i_a(\Gamma_a(M)) \equiv H^i_m(\Gamma_a(M))$ for all $i \geq 0$. So, if $j < f$.grade$_S(a, M)$, then $f^j_a(\Gamma_a(M)) \equiv H^j_m(\Gamma_a(M)) \in S$ and $\text{Ext}^k_R \left( \frac{R}{m}, H^j_m(\Gamma_a(M)) \right) \in S$ for all $k \geq 0$ and $j < f$.grade$_S(a, M)$. Moreover $\text{Ext}^k_R \left( \frac{R}{m}, \Gamma_a(M) \right) \in S$, because $\Gamma_a(M) \in S$. Consequently, according to [7, Theorem 2.2],

\[
\text{Hom}_R \left( \frac{R}{m}, H^i_m(\Gamma_a(M)) \right) \in S, \text{ where } t = f$.grade$_S(a, M).
\]

Corollary 2.9 With the same notations as Theorem 2.8, let $X \in S$ be a submodule of $f^i_a(\Gamma_a(M))$, where $t = f$.grade$_S(a, M)$. Then $\text{Hom}_R \left( \frac{R}{m}, f^i_a(\Gamma_a(M)) \right) \in S$.

Proof. Consider the long exact sequence:
In accordance with the previous theorem $\text{Hom}_R\left( \frac{R}{m}, f_a^t(\Gamma_a(M)) \right) \rightarrow \text{Hom}_R\left( \frac{R}{m}, \frac{f_a^t(\Gamma_a(M))}{x} \right) \rightarrow \text{Ext}_R^1\left( \frac{R}{m}, X \right) \cdot (*)$

Moreover $\text{Ext}_R^1\left( \frac{R}{m}, X \right) \in S$. It follows from the exact sequence (*) that $\text{Hom}_R\left( \frac{R}{m}, \frac{f_a^t(\Gamma_a(M))}{x} \right) \in S$.

**Theorem 2.10.** Suppose that $a$ is an ideal of $(R,m)$ and $M \in S$ is a finitely generated $R$-module such that $\Gamma_a(M)$ is a pure submodule of $M$. Then $\text{Hom}_R\left( \frac{R}{m}, f_a^{t-1}\left( \frac{M}{\Gamma_a(M)} \right) \right) \in S$, where $t = f.\text{grade}_S(a,M)$.

**Proof.** One has $f.\text{grade}_S(a,\Gamma_a(M)) \geq f.\text{grade}_S(a,M)$, by Theorem 2.6. Now, the exact sequence $0 \rightarrow \Gamma_a(M) \rightarrow M \rightarrow \frac{M}{\Gamma_a(M)} \rightarrow 0$ induces the following long exact sequence:

$$
\cdots \rightarrow f_a^{t-1}(\Gamma_a(M)) \rightarrow \beta f_a^{t-1}(M) \rightarrow \gamma f_a^{t-1}\left( \frac{M}{\Gamma_a(M)} \right) \rightarrow \delta f_a^{t-1}(\Gamma_a(M)) \rightarrow \gamma f_a^{t-1}(M) \rightarrow \cdots \cdot (*)
$$

Using the exact sequence (*), we obtain the short exact sequence $0 \rightarrow \text{Im}(\beta) \rightarrow f_a^{t-1}(M) \rightarrow \text{Im}(\gamma) \rightarrow 0$. Since $f_a^{t-1}(M) \in S$, $\text{Im}(\beta) \in S$ and $\text{Im}(\gamma) \in S$.

Furthermore, we have the exact sequence $0 \rightarrow \text{Im}(\xi) \rightarrow H^t_m(\Gamma_a(M)) \rightarrow \text{Im}(\phi) \rightarrow 0$ which induces the following long exact sequence:

$$
0 \rightarrow \text{Hom}_R\left( \frac{R}{m}, \text{Im}(\xi) \right) \rightarrow \text{Hom}_R\left( \frac{R}{m}, H^t_m(\Gamma_a(M)) \right) \rightarrow \cdots.
$$

Thus $\text{Hom}_R\left( \frac{R}{m}, \text{Im}(\xi) \right) \in S$. Finally, by considering the short exact sequence $0 \rightarrow \text{Im}(\gamma) \rightarrow f_a^{t-1}\left( \frac{M}{\Gamma_a(M)} \right) \rightarrow \text{Im}(\xi) \rightarrow 0$ we can conclude that $\text{Hom}_R\left( \frac{R}{m}, f_a^{t-1}\left( \frac{M}{\Gamma_a(M)} \right) \right) \in S$.

**Theorem 2.11.** Suppose that $R$ is complete with respect to the $a$–adic topology and $M \in S$ be a finitely generated $R$-module and $t$ a positive integer such that $f_a^i(M) \in S$ for all $i < t$. Then $\text{Hom}_R\left( \frac{R}{m}, f_a^i(M) \right) \in S$.

**Proof.** We use induction on $t$. Let $t=0$. Consider the following isomorphisms.

$$
\text{Hom}_R\left( \frac{R}{m}, f_a^0(M) \right) \cong \lim_{\rightarrow m} \text{Hom}_R\left( \frac{R}{m}, H^0_a(M) \right) \cong \lim_{\rightarrow m} \text{Hom}_R\left( \frac{R}{m}, M_a^0 M \right) 
$$

$$
\cong \text{Hom}_R\left( \frac{R}{m}, \lim_{\rightarrow m} M_a^0 M \right) \cong \text{Hom}_R\left( \frac{R}{m}, M^{\infty} \right). 
$$
It is clear that $\text{Hom}_R\left(\frac{R}{m}, M\right) \in S$. So by the above isomorphisms, we deduce that $\text{Hom}_R\left(\frac{R}{m}, f^i_a(M)\right) \in S$.

Suppose that $t > 0$ and the result is true for all integer $i$ less than $t$. Set $N := f_m^t(M)$. Then $f_a^i(M) \cong f_a^i(M/N)$ for all $i > 0$, and so we may assume that $\text{depth}_R(M) > 0$. There is an $M$-regular element $x \in m$. The exact sequence $0 \to M \xrightarrow{x} M \to \frac{M}{xM} \to 0$ induces the following long exact sequence:

$$\cdots \to f_a^{t-2}(M) \xrightarrow{x} f_a^{t-2}(M) \xrightarrow{f} f_a^{t-2}\left(\frac{M}{xM}\right)$$

$$\to f_a^{t-1}(M) \xrightarrow{x} f_a^{t-1}(M) \xrightarrow{g} f_a^{t-1}\left(\frac{M}{xM}\right)$$

$$\to f_a^t(M) \xrightarrow{x} f_a^t(M) \xrightarrow{h} \cdots.$$  \hspace{1cm} (*)

Using the exact sequence (\ast) we obtain the short exact sequence

$$0 \to f_a^{t-1}(M) \to f_a^{t-1}\left(\frac{M}{xM}\right) \to \left(0 : x\right) \to 0.$$

Now, this exact sequence induces the following long exact sequence:

$$0 \to \text{Hom}_R\left(\frac{R}{m}, f_a^{t-1}(M)\right) \to \text{Hom}_R\left(\frac{R}{m}, f_a^{t-1}\left(\frac{M}{xM}\right)\right) \to \text{Hom}_R\left(\frac{R}{m}, \left(0 : x\right)\right) \to \text{Ext}^1_R\left(\frac{R}{m}, f_a^{t-1}(M)\right) \to \cdots.$$ \hspace{1cm} (**)

By using (\ast), $f_a^i\left(\frac{M}{xM}\right) \in S$ for all $i < t - 1$. Therefore by the induction hypothesis $\text{Hom}_R\left(\frac{R}{m}, f_a^{t-1}\left(\frac{M}{xM}\right)\right) \in S$. Furthermore $\text{Ext}^1_R\left(\frac{R}{m}, f_a^{t-1}(M)\right) \in S$ because $f_a^{t-1}(M) \in S$. Thus in accordance with (\ast\ast), $\text{Hom}_R\left(\frac{R}{m}, \left(0 : x\right)\right) \in S$. Since $x \in m$ according to [9,10.86] we have the following isomorphisms.

$$\text{Hom}_R\left(\frac{R}{m}, \left(0 : x\right)\right) \cong \text{Hom}_R\left(\frac{R}{m}, \text{Hom}_R\left(\frac{R}{xR}, f_a^i(M)\right)\right) \cong$$

$$\text{Hom}_R\left(\frac{R}{m} \otimes_R \frac{R}{xR}, f_a^i(M)\right) \cong \text{Hom}_R\left(\frac{R}{m}, f_a^i(M)\right).$$

Consequently $\text{Hom}_R\left(\frac{R}{m}, f_a^i(M)\right) \in S$.
3. The formal cohomological dimension in a Serre subcategory

We recall from [3,Theorem 1.1] that for a finitely generated $R$-module $M$, $\sup\{i \in \mathbb{N}_0 | f^i_a(M) \neq 0\} = \dim \left( \frac{M}{aM} \right)$.

**Definition 3.1.** The formal cohomological dimension of $M$ with respect to $a$ in $S$ is the supremum of the integers $i$ such that $f^i_a(M) \notin S$ and is denoted by $f.cd_S(a, M)$.

**Theorem 3.2.** Suppose that $S$ is a Serre subcategory of the category of $R$-modules and $R$-homomorphisms and $L$ and $N$ are two finitely generated $R$-modules such that $\text{Supp}_R(L) \subseteq \text{Supp}_R(N)$. Then $f.cd_S(a, L) \leq f.cd_S(a, N)$.

**Proof.** It is enough to prove that $f^i_a(L) \in S$ for all $i > f.cd_S(a, N)$ and all finitely generated $R$-module $L$ such that $\text{Supp}_R(L) \subseteq \text{Supp}_R(N)$. We use descending induction on $i$. For all $i > \dim \left( \frac{L}{aL} \right) + f.cd_S(a, N)$, $f^i_a(L) = 0 \in S$. Let $i > f.cd_S(a, N)$ and the result is proved for $i + 1$. By Gruson’s theorem, there is a chain $0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_i = L$ of submodules of $L$ such that $\frac{L_i}{L_{i-1}}$ is a homomorphic image of a direct sum of finitely many copies of $N$. Consider the exact sequence $0 \to L_{i-1} \to L_i \to \frac{L_i}{L_{i-1}} \to 0$ $(i = 0, 1, \ldots, l)$. We may assume that $l = l$. The exact sequence $0 \to K \to \bigoplus_{j=1}^{l} N \to L \to 0$ where $K$ is a finitely generated $R$-module induces the following long exact sequence:

$$\cdots \to f^i_a \left( \bigoplus_{j=1}^{l} N \right) \to f^i_a(L) \to f^{i+1}_a(K) \to \cdots \left( \ast \right)$$

Based on the induction hypothesis $f^{i+1}_a(K) \in S$. Moreover $f^i_a \left( \bigoplus_{j=1}^{l} N \right) = \bigoplus_{j=1}^{l} f^i_a(N) \in S$ for all $i > f.cd_S(a, N)$. Hence it follows from the exact sequence $(\ast)$ that $f^i_a(L) \in S$.

The next example shows that even if $\text{Supp}_R(M) = \text{Supp}_R(N)$, then it may not true that $f.grade_S(a, M) = f.grade_S(a, N)$.

**Example 3.3.** (See [4, Example 4.3 (i)]) Let $(R, m)$ be a 2 dimensional complete regular local ring, $S = 0$ and $a$ be an ideal of $R$ with $\dim \left( \frac{R}{a} \right) = l$. Then by using [5,Theorem 1.1], $f.grade_S(a, R) = 1$ and $f.grade_S \left( a, \frac{R}{m} \right) = 0$. Set $M := R \oplus \frac{R}{m}$. Then $\text{Supp}_R(M) = \text{Supp}_R(R)$. But

$$f.grade_S(a, M) = \inf \left\{ f.grade_S(a, R), f.grade_S \left( a, \frac{R}{m} \right) \right\} = 0.$$ 

**Corollary 3.4.** For all $x \in a$, $f.cd_S(a, M) \geq f.cd_S \left( a, \frac{M}{xM} \right)$.

**Corollary 3.5.** Suppose that $0 \to L \to M \to N \to 0$ is an exact sequence of finitely generated $R$-modules. Then $f.cd_S(a, M) = \max \{ f.cd_S(a, L), f.cd_S(a, N) \}$. 

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**Proof.** Since $\text{Supp}_R(M) = \text{Supp}_R(L) \cup \text{Supp}_R(N)$ by referring to Theorem 3.2 we deduce that $f \cdot \text{cd}_S(a, M) \geq f \cdot \text{cd}_S(a, L)$ and $f \cdot \text{cd}_S(a, M) \geq f \cdot \text{cd}_S(a, N)$. Therefore $f \cdot \text{cd}_S(a, M) \geq \max \{f \cdot \text{cd}_S(a, L), f \cdot \text{cd}_S(a, N)\}$.

Next we prove that $\max \{f \cdot \text{cd}_S(a, L), f \cdot \text{cd}_S(a, N)\} \geq f \cdot \text{cd}_S(a, M)$.

Let $i > \max \{f \cdot \text{cd}_S(a, L), f \cdot \text{cd}_S(a, N)\}$. Then $f^i_M(N), f^i_M(L) \in S$ and from the exact sequence $f^i_M(L) \to f^i_M(M) \to f^i_M(N)$ we conclude that $f^i_M(M) \in S$. Thus, $\max \{f \cdot \text{cd}_S(a, L), f \cdot \text{cd}_S(a, N)\} \geq f \cdot \text{cd}_S(a, M)$.

We recall that the cohomological dimension of an $R$-module $M$ with respect to an ideal $a$ of $R$ in $S$ is defined as

$$\text{cd}_S(a, M) := \sup \{i \in \mathbb{N}_0 | H^i_M(M) \not\in S\}.$$ 

The following lemma shows that when we considering the Artinianness of $f^i_M(M)$, we can assume that $M$ is $a$-torsion-free.

**Lemma 3.6.** Suppose that $a$ is an ideal of a local ring $(R, m)$ and $t$ be a non-negative integer. If $H^i_m(M) \in S$ for all $i \geq t$, then the following are equivalent:

(a) $f^i_a(M) \in S$ for all $i \geq t$.

(b) $f^i_a(M) \in S$ for all $i \geq t$.

**Proof.** According to the hypothesis $t > \text{cd}_S(m, M)$. On the other hand $\text{Supp}_R(M) \subseteq \text{Supp}_R(M)$. So by referring to [7, Theorem 3.5], $\text{cd}_S(m, \Gamma_a(M)) \leq \text{cd}_S(m, M)$. Thus, $t > \text{cd}_S(m, \Gamma_a(M))$ and $H^i_m(M) \in S$ for all $i \geq t$. Now, consider the following long exact sequence:

$$\cdots \to f^i_a(\Gamma_a(M)) \to f^i_a(M) \to f^i_a\left(\frac{M}{\Gamma_a(M)}\right) \to f^{i+1}_a(\Gamma_a(M)) \to \cdots \ast$$

According to [4, Lemma 2.3] $f^i_a(\Gamma_a(M)) \cong H^i_m(\Gamma_a(M))$. By using the hypothesis $f^i_a(\Gamma_a(M)) \in S$ for all $i \geq t$. So it follows from the exact sequence $\ast$ that $f^i_a(M) \in S$ if and only if $f^i_a\left(\frac{M}{\Gamma_a(M)}\right) \in S$ for all $i \geq t$.

**Theorem 3.7.** Let $(R, m)$ be a local ring and $M \in S$ be a finitely generated $R$-module of dimension $d$ such that $\text{cd}_S(m, M) \\leq f \cdot \text{cd}_S(a, M)$. Then $f^i_a(M) a^{f^i_a(M)} \in S$ where $t = f \cdot \text{cd}_S(a, M)$.

**Proof.** We use induction on $d = \dim(M)$. If $d = 0$, then $\dim\left(\frac{M}{aM}\right) = 0$. Accordingly to [3, Theorem 1.1], $f^i_a(M) = 0$ for all $i > 0$. 

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Moreover \( f_a^0(M) \cong M \in S \). By definition \( H^i_m(M) \in S \) for all \( i > t \). Therefore from the above lemma we can assume that \( M \) is \( a \)-torsion-free and there is an \( M \)-regular element \( \alpha \in a \). Consider the long exact sequence:

\[
\cdots \to f_a^i(M) \xrightarrow{x} f_a^i(M) \xrightarrow{f} f_a^i\left(\frac{M}{\alpha M}\right) \xrightarrow{g} f_a^{i+1}(M) \xrightarrow{h} \cdots \ (\ast)
\]

By using the hypothesis \( f_a^i(M) \in S \) for all \( i > t \) (because \( t = f.cd_S(a,M) \)). So using the above long exact sequence \( f_a^i\left(\frac{M}{\alpha M}\right) \in S \) for all \( i > t \). By induction hypothesis, \( \frac{f_a^i(M)}{af_a^i(M)} \in S \) because \( \dim\left(\frac{M}{\alpha M}\right) = \dim(M) - 1 \).

Afterwards from the exact sequence (\( \ast \)) we get the following short exact sequence. 

\[
0 \to \text{Im}(f) \to f_a^i\left(\frac{M}{\alpha M}\right) \to \text{Im}(g) \to 0.
\]

So we obtain the following long exact sequence.

\[
\cdots \to \text{Tor}^R_x\left(\frac{R}{a},\text{Im}(g)\right) \to \text{Im}(f) \xrightarrow{af_a^i(M)} \text{Im}(g) \to \text{Im}(f) \to 0.
\]

Since \( f_a^i(M) \in S \) and \( \text{Im}(g) \) is a submodule of \( f_a^{i+1}(M) \), we deduce that \( \text{Tor}^R_x\left(\frac{R}{a},\text{Im}(g)\right) \in S \). On the other hand, \( \frac{f_a^i(M)}{af_a^i(M)} \in S \). Therefore, \( \frac{\text{Im}(f)}{\text{Im}(f)} \in S \) by the above long exact sequence.

Now, consider the following long exact sequence.

\[
\frac{f_a^i(M)}{af_a^i(M)} \xrightarrow{x} f_a^i(M) \xrightarrow{f} f_a^i\left(\frac{M}{\alpha M}\right) \to \frac{\text{Im}(f)}{\text{Im}(f)} \to 0.
\]

So, \( \frac{f_a^i(M)}{af_a^i(M)} \cong \frac{\text{Im}(f)}{\text{Im}(f)} \) because \( \alpha \in a \). Consequently, \( \frac{f_a^i(M)}{af_a^i(M)} \in S \).

**Proposition 3.8.** For a finitely generated \( R \)-module \( M \),

\[
f.cd_S(a,M) = \max \{ f.cd_S\left(a,\frac{R}{p}\right) | P \in \text{Ass}_R(M) \}.
\]

**Proof.** Set \( N := \bigoplus_{p \in \text{Ass}_R(M)} \frac{R}{p} \). Then \( \text{Supp}_R(M) = \text{Supp}_R(N) \). So, by Theorem 3.2 and Corollary 3.5, \( f.cd_S(a,M) = f.cd_S(a,N) = \max \{ f.cd_S\left(a,\frac{R}{p}\right) | P \in \text{Ass}_R(M) \} \).

**Proposition 3.9.** Assume that \( a \) is an ideal of the local ring \((R, m)\). Then \( \text{Hom}_R(\frac{R}{m}, f_a^0(M)) \in S \) if and only if \( \text{Hom}_R(\frac{R}{m}, \tilde{M}^a) \in S \).

**Proof.** It is enough to consider the following isomorphisms

\[
\text{Hom}_R\left(\frac{R}{m}, f_a^0(M)\right) \cong \text{lim}_{n \in \mathbb{N}} \text{Hom}_R\left(\frac{R}{m}, h_0^m\left(\frac{M}{a^nM}\right)\right) \cong \text{lim}_{n \in \mathbb{N}} \text{Hom}_R\left(\frac{R}{m}, \frac{M}{a^nM}\right) \cong \text{Hom}_R\left(\frac{R}{m}, \tilde{M}^a\right).
\]
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