Formal Local Cohomology Modules and Serre Subcategories

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Abstract

Let \((R, m)\) be a Noetherian local ring, \(a\) an ideal of \(R\) and \(M\) a finitely generated \(R\)-module. We investigate some properties of formal local cohomology modules with respect to a Serre subcategory. We provide a common language to indicate some properties of formal local cohomology modules.

1. Introduction

Throughout this paper \((R, m)\) is a commutative Noetherian local ring, \(a\) an ideal of \(R\) and \(M\) is a finitely generated \(R\)-module. For an integer \(i \in \mathbb{N}_0\), \(H^i_a(N)\) denotes the \(i\)-th local cohomology module of \(M\) with respect to \(a\) as introduced by Grothendieck (cf. [1], [2]).

We shall consider the family of local cohomology modules \(\{H^i_m \left( \frac{M}{a^n M} \right) \}_{n \in \mathbb{N}}\) for a non-negative integer \(i \in \mathbb{N}_0\). With natural homomorphisms; this family forms an inverse system. Schenzel introduced the \(i\)-th formal local cohomology of \(M\) with respect to \(a\) in the form of \(f^i_a(M) = \lim_{n \to \infty} H^i_m \left( \frac{M}{a^n M} \right)\), which is the \(i\)-th cohomology module of the \(a\)-adic completion of the Čech complex \(\tilde{\mathcal{C}}_x \otimes_R M\), where \(x\) denotes a system of elements of \(R\) such that \(\text{Rad} \left( \frac{R}{x} \right) = m\) (see [3, Definition 3.1]). He defines the formal grade as \(f.\text{grade} \left( a, M \right) = \inf \{i \in \mathbb{N}_0 \mid f^i_a(M) \neq 0\}\). For any ideal \(a\) of \(R\) and finitely generated \(R\)-module \(M\) the following statements hold:

(i) (See [3, Theorem 3.11]). If \(0 \to M' \to M \to M'' \to 0\) is a short exact sequence of finitely generated \(R\)-modules, then there is the following long exact sequence:

\[
\cdots \to f^1_a(M') \to f^1_a(M) \to f^1_a(M'') \to \cdots
\]

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(ii) (See [3, Theorem 1.3]). \( f. \text{grade} (a, M) \leq \dim (M) - \text{cd} (a, M) \); some properties of formal local cohomology have been presented in [3, 4, 5 and 6].

Throughout this paper \( S \) denotes a Serre subcategory of the category of \( R \)-modules and \( R \)-homomorphisms (we recall that a class \( S \) of \( R \)-modules is a Serre subcategory of the category of \( R \)-modules and \( R \)-homomorphisms if \( S \) is closed under taking submodules, quotients and extensions).

Our paper contains three sections. In Section 2, we shall define the formal grade of \( a \) with respect to \( M \) in \( S \) as the infimum of the integers \( i \) such that \( f_a^i (M) \notin S \) and is denoted by \( f. \text{grade}_S (a, M) \). (See definition 2.1). Then we shall obtain some properties of this notion. We show that if \( \Gamma_a (M) \) is a pure submodule of \( M \), then \( \text{Hom}_R \left( \frac{R}{m}, f_a^t \left( \Gamma_a (M) \right) \right) \) and \( \text{Hom}_R \left( \frac{R}{m}, f_a^{t-1} \left( \frac{M}{\Gamma_a (M)} \right) \right) \) belong to \( S \), where \( t = f. \text{grade}_S (a, M) \).

In Section 3, we shall define the formal cohomological dimension of \( a \) with respect to \( M \) in \( S \) as the supremum of the integers \( i \) such that \( f_a^i (M) \notin S \) and is denoted by \( f. \text{cd}_S (a, M) \). (See definition 3.1). The main result of this section is that if \( f_a^i (M) \in S \) and \( H_m^i (M) \in S \) for all \( i > t \), then \( \frac{R}{a} \otimes_R f_a^t (M) \) belongs to \( S \).

2. The formal grade of a module in a Serre subcategory

**Definition 2.1.** The formal grade of \( a \) with respect to \( M \) in \( S \) is the infimum of the integers \( i \) such that \( f_a^i (M) \notin S \) and is denoted by \( f. \text{grade}_S (a, M) \).

**Proposition 2.2.** Let \((R, m)\) be a local ring and \( a \) be an ideal of \( R \). If \( 0 \to L \to M \to N \to 0 \) is an exact sequence of finitely generated \( R \)-modules, then the following statements hold.

\[
\begin{align*}
(a) & \quad f. \text{grade}_S (a, M) \geq \min \{ f. \text{grade}_S (a, L), f. \text{grade}_S (a, N) \}. \\
(b) & \quad f. \text{grade}_S (a, L) \geq \min \{ f. \text{grade}_S (a, M), f. \text{grade}_S (a, N) + 1 \}. \\
(c) & \quad f. \text{grade}_S (a, N) \geq \min \{ f. \text{grade}_S (a, L) - 1, f. \text{grade}_S (a, M) \}.
\end{align*}
\]

**Proof.** According to [3, Theorem 3.11], the above short exact sequence induces the following long exact sequence.

\[
\cdots \to f_a^{i-1} (N) \to f_a^i (L) \to f_a^i (M) \to f_a^i (N) \to f_a^{i+1} (L) \to \cdots
\]

So, the result follows.
Corollary 2.3. If \( x = x_1, \ldots, x_n \) is a regular \( M \)-sequence, then \( f.\text{grade}_S \left( a, \frac{M}{\sum M} \right) \geq f.\text{grade}_S (a, M) - n \).

**Proof.** Consider the following exact sequence \((n \in \mathbb{N})\)

\[
0 \to \frac{M}{(x_1, \ldots, x_{n-1})M} \xrightarrow{x_n} \frac{M}{(x_1, \ldots, x_n)M} \xrightarrow{\text{nat.}} \frac{M}{(x_1, \ldots, x_n)M} \to 0
\]

whenever \( n = 1 \) by \((x_1, \ldots, x_{n-1})M\) we means 0.

Corollary 2.4. Let \( a \) and \( b \) be ideals of \( R \). Then

(a) \( f.\text{grade}_S (a \cap b, M) \geq \min \{ f.\text{grade}_S (a, M), f.\text{grade}_S (b, M), f.\text{grade}_S ((a, b), M) + 1 \} \).

(b) \( f.\text{grade}_S ((a, b), M) \geq \min \{ f.\text{grade}_S (a \cap b, M) - 1, f.\text{grade}_S (a, M), f.\text{grade}_S (b, M) \} \).

**Proof.** For all \( n \in \mathbb{N} \) there is a short exact sequence as follows:

\[
0 \to \frac{M}{a^n M \cap b^n M} \to \frac{M}{a^n M} \oplus \frac{M}{b^n M} \to \frac{M}{(a^n, b^n) M} \to 0.
\]

By using [3, Theorem 5.1], the above exact sequence induces the following long exact sequence.

\[
\ldots \to \lim_{n \in \mathbb{N}} H^i_m \left( \frac{M}{(a^n b^n) M} \right) \to \lim_{n \in \mathbb{N}} H^i_m \left( \frac{M}{a^n M} \right) \oplus \lim_{n \in \mathbb{N}} H^i_m \left( \frac{M}{b^n M} \right) \to \lim_{n \in \mathbb{N}} H^i_m \left( \frac{M}{(a^n, b^n) M} \right) \to \ldots
\]

So by using an argument similar to that of Proposition 2.2, the result follows.

Corollary 2.5. Assume that \( M \) is a finitely generated \( R \)-module and \( N_1 \) and \( N_2 \) are submodules of \( M \). Then considering the exact sequence \( 0 \to \frac{M}{N_1 \cap N_2} \to \frac{M}{N_1} \oplus \frac{M}{N_2} \to \frac{M}{N_1 + N_2} \to 0 \) we shall have

(a) \( f.\text{grade}_S \left( \frac{M}{N_1 \cap N_2} \right) \geq \min \{ f.\text{grade}_S \left( \frac{M}{N_1} \right), f.\text{grade}_S \left( \frac{M}{N_2} \right), f.\text{grade}_S \left( \frac{M}{N_1 + N_2} \right) \} \).

(b) \( f.\text{grade}_S \left( \frac{M}{N_1 \cap N_2} \right) \geq \min \{ f.\text{grade}_S \left( \frac{M}{N_1} \right) - 1, f.\text{grade}_S \left( \frac{M}{N_2} \right), f.\text{grade}_S \left( \frac{M}{N_1 + N_2} \right) \} \).

Theorem 2.6. Let \( a \) be an ideal of a local ring \((R, m)\), \( M \) be a finitely generated \( R \)-module and \( L \) be a pure submodule of \( M \). Then \( f.\text{grade}_S (a, L) \geq f.\text{grade}_S (a, M) \) where \( S \) is a Serre subcategory of the category of \( R \)-modules and \( R \)-homomorphisms. In particular, \( \inf \{ i | H^i_m (L) \not\in S \} \geq \inf \{ i | H^i_m (M) \not\in S \} \).
Proof. Let \( L \) be a pure submodule of \( M \). So \(
abla^aM \rightarrow \nabla^aM \) is pure for each \( n \in \mathbb{N} \). Now according to [8, Corollary 3.2 (a)], \( H_i^a\left(\frac{L}{a^nL}\right) \rightarrow H_i^a\left(\frac{M}{a^nM}\right) \) is injective. Since inverse limit is a left exact functor, \( f_i^a(L) \) is isomorphic to a submodule of \( f_i^a(M) \). Consequently, \( f_\text{grade}_S(a, L) \geq f_\text{grade}_S(a, M) \). If \( a = 0 \) then, \( f_\text{grade}_S(0, M) = \inf \{ i| H_i^a(M) \notin S \} \) and the result follows.

Corollary 2.7. If \( 0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 \) is a pure exact sequence of finitely generated \( R \)-modules, then \( \min \{ f_\text{grade}_S(a, L), f_\text{grade}_S(a, N) + 1 \} \geq f_\text{grade}_S(a, M) \).

Proof. Since \( L \) is a pure submodules of \( M \), as a result of the previous theorem, \( f_\text{grade}_S(a, L) \geq f_\text{grade}_S(a, M) \). Hence we must prove that \( f_\text{grade}_S(a, N) + 1 \geq f_\text{grade}_S(a, M) \). We assume that \( i < f_\text{grade}_S(a, M) \) and we show that \( i < f_\text{grade}_S(a, N) + 1 \). Consider the following long exact sequence.

\[
\cdots \rightarrow f_{i-1}^a(M) \rightarrow f_i^a(N) \rightarrow f_i^a(L) \rightarrow f_i^a(M) \rightarrow f_{i-1}^a(N) \rightarrow \cdots \ (**)
\]

If \( i < f_\text{grade}_S(a, M) \), then \( f_i^a(M), f_i^a(M), ..., f_i^a(M), f_i^a(M) \in S \). On the other hand, since \( i < f_\text{grade}_S(a, M) \leq f_\text{grade}_S(a, L), f_i^a(L), ..., f_i^a(L) \in S \). Hence, it follows from \((**)\) that \( f_i^a(N), ..., f_i^a(N) \in S \) and so \( i - 1 < f_\text{grade}_S(a, N) \).

Theorem 2.8. Let \( (R, m) \) be a local ring, \( a \) be an ideal of \( R \), \( S \) be a Serre subcategory of the category of \( R \)-modules and \( R \_{}\text{hom} \)omomorphisms and \( M \in S \) be a finitely generated \( R \)-module such that \( \Gamma_a(M) \) is a pure submodule of \( M \). Then \( Hom_R\left(\frac{R}{a}, f_i^a(\Gamma_a(M))\right) \in S \), where \( t = f_\text{grade}_S(a, M) \).

Proof. Due to the previous theorem, \( f_\text{grade}_S(a, \Gamma_a(M)) \geq f_\text{grade}_S(a, M) \). If \( f_\text{grade}_S(a, \Gamma_a(M)) > f_\text{grade}_S(a, M) \), then the result is obvious. Accordingly, we assume that \( f_\text{grade}_S(a, \Gamma_a(M)) = f_\text{grade}_S(a, M) \). We know that \( Supp(\Gamma_a(M)) \subseteq Var(\mathfrak{a}) \). By using [4, Lemma 2.3], \( f_i^a(\Gamma_a(M)) \equiv H_i^a(M) \) for all \( i \geq 0 \). So, if \( j < f_\text{grade}_S(a, M) \), then \( f_j(\Gamma_a(M)) \equiv H_j^a(M) \) for all \( k \geq 0 \) and \( j < f_\text{grade}_S(a, M) \). Moreover, \( ext_k^R\left(R\frac{m}{m}, \Gamma_a(M)\right) \in S \), because \( \Gamma_a(M) \in S \). Consequently, according to [7, Theorem 2.2],

\[
Hom_R\left(\frac{R}{m}, f_i(\Gamma_a(M))\right) \in S \), where \( t = f_\text{grade}_S(a, M) \).

Corollary 2.9 With the same notations as Theorem 2.8, let \( X \in S \) be a submodule of \( f_i^a(\Gamma_a(M)) \), where \( t = f_\text{grade}_S(a, M) \). Then \( Hom_R\left(\frac{R}{m}, \frac{f_i(\Gamma_a(M))}{X}\right) \in S \).

Proof. Consider the long exact sequence:
In accordance with the previous theorem \( \text{Hom}_R \left( \frac{R}{m}, f_{a}^{t}(\Gamma_a(M)) \right) \rightarrow \text{Hom}_R \left( \frac{R}{m}, f_{a}^{t}(\Gamma_a(M)) \right) \rightarrow \text{Ext}_R^{1} \left( \frac{R}{m}, X \right), (*) \). Moreover, \( \text{Ext}_R^{1} \left( \frac{R}{m}, X \right) \in S \). It follows from the exact sequence (*) that \( \text{Hom}_R \left( \frac{R}{m}, f_{a}^{t}(\Gamma_a(M)) \right) \in S \).

**Theorem 2.10.** Suppose that \( a \) is an ideal of \((R, m)\) and \( M \in S \) is a finitely generated \( R \)-module such that \( \Gamma_a(M) \) is a pure submodule of \( M \). Then

\[
\text{Hom}_R \left( \frac{R}{m}, f_{a}^{t-1} \left( \frac{M}{\Gamma_a(M)} \right) \right) \in S, \text{ where } t = f. \text{grade}_S(a, M).
\]

**Proof.** One has \( f. \text{grade}_S(a, \Gamma_a(M)) \geq f. \text{grade}_S(a, M) \), by Theorem 2.6. Now, the exact sequence \( 0 \rightarrow \Gamma_a(M) \rightarrow M \rightarrow \frac{M}{\Gamma_a(M)} \rightarrow 0 \) induces the following long exact sequence:

\[
\ldots \longrightarrow f_{a}^{t-1} \left( \Gamma_a(M) \right) \xrightarrow{\alpha} f_{a}^{t-1} \left( M \right) \xrightarrow{\beta} f_{a}^{t-1} \left( \frac{M}{\Gamma_a(M)} \right) \xrightarrow{\gamma} f_{a}^{t-1} \left( \Gamma_a(M) \right) \xrightarrow{\delta} f_{a}^{t-1} \left( M \right) \xrightarrow{\varphi} \ldots. (*).
\]

Using the exact sequence (*), we obtain the short exact sequence \( 0 \rightarrow \text{Im}(\beta) \rightarrow f_{a}^{t-1} \left( M \right) \rightarrow \text{Im}(\gamma) \rightarrow 0 \). Since \( f_{a}^{t-1} \left( M \right) \in S \), \( \text{Im}(\beta) \in S \) and \( \text{Im}(\gamma) \in S \).

Furthermore, we have the exact sequence \( 0 \rightarrow \text{Im}(\xi) \rightarrow M \rightarrow \frac{M}{\Gamma_a(M)} \rightarrow 0 \) which induces the following long exact sequence:

\[
0 \rightarrow \text{Hom}_R \left( \frac{R}{m}, \text{Im}(\xi) \right) \rightarrow \text{Hom}_R \left( \frac{R}{m}, H_{a}^{t} (\Gamma_a(M)) \right) \rightarrow \ldots.
\]

Thus \( \text{Hom}_R \left( \frac{R}{m}, \text{Im}(\xi) \right) \in S \). Finally, by considering the short exact sequence \( 0 \rightarrow \text{Im}(\gamma) \rightarrow f_{a}^{t-1} \left( \frac{M}{\Gamma_a(M)} \right) \rightarrow \text{Im}(\xi) \rightarrow 0 \) we can conclude that \( \text{Hom}_R \left( \frac{R}{m}, f_{a}^{t-1} \left( \frac{M}{\Gamma_a(M)} \right) \right) \in S \).

**Theorem 2.11.** Suppose that \( R \) is complete with respect to the \( a \)-adic topology and \( M \in S \) be a finitely generated \( R \)-module and \( t \) a positive integer such that \( f_{a}^{t} (M) \in S \) for all \( i < t \). Then \( \text{Hom}_R \left( \frac{R}{m}, f_{a}^{t} (M) \right) \in S \).

**Proof.** We use induction on \( t \). Let \( t=0 \). Consider the following isomorphisms.

\[
\text{Hom}_R \left( \frac{R}{m}, f_{a}^{0} (M) \right) \cong \lim \text{Hom}_R \left( \frac{R}{m}, H_{a}^{0} \left( \frac{M}{a^{i} M} \right) \right) \cong \lim \text{Hom}_R \left( \frac{R}{m}, \frac{M}{a^{i} M} \right) \cong \text{Hom}_R \left( \frac{R}{m}, M \right).
\]

\[
\lim \text{Hom}_R \left( \frac{R}{m}, \frac{M}{a^{i} M} \right) \cong \text{Hom}_R \left( \frac{R}{m}, M \right).
\]
It is clear that $\text{Hom}_R \left( \frac{R}{m}, M \right) \in S$. So by the above isomorphisms, we deduce that 

$$\text{Hom}_R \left( \frac{R}{m}, f^i_a(M) \right) \in S.$$ 

Suppose that $t > 0$ and the result is true for all integer $i$ less than $t$. Set $N = \mathfrak{r}_m(M)$. Then $f^i_a(M) \cong f^i_a \left( \frac{M}{N} \right)$ for all $i > 0$, and so we may assume that $\text{depth}_R(M) > 0$. There is an $M$-regular element $x \in m$. The exact sequence $0 \to M \to M \to \frac{M}{xM} \to 0$ induces the following long exact sequence:

$$\cdots \to f^{t-2}_a(M) \xrightarrow{x} f^{t-2}_a(M) \xrightarrow{f} f^{t-2}_a \left( \frac{M}{xM} \right) \to f^{t-1}_a(M) \xrightarrow{x} f^{t-1}_a(M) \xrightarrow{g} f^{t-1}_a \left( \frac{M}{xM} \right) \to f^t_a(M) \xrightarrow{h} f^t_a(M) \to \cdots.$$ 

Using the exact sequence $(*)$ we obtain the short exact sequence

$$0 \to f^{t-1}_a(M) \to f^t_a \left( \frac{M}{xM} \right) \to (0 : x)_{\mathfrak{r}(M)} \to 0.$$

Now, this exact sequence induces the following long exact sequence:

$$0 \to \text{Hom}_R \left( \frac{R}{m}, f^{t-1}_a(M) \right) \to \text{Hom}_R \left( \frac{R}{m}, f^t_a \left( \frac{M}{xM} \right) \right) \to \text{Hom}_R \left( \frac{R}{m}, (0 : x)_{\mathfrak{r}(M)} \right) \to \text{Ext}^1_R \left( \frac{R}{m}, f^{t-1}_a(M) \right) \to \cdots.$$ 

By using $(*)$, $f^i_a \left( \frac{M}{xM} \right) \in S$ for all $i < t - 1$. Therefore by the induction hypothesis $\text{Hom}_R \left( \frac{R}{m}, f^{t-1}_a(M) \right) \in S$. Furthermore $\text{Ext}^1_R \left( \frac{R}{m}, f^{t-1}_a(M) \right) \in S$ because $f^{t-1}_a(M) \in S$. Thus in accordance with $(**)$, $\text{Hom}_R \left( \frac{R}{m}, (0 : x)_{\mathfrak{r}(M)} \right) \in S$. Since $x \in m$ according to [9,10.86] we have the following isomorphisms.

$$\text{Hom}_R \left( \frac{R}{m}, (0 : x)_{\mathfrak{r}(M)} \right) \cong \text{Hom}_R \left( \frac{R}{m}, \text{Hom}_R \left( \frac{R}{xR}, f^t_a(M) \right) \right) \cong \text{Hom}_R \left( \frac{R}{m}, \text{Hom}_R \left( \frac{R}{xR}, f^t_a(M) \right) \right) \cong \text{Hom}_R \left( \frac{R}{m}, f^t_a(M) \right).$$

Consequently $\text{Hom}_R \left( \frac{R}{m}, f^t_a(M) \right) \in S$.

\[342\]
3. The formal cohomological dimension in a Serre subcategory

We recall from [3, Theorem 1.1] that for a finitely generated $R$-module $M$, $\sup\{i \in \mathbb{N}_0 | f^i_a(M) \neq 0 \} = \dim \left( \frac{M}{aM} \right)$.

**Definition 3.1.** The formal cohomological dimension of $M$ with respect to $a$ in $S$ is the supremum of the integers $i$ such that $f^i_a(M) \notin S$ and is denoted by $f.c.d_S(a, M)$.

**Theorem 3.2.** Suppose that $S$ is a Serre subcategory of the category of $R$-modules and $R$-homomorphisms and $L$ and $N$ are two finitely generated $R$-modules such that $\text{Supp}_R(L) \subseteq \text{Supp}_R(N)$. Then $f.c.d_S(a, L) \leq f.c.d_S(a, N)$.

**Proof.** It is enough to prove that $f^i_a(L) \in S$ for all $i > f.c.d_S(a, N)$ and all finitely generated $R$-module $L$ such that $\text{Supp}_R(L) \subseteq \text{Supp}_R(N)$. We use descending induction on $i$. For all $i > \dim \left( \frac{L}{aL} \right)$, $f^i_a(L) = 0 \in S$. Let $i > f.c.d_S(a, N)$ and the result is proved for $i + 1$. By Gruson’s theorem, there is a chain $0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_l = L$ of submodules of $L$ such that $\frac{L_i}{L_{i-1}}$ is a homormorphic image of a direct sum of finitely many copies of $N$. Consider the exact sequence $0 \to L_{i-1} \to L_i \to \frac{L_i}{L_{i-1}} \to 0$ ($i = 0, l, \ldots, l$). We may assume that $l = 1$. The exact sequence $0 \to K \to \bigoplus_{j=1}^t N \to L \to 0$ where $K$ is a finitely generated $R$-module induces the following long exact sequence:

$$\cdots \to f^i_a\left( \bigoplus_{j=1}^t N \right) \to f^i_a(L) \to f^{i+1}_a(K) \to \cdots \text{(*)}$$

Based on the induction hypothesis $f^{i+1}_a(K) \in S$. Moreover $f^i_a\left( \bigoplus_{j=1}^t N \right) = \bigoplus_{j=1}^t f^i_a(N) \in S$ for all $i > f.c.d_S(a, N)$. Hence it follows from the exact sequence (*) that $f^i_a(L) \in S$.

The next example shows that even if $\text{Supp}_R(M) = \text{Supp}_R(N)$, then it may not true that $f.g.s.e_S(a, M) = f.g.s.e_S(a, N)$.

**Example 3.3.** (See [4, Example 4.3 (i)]) Let $(R, m)$ be a 2 dimensional complete regular local ring, $S = 0$ and $a$ be an ideal of $R$ with $\dim \left( \frac{R}{a} \right) = 1$. Then by using [5, Theorem 1.1], $f.g.s.e_S(a, R) = 1$ and $f.g.s.e_S\left( a, \frac{R}{m} \right) = 0$. Set $M := R \oplus \frac{R}{m}$.

Then $\text{Supp}_R(M) = \text{Supp}_R(R)$. But $f.g.s.e_S(a, M) = \inf \{ f.g.s.e_S(a, R), f.g.s.e_S\left( a, \frac{R}{m} \right) \} = 0$.

**Corollary 3.4.** For all $x \in a \cdot f.c.d_S(a, M) \geq f.c.d_S\left( a, \frac{M}{xM} \right)$.

**Corollary 3.5.** Suppose that $0 \to L \to M \to N \to 0$ is an exact sequence of finitely generated $R$-modules. Then $f.c.d_S(a, M) = \max \{ f.c.d_S(a, L), f.c.d_S(a, N) \}$. 

343
Proof. Since $\text{Supp}_R(M) = \text{Supp}_R(L) \cup \text{Supp}_R(N)$ by referring to Theorem 3.2 we deduce that $f. \text{cd}_S(\mathfrak{a}, M) \geq f. \text{cd}_S(\mathfrak{a}, L)$ and $f. \text{cd}_S(\mathfrak{a}, M) \geq f. \text{cd}_S(\mathfrak{a}, N)$. Therefore $f. \text{cd}_S(\mathfrak{a}, M) \geq \max \{f. \text{cd}_S(\mathfrak{a}, L), f. \text{cd}_S(\mathfrak{a}, N)\}$.

Next we prove that $\max \{f. \text{cd}_S(\mathfrak{a}, L), f. \text{cd}_S(\mathfrak{a}, N)\} \geq f. \text{cd}_S(\mathfrak{a}, M)$.

Let $i > \max \{f. \text{cd}_S(\mathfrak{a}, L), f. \text{cd}_S(\mathfrak{a}, N)\}$. Then $f^i_\mathfrak{a}(N), f^i_\mathfrak{a}(L) \in S$ and from the exact sequence $f^i_\mathfrak{a}(L) \to f^i_\mathfrak{a}(M) \to f^i_\mathfrak{a}(N)$ we conclude that $f^i_\mathfrak{a}(M) \in S$. Thus, $\max \{f. \text{cd}_S(\mathfrak{a}, L), f. \text{cd}_S(\mathfrak{a}, N)\} \geq f. \text{cd}_S(\mathfrak{a}, M)$.

We recall that the cohomological dimension of an $R$-module $M$ with respect to an ideal $\mathfrak{a}$ of $R$ in $\mathcal{S}$ is defined as

$$\text{cd}_S(\mathfrak{a}, M) := \sup \{i \in \mathbb{N}_0 | H^i_\mathfrak{a}(M) \notin S\}.$$  

The following lemma shows that when we considering the Artinianness of $f^i_\mathfrak{a}(M)$, we can assume that $M$ is $\mathfrak{a}$-torsion-free.

Lemma 3.6. Suppose that $\mathfrak{a}$ is an ideal of a local ring $(R, \mathfrak{m})$ and $t$ be a non-negative integer. If $H^i_\mathfrak{m}(M) \in S$ for all $i \geq t$, then the following are equivalent:

(a) $f^i_\mathfrak{a}(M) \in S$ for all $i \geq t$.

(b) $f^i_\mathfrak{a}\left(\frac{M}{\Gamma_\mathfrak{a}(M)}\right) \in S$ for all $i \geq t$.

Proof. According to the hypothesis $t > \text{cd}_S(\mathfrak{m}, M)$. On the other hand $\text{Supp}_R(\Gamma_\mathfrak{a}(M)) \subseteq \text{Supp}_R(M)$. So by referring to [7,Theorem 3.5], $\text{cd}_S(\mathfrak{m}, \Gamma_\mathfrak{a}(M)) \leq \text{cd}_S(\mathfrak{m}, M)$. Thus, $t > \text{cd}_S(\mathfrak{m}, \Gamma_\mathfrak{a}(M))$ and $H^i_\mathfrak{m}(\Gamma_\mathfrak{a}(M)) \in S$ for all $i \geq t$. Now, consider the following exact sequence:

$$\cdots \to f^i_\mathfrak{a}(\Gamma_\mathfrak{a}(M)) \to f^i_\mathfrak{a}(M) \to f^i_\mathfrak{a}\left(\frac{M}{\Gamma_\mathfrak{a}(M)}\right) \to f^{i+1}_\mathfrak{a}(\Gamma_\mathfrak{a}(M)) \to \cdots. \ (*)$$

According to [4, Lemma 2.3] $f^i_\mathfrak{a}(\Gamma_\mathfrak{a}(M)) \cong H^i_\mathfrak{m}(\Gamma_\mathfrak{a}(M))$. By using the hypothesis $f^i_\mathfrak{a}(\Gamma_\mathfrak{a}(M)) \in S$ for all $i \geq t$. So it follows from the exact sequence $(\ast)$ that $f^i_\mathfrak{a}(M) \in S$ if and only if $f^i_\mathfrak{a}\left(\frac{M}{\Gamma_\mathfrak{a}(M)}\right) \in S$ for all $i \geq t$.

Theorem 3.7. Let $(R, \mathfrak{m})$ be a local ring and $M \in S$ be a finitely generated $R$-module of dimension $d$ such that $\text{cd}_S(\mathfrak{m}, M) \leq f. \text{cd}_S(\mathfrak{a}, M)$. Then $f^i_\mathfrak{a}(M) \in S$ where $t = f. \text{cd}_S(\mathfrak{a}, M)$.

Proof. We use induction on $d = \dim (M)$. If $d = 0$, then $\dim \left(\frac{M}{\mathfrak{a}M}\right) = 0$. Accordingly to [3, Theorem 1.1], $f^i_\mathfrak{a}(M) = 0$ for all $i > 0$. 

344
Moreover $f_a^0(M) \equiv M \in S$. By definition $H_m^i(M) \in S$ for all $i > t$. Therefore from the above lemma we can assume that $M$ is $a$-torsion-free and there is an $M$-regular element $x \in a$. Consider the long exact sequence:

$$\cdots \to f_a^i(M) \xrightarrow{x} f_a^i(M) \xrightarrow{f} f_a^i \left( \frac{M}{xM} \right) \xrightarrow{g} f_a^{i+1}(M) \xrightarrow{h} \cdots. (*)$$

By using the hypothesis $f_a^i(M) \in S$ for all $i > t$ (because $t = f.cd_S(a,M)$). So using the above long exact sequence $f_a^i \left( \frac{M}{xM} \right) \in S$ for all $i > t$. By induction hypothesis, $\frac{f_a^i(M)}{af_a^i(M)} \in S$ because $\dim \left( \frac{M}{xM} \right) = \dim(M) - 1$.

Afterwards from the exact sequence $(*)$ we get the following short exact sequence:

$$0 \to \text{Im}(f) \to f_a^t \left( \frac{M}{xM} \right) \to \text{Im}(g) \to 0$$

So we obtain the following long exact sequence.

$$\cdots \to Tor^R \left( \frac{R}{a}, \text{Im}(g) \right) \to \text{Im}(f) \to \frac{f_a^t(M)}{af_a^t(M)} \xrightarrow{\text{Im}(g)} 0.$$

Since $f_a^t(M) \in S$ and $\text{Im}(g)$ is a submodule of $f_a^{t+1}(M)$, we deduce that $Tor^R \left( \frac{R}{a}, \text{Im}(g) \right) \in S$. On the other hand, $\frac{f_a^t(M)}{af_a^t(M)} \in S$. Therefore, $\frac{\text{Im}(f)}{\text{Im}(g)} \in S$ by the above long exact sequence.

Now consider the following long exact sequence.

$$\frac{f_a^t(M)}{af_a^t(M)} \xrightarrow{x} \frac{f_a^t(M)}{af_a^t(M)} \xrightarrow{\text{Im}(f)} 0.$$

So, $\frac{f_a^t(M)}{af_a^t(M)} \equiv \frac{\text{Im}(f)}{\text{Im}(g)}$ because $x \in a$. Consequently, $\frac{f_a^t(M)}{af_a^t(M)} \equiv S$.

**Proposition 3.8.** For a finitely generated $R$-module $M$,

$$f.cd_S(a,M) = \max \{ f.cd_S(a, \frac{R}{p}) | P \in \text{Ass}_R(M) \}.$$

**Proof.** Set $N := \bigoplus_{p \in \text{Ass}_R(M) \cap P} \frac{R}{p}$. Then $\text{Supp}_R(M) = \text{Supp}_R(N)$. So, by Theorem 3.2 and Corollary 3.5, $f.cd_S(a,M) = f.cd_S(a,N) = \max \{ f.cd_S(a, \frac{R}{p}) | P \in \text{Ass}_R(M) \}$.

**Proposition 3.9.** Assume that $a$ is an ideal of the local ring $(R, m)$. Then $\text{Hom}_R(\frac{R}{m}, f_a^0(M)) \in S$ if and only if $\text{Hom}_R(\frac{R}{m}, M)$.

**Proof.** It is enough to consider the following isomorphisms

$$\text{Hom}_R \left( \frac{R}{m}, f_a^0(M) \right) \equiv \lim_{m \in \mathbb{N}} \text{Hom}_R \left( \frac{R}{m}, M \right) \equiv \lim_{m \in \mathbb{N}} \text{Hom}_R \left( \frac{R}{m}, \frac{M}{a^mM} \right) \equiv \text{Hom}_R \left( \frac{R}{m}, \hat{M}^a \right).$$
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