Formal Local Cohomology Modules and Serre Subcategories

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Abstract

Let \((R, \mathfrak{m})\) be a Noetherian local ring, \(\mathfrak{a}\) an ideal of \(R\) and \(M\) a finitely generated \(R\)-module. We investigate some properties of formal local cohomology modules with respect to a Serre subcategory. We provide a common language to indicate some properties of formal local cohomology modules.

1. Introduction

Throughout this paper \((R, \mathfrak{m})\) is a commutative Noetherian local ring, \(\mathfrak{a}\) an ideal of \(R\) and \(M\) is a finitely generated \(R\)-module. For an integer \(i \in \mathbb{N}_0\), \(H^i_{\mathfrak{a}}(N)\) denotes the \(i\)-th local cohomology module of \(M\) with respect to \(\mathfrak{a}\) as introduced by Grothendieck (cf. [1], [2]).

We shall consider the family of local cohomology modules \(\{H^i_{\mathfrak{m}}(\frac{M}{\mathfrak{a}^nM})\}_{n \in \mathbb{N}}\) for a non-negative integer \(i \in \mathbb{N}_0\). With natural homomorphisms; this family forms an inverse system. Schenzel introduced the \(i\)-th formal local cohomology of \(M\) with respect to \(\mathfrak{a}\) in the form of \(f^i_{\mathfrak{a}}(M) := \lim_{n \in \mathbb{N}} H^i_{\mathfrak{m}}(\frac{M}{\mathfrak{a}^nM})\), which is the \(i\)-th cohomology module of the \(\mathfrak{a}\)-adic completion of the Čech complex \(\check{\mathcal{C}}^* \otimes_R M\), where \(\check{\mathcal{C}}^*\) denotes a system of elements of \(R\) such that \(\text{Rad}\left(\check{\mathcal{C}}^*, R\right) = \mathfrak{m}\) (see [3, Definition 3.1]). He defines the formal grade as \(f.\text{grade}(\mathfrak{a}, M) = \inf \{i \in \mathbb{N}_0 \mid f^i_{\mathfrak{a}}(M) \neq 0\}\). For any ideal \(\mathfrak{a}\) of \(R\) and finitely generated \(R\)-module \(M\) the following statements hold:

(i) (See [3, Theorem 3.11]). If \(0 \to M' \to M \to M'' \to 0\) is a short exact sequence of finitely generated \(R\)-modules, then there is the following long exact sequence:

\[ \cdots \to f^i_{\mathfrak{a}}(M') \to f^i_{\mathfrak{a}}(M) \to f^i_{\mathfrak{a}}(M'') \to \cdots \]

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(ii) (See [3, Theorem 1.3]). $f.\text{grade}(a, M) \leq \dim(M) - cd(a, M)$; some properties of formal local cohomology have been presented in [3, 4, 5 and 6].

Throughout this paper $S$ denotes a Serre subcategory of the category of $R$-modules and $R-$homomorphisms (we recall that a class $S$ of $R$-modules is a Serre subcategory of the category of $R$-modules and $R$-homomorphisms if $S$ is closed under taking submodules, quotients and extensions).

Our paper contains three sections. In Section 2, we shall define the formal grade of $a$ with respect to $M$ in $S$ as the infimum of the integers $i$ such that $f^{i}_{a}(M) \notin S$ and is denoted by $f.\text{grade}_{S}(a, M)$. (See definition 2.1). Then we shall obtain some properties of this notion. We show that if $\Gamma_{a}(M)$ is a pure submodule of $M$, then $\text{Hom}_{R}(\frac{R}{m}, f^{i}_{a}(\Gamma_{a}(M)))$ and $\text{Hom}_{R}(\frac{R}{m}, f^{t}_{a}(\frac{M}{\Gamma_{a}(M)}))$ belong to $S$, where $t = f.\text{grade}_{S}(a, M)$.

In Section 3, we shall define the formal cohomological dimension of $a$ with respect to $M$ in $S$ as the supremum of the integers $i$ such that $f^{i}_{a}(M) \notin S$ and is denoted by $f.\text{cd}_{S}(a, M)$. (See definition 3.1). The main result of this section is that if $f^{i}_{a}(M) \in S$ and $H^{i}_{m}(M) \in S$ for all $i > t$, then $\frac{R}{a} \otimes_{R} f^{i}_{a}(M)$ belongs to $S$.

2. The formal grade of a module in a Serre subcategory

**Definition 2.1.** The formal grade of $a$ with respect to $M$ in $S$ is the infimum of the integers $i$ such that $f^{i}_{a}(M) \notin S$ and is denoted by $f.\text{grade}_{S}(a, M)$.

**Proposition 2.2.** Let $(R, m)$ be a local ring and $a$ be an ideal of $R$. If $0 \to L \to M \to N \to 0$ is an exact sequence of finitely generated $R$-modules, then the following statements hold.

(a) $f.\text{grade}_{S}(a, M) \geq \min\{f.\text{grade}_{S}(a, L), f.\text{grade}_{S}(a, N)\}$.

(b) $f.\text{grade}_{S}(a, L) \geq \min\{f.\text{grade}_{S}(a, M), f.\text{grade}_{S}(a, N) + 1\}$.

(c) $f.\text{grade}_{S}(a, N) \geq \min\{f.\text{grade}_{S}(a, L) - 1, f.\text{grade}_{S}(a, M)\}$.

**Proof.** According to [3, Theorem 3.11], the above short exact sequence induces the following long exact sequence.

$$\cdots \to f^{i-1}_{a}(N) \to f^{i}_{a}(L) \to f^{i}_{a}(M) \to f^{i}_{a}(N) \to f^{i+1}_{a}(L) \to \cdots$$

So, the result follows.
Corollary 2.3. If \( x = x_1, \ldots, x_n \) is a regular \( M \)-sequence, then \( f. \text{grade}_S \left( a, \frac{M}{\sum M} \right) \geq f. \text{grade}_S (a, M) - n \).

**Proof.** Consider the following exact sequence \((n \in \mathbb{N})\)

\[
0 \rightarrow \frac{M}{(x_1, \ldots, x_{n-1})M} \rightarrow \frac{M}{(x_1, \ldots, x_n)M} \rightarrow \frac{M}{(x_1, \ldots, x_n)M} \rightarrow 0
\]
whenever \( n = 1 \) by \((x_1, \ldots, x_{n-1})M\) we mean 0.

Corollary 2.4. Let \( a \) and \( b \) be ideals of \( R \). Then
(a) \( f. \text{grade}_S (a \cap b, M) \geq \min \{ f. \text{grade}_S (a, M), f. \text{grade}_S (b, M), f. \text{grade}_S ((a, b), M) + 1 \} \).

(b) \( f. \text{grade}_S ((a, b), M) \geq \min \{ f. \text{grade}_S (a \cap b, M) - 1, f. \text{grade}_S (a, M), f. \text{grade}_S (b, M) \} \).

**Proof.** For all \( n \in \mathbb{N} \) there is a short exact sequence as follows:

\[
0 \rightarrow \frac{M}{a^nM} \rightarrow \frac{M}{a^nM} \oplus \frac{M}{b^nM} \rightarrow \frac{M}{(a^n, b^n)M} \rightarrow 0.
\]

By using [3, Theorem 5.1], the above exact sequence induces the following long exact sequence.

\[
\cdots \rightarrow \lim_{n \in \mathbb{N}} H^i_m \left( \frac{M}{(a^n b^n)M} \right) \rightarrow \lim_{n \in \mathbb{N}} H^i_m \left( \frac{M}{a^nM} \right) \oplus \lim_{n \in \mathbb{N}} H^i_m \left( \frac{M}{b^nM} \right) \rightarrow \lim_{n \in \mathbb{N}} H^i_m \left( \frac{M}{(a b)^nM} \right) \rightarrow \cdots
\]

So by using an argument similar to that of Proposition 2.2, the result follows.

Corollary 2.5. Assume that \( M \) is a finitely generated \( R \)-module and \( N_1 \) and \( N_2 \) are submodules of \( M \). Then considering the exact sequence \( 0 \rightarrow \frac{M}{N_1 \cap N_2} \rightarrow \frac{M}{N_1} \oplus \frac{M}{N_2} \rightarrow \frac{M}{N_1 + N_2} \rightarrow 0 \) we shall have

(a) \( f. \text{grade}_S \left( a, \frac{M}{N_1 \cap N_2} \right) \geq \min \{ f. \text{grade}_S (a, \frac{M}{N_1}), f. \text{grade}_S (a, \frac{M}{N_2}), f. \text{grade}_S (a, MN2), f. \text{grade}_S (a, MN1), f. \text{grade}_S (a, MN2 + 1) \} \).

(b) \( f. \text{grade}_S \left( a, \frac{M}{N_1 + N_2} \right) \geq \min \{ f. \text{grade}_S (a, \frac{M}{N_1}), f. \text{grade}_S (a, \frac{M}{N_2}) - 1, f. \text{grade}_S (a, MN1), f. \text{grade}_S (a, MN2) \} \).

Theorem 2.6. Let \( a \) be an ideal of a local ring \((R, m)\), \( M \) be a finitely generated \( R \)-module and \( L \) be a pure submodule of \( M \). Then \( f. \text{grade}_S (a, L) \geq f. \text{grade}_S (a, M) \) where \( S \) is a Serre subcategory of the category of \( R \)-modules and \( R \)-homomorphisms. In particular, \( \inf \{ i | H^i_m(L) \notin S \} \geq \inf \{ i | H^i_m (M) \notin S \} \).
Proof. Let $L$ be a pure submodule of $M$. So $\frac{L}{a^nL} \to \frac{M}{a^nM}$ is pure for each $n \in \mathbb{N}$. Now according to [8, Corollary 3.2 (a)], $H^i_m \left( \frac{L}{a^nL} \right) \to H^i_m \left( \frac{M}{a^nM} \right)$ is injective. Since inverse limit is a left exact functor, $f^i_a(L)$ is isomorphic to a submodule of $f^i_a(M)$. Consequently, $f.\text{grade}_S(a, L) \geq f.\text{grade}_S(a, M)$. If $a = 0$ then, $f.\text{grade}_S(0, M) = \inf \{ i | H^i_m(M) \not\in S \}$ and the result follows.

Corollary 2.7. If $0 \to L \to M \to N \to 0$ is a pure exact sequence of finitely generated $R$-modules, then $\min \{ f.\text{grade}_S(a, L), f.\text{grade}_S(a, N) + 1 \} \geq f.\text{grade}_S(a, M)$.

Proof. Since $L$ is a pure submodules of $M$, as a result of the previous theorem, $f.\text{grade}_S(a, L) \geq f.\text{grade}_S(a, M)$. Hence we must prove that $f.\text{grade}_S(a, N) + 1 \geq f.\text{grade}_S(a, M)$. We assume that $i < f.\text{grade}_S(a, M)$ and we show that $i < f.\text{grade}_S(a, N) + 1$. Consider the following long exact sequence.

$$\cdots \to f^{i-1}_a(M) \to f^i_a(N) \to f^i_a(L) \to f^i_a(M) \to f^i_a(N) \to \cdots \ (*)$$

If $i < f.\text{grade}_S(a, M)$, then $f^i_a(M), f^{i-1}_a(M), \ldots, f^0_a(M), f^0_a(L) \in S$. On the other hand, since $i < f.\text{grade}_S(a, M) \leq f.\text{grade}_S(a, L), f^0_a(L) \in S$. Hence, it follows from $(*)$ that $f^i_a(N), \ldots, f^{i-1}_a(N) \in S$ and so $i - 1 < f.\text{grade}_S(a, N)$.

Theorem 2.8. Let $(R, m)$ be a local ring, $a$ be an ideal of $R$, $S$ be a Serre subcategory of the category of $R$-modules and $R$-homomorphisms and $M \in S$ be a finitely generated $R$-module such that $\Gamma_a(M)$ is a pure submodule of $M$. Then $\text{Hom}_R \left( \frac{R}{a}, f^i_a(\Gamma_a(M)) \right) \in S$, where $t = f.\text{grade}_S(a, M)$.

Proof. Due to the previous theorem, $f.\text{grade}_S(a, \Gamma_a(M)) \geq f.\text{grade}_S(a, M)$. If $f.\text{grade}_S(a, \Gamma_a(M)) > f.\text{grade}_S(a, M)$, then the result is obvious. Accordingly, we assume that $f.\text{grade}_S(a, \Gamma_a(M)) = f.\text{grade}_S(a, M)$. We know that $\text{Supp}(\Gamma_a(M)) \subseteq \text{Var}(a)$. By using [4, Lemma 2.3], $f^i_a(\Gamma_a(M)) \cong H^i_m(\Gamma_a(M))$ for all $i \geq 0$. So, if $j < f.\text{grade}_S(a, M)$, then $f^j_a(\Gamma_a(M)) \cong H^j_m(\Gamma_a(M)) \in S$ and $\text{Ext}_R^k \left( \frac{R}{m}, H^j_m(\Gamma_a(M)) \right) \in S$ for all $k \geq 0$ and $j < f.\text{grade}_S(a, M)$. Moreover $\text{Ext}_R^i \left( \frac{R}{m}, \Gamma_a(M) \right) \in S$, because $\Gamma_a(M) \in S$. Consequently, according to [7, Theorem 2.2], $\text{Hom}_R \left( \frac{R}{m}, H^i_m(\Gamma_a(M)) \right) \in S$, where $t = f.\text{grade}_S(a, M)$.

Corollary 2.9 With the same notations as Theorem 2.8, let $X \subseteq S$ be a submodule of $f^i_a(\Gamma_a(M))$, where $t = f.\text{grade}_S(a, M)$. Then $\text{Hom}_R \left( \frac{R}{m}, \frac{f^i_a(\Gamma_a(M))}{X} \right) \in S$.

Proof. Consider the long exact sequence:
In accordance with the previous theorem $\text{Hom}_R\left(\frac{R}{m}, f^t_\alpha(\Gamma_a(M))\right) \to \text{Hom}_R\left(\frac{R}{m}, \frac{f^t_\alpha(\Gamma_a(M))}{X}\right) \to \text{Ext}^1_R\left(\frac{R}{m}, X\right)$. (*) 

Theorem 2.10. Suppose that $a$ is an ideal of $(R, m)$ and $M \in S$ is a finitely generated $R$-module such that $\Gamma_a(M)$ is a pure submodule of $M$. Then $\text{Hom}_R\left(\frac{R}{m}, f^{t-1}_\alpha\left(\frac{M}{\Gamma_a(M)}\right)\right) \in S$, where $t = f.\, \text{grade}_S(a, M)$.

Proof. One has $f.\, \text{grade}_S(a, \Gamma_a(M)) \geq f.\, \text{grade}_S(a, M)$, by Theorem 2.6. Now, the exact sequence $0 \to \Gamma_a(M) \to M \to \frac{M}{\Gamma_a(M)} \to 0$ induces the following long exact sequence:

$$\cdots \to f^{t-1}_\alpha\left(\Gamma_a(M)\right) \xrightarrow{\alpha} f^{t-1}_\alpha(M) \xrightarrow{\beta} \frac{M}{\Gamma_a(M)} \xrightarrow{\gamma} f^{t-1}_\alpha(M) \xrightarrow{\xi} f^{t-1}_\alpha\left(\Gamma_a(M)\right) \xrightarrow{\varphi} \cdots \tag{*}$$

Using the exact sequence (*), we obtain the short exact sequence $0 \to \text{Im}(\beta) \to f^{t-1}_\alpha(M) \to \text{Im}(\gamma) \to 0$. Since $f^{t-1}_\alpha(M) \in S$, $\text{Im}(\beta) \in S$ and $\text{Im}(\gamma) \in S$. Furthermore, we have the exact sequence $0 \to \text{Im}(\xi) \to H^t_m(\Gamma_a(M)) \to \text{Im}(\varphi) \to 0$ which induces the following long exact sequence:

$$0 \to \text{Hom}_R\left(\frac{R}{m}, \text{Im}(\xi)\right) \to \text{Hom}_R\left(\frac{R}{m}, H^t_m(\Gamma_a(M))\right) \to \cdots.$$ 

Thus $\text{Hom}_R\left(\frac{R}{m}, \text{Im}(\xi)\right) \in S$. Finally, by considering the short exact sequence $0 \to \text{Im}(\gamma) \to f^{t-1}_\alpha\left(\frac{M}{\Gamma_a(M)}\right) \to \text{Im}(\xi) \to 0$ we can conclude that $\text{Hom}_R\left(\frac{R}{m}, f^{t-1}_\alpha\left(\frac{M}{\Gamma_a(M)}\right)\right) \in S$.

Theorem 2.11. Suppose that $R$ is complete with respect to the $a$-adic topology and $M \in S$ be a finitely generated $R$-module and $t$ a positive integer such that $f^t_a(M) \in S$ for all $i < t$. Then $\text{Hom}_R\left(\frac{R}{m}, f^t_a(M)\right) \in S$.

Proof. We use induction on $t$. Let $t=0$. Consider the following isomorphisms.

$$\text{Hom}_R\left(\frac{R}{m}, f^0_a(M)\right) \cong \lim_{\leftarrow n} \text{Hom}_R\left(\frac{R}{m}, H^0_a\left(\frac{M}{a^nM}\right)\right) \cong \lim_{\leftarrow n} \text{Hom}_R\left(\frac{R}{m}, \frac{M}{a^nM}\right),$$

$$\cong \text{Hom}_R\left(\frac{R}{m}, \lim_{\leftarrow n} \frac{M}{a^nM}\right) \cong \text{Hom}_R\left(\frac{R}{m}, M^\wedge\right) \cong \text{Hom}_R\left(\frac{R}{m}, M\right).$$
It is clear that $\text{Hom}_R \left( \frac{M}{m}, M \right) \in S$. So by the above isomorphisms, we deduce that $\text{Hom}_R \left( \frac{M}{m}, f^i_a(M) \right) \in S$.

Suppose that $t > 0$ and the result is true for all integer $i$ less than $t$. Set $N := f^i_a(M)$. Then $f^i_a(M) \cong f^i_a \left( \frac{M}{N} \right)$ for all $i > 0$, and so we may assume that $\text{depth}_R(M) > 0$. There is an $M$-regular element $x \in m$. The exact sequence $0 \to M \to M \to \frac{M}{xM} \to 0$ induces the following long exact sequence:

$$\cdots \to f^{t-2}_a(M) \xrightarrow{x} f^{t-2}_a(M) \xrightarrow{f} f^{t-2}_a \left( \frac{M}{xM} \right) \to f^{t-1}_a(M) \xrightarrow{x} f^{t-1}_a(M) \xrightarrow{g} f^{t-1}_a \left( \frac{M}{xM} \right) \to f^t_a(M) \xrightarrow{h} \cdots. \ (*)$$

Using the exact sequence $(*)$ we obtain the short exact sequence

$$0 \to f^{t-1}_a \left( \frac{M}{x f^{t-1}_a(M)} \right) \to f^t_a \left( \frac{M}{xM} \right) \to (0 : x) \to 0.$$

Now, this exact sequence induces the following long exact sequence:

$$0 \to \text{Hom}_R \left( \frac{R}{m}, f^{t-1}_a(M) \right) \to \text{Hom}_R \left( \frac{R}{m}, f^t_a \left( \frac{M}{xM} \right) \right) \to \text{Hom}_R \left( \frac{R}{m}, (0 : x) \right) \to \text{Ext}_R^1 \left( \frac{R}{m}, f^{t-1}_a(M) \right) \to \cdots. (**)$$

By using $(*)$, $f^i_a \left( \frac{M}{xM} \right) \in S$ for all $i < t - 1$. Therefore by the induction hypothesis $\text{Hom}_R \left( \frac{R}{m}, f^{t-1}_a(M) \right) \in S$. Furthermore $\text{Ext}_R^1 \left( \frac{R}{m}, f^{t-1}_a(M) \right) \in S$ because $f^{t-1}_a(M) \in S$. Thus in accordance with $(**)$, $\text{Hom}_R \left( \frac{R}{m}, (0 : x) \right) \in S$. Since $x \in m$ according to [9,10.86] we have the following isomorphisms.

$$\text{Hom}_R \left( \frac{R}{m}, (0 : x) \right) \cong \text{Hom}_R \left( \frac{R}{m}, \text{Hom}_R \left( \frac{R}{xM}, f^t_a(M) \right) \right) \cong \text{Hom}_R \left( \frac{R}{m} \otimes_R \frac{R}{xM}, f^t_a(M) \right) \cong \text{Hom}_R \left( \frac{R}{m}, f^t_a(M) \right).$$

Consequently $\text{Hom}_R \left( \frac{R}{m}, f^t_a(M) \right) \in S$. 

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3. The formal cohomological dimension in a Serre subcategory

We recall from [3,Theorem 1.1] that for a finitely generated $R$-module $M$, \[ \text{sup} \{ i \in \mathbb{N}_0 | f^i_a(M) \neq 0 \} = \text{dim} \left( \frac{M}{aM} \right). \]

**Definition 3.1.** The formal cohomological dimension of $M$ with respect to $a$ in $S$ is the supremum of the integers $i$ such that $f^i_a(M) \notin S$ and is denoted by $f \cdot \text{cd}_S(a, M)$.

**Theorem 3.2.** Suppose that $S$ is a Serre subcategory of the category of $R$-modules and $R$-homomorphisms and $L$ and $N$ are two finitely generated $R$-modules such that $\text{Supp}_R(L) \subseteq \text{Supp}_R(N)$. Then $f \cdot \text{cd}_S(a, L) \leq f \cdot \text{cd}_S(a, N)$.

**Proof.** It is enough to prove that $f^i_a(L) \in S$ for all $i > f \cdot \text{cd}_S(a, N)$ and all finitely generated $R$-module $L$ such that $\text{Supp}_R(L) \subseteq \text{Supp}_R(N)$. We use descending induction on $i$. For all $i > \text{dim} \left( \frac{L}{aL} \right) + f \cdot \text{cd}_S(a, N)$, $f^i_a(L) = 0 \notin S$. Let $i > f \cdot \text{cd}_S(a, N)$ and the result is proved for $i + l$. By Gruson’s theorem, there is a chain $0 = L_0 \subset L_1 \subset \ldots \subset L_l = L$ of submodules of $L$ such that $\frac{L_i}{L_{i-1}}$ is a homomorphic image of a direct sum of finitely many copies of $N$. Consider the exact sequence $0 \to L_{i-1} \to L_i \to \frac{L_i}{L_{i-1}} \to 0$ ($i = 0, l, \ldots, l$). We may assume that $l = l$. The exact sequence $0 \to K \to \bigoplus_{j=i}^l N \to L \to 0$ where $K$ is a finitely generated $R$-module induces the following long exact sequence:

\[ \cdots \to f^i_a(\bigoplus_{j=i}^l N) \to f^i_a(L) \to f^{i+1}_a(K) \to \cdots \text{(*)} \]

Based on the induction hypothesis $f^{i+1}_a(K) \in S$. Moreover $f^i_a(\bigoplus_{j=i}^l N) = a f^i_a(N) \in S$ for all $i > f \cdot \text{cd}_S(a, N)$. Hence it follows from the exact sequence (*) that $f^i_a(L) \in S$.

The next example shows that even if $\text{Supp}_R(M) = \text{Supp}_R(N)$, then it may not true that $f \cdot \text{grade}_S(a, M) = f \cdot \text{grade}_S(a, N)$.

**Example 3.3.** (See [4, Example 4.3 (i)]) Let $(R, m)$ be a 2 dimensional complete regular local ring, $S = 0$ and $a$ be an ideal of $R$ with $\text{dim} \left( \frac{R}{a} \right) = 1$. Then by using [5,Theorem 1.1], $f \cdot \text{grade}_S(a, R) = 1$ and $f \cdot \text{grade}_S \left( a, \frac{R}{m} \right) = 0$. Set $M := R \oplus \frac{R}{m}$. Then $\text{Supp}_R(M) = \text{Supp}_R(R)$.

Then $f \cdot \text{grade}_S(a, M) = \inf \left\{ f \cdot \text{grade}_S(a, R), f \cdot \text{grade}_S \left( a, \frac{R}{m} \right) \right\} = 0$.

**Corollary 3.4.** For all $x \in a \cdot f \cdot \text{cd}_S(a, M) \geq f \cdot \text{cd}_S \left( a, \frac{M}{xM} \right)$.

**Corollary 3.5.** Suppose that $0 \to L \to M \to N \to 0$ is an exact sequence of finitely generated $R$-modules. Then $f \cdot \text{cd}_S(a, M) = \max \left\{ f \cdot \text{cd}_S(a, L), f \cdot \text{cd}_S(a, N) \right\}$. 

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Proof. Since $\text{Supp}_R(M) = \text{Supp}_R(L) \cup \text{Supp}_R(N)$ by referring to Theorem 3.2 we deduce that $f \cdot \text{cd}_S(\mathfrak{a}, M) \geq f \cdot \text{cd}_S(\mathfrak{a}, L)$ and $f \cdot \text{cd}_S(\mathfrak{a}, M) \geq f \cdot \text{cd}_S(\mathfrak{a}, N)$. Therefore $f \cdot \text{cd}_S(\mathfrak{a}, M) \geq \max \{ f \cdot \text{cd}_S(\mathfrak{a}, L), f \cdot \text{cd}_S(\mathfrak{a}, N) \}$.

Next we prove that $\max \{ f \cdot \text{cd}_S(\mathfrak{a}, L), f \cdot \text{cd}_S(\mathfrak{a}, N) \} \geq f \cdot \text{cd}_S(\mathfrak{a}, M)$.

Let $i > \max \{ f \cdot \text{cd}_S(\mathfrak{a}, L), f \cdot \text{cd}_S(\mathfrak{a}, N) \}$. Then $f^i_\mathfrak{a}(N), f^i_\mathfrak{a}(L) \in S$ and from the exact sequence $f^i_\mathfrak{a}(L) \rightarrow f^i_\mathfrak{a}(M) \rightarrow f^i_\mathfrak{a}(N)$ we conclude that $f^i_\mathfrak{a}(M) \in S$. Thus, $\max \{ f \cdot \text{cd}_S(\mathfrak{a}, L), f \cdot \text{cd}_S(\mathfrak{a}, N) \} \geq f \cdot \text{cd}_S(\mathfrak{a}, M)$.

We recall that the cohomological dimension of an $R$-module $M$ with respect to an ideal $\mathfrak{a}$ of $R$ in $S$ is defined as

$$\text{cd}_S(\mathfrak{a}, M) := \sup \{ i \in \mathbb{N} \mid H^i_\mathfrak{a}(M) \notin S \}.$$

The following lemma shows that when we considering the Artinianness of $f^i_\mathfrak{a}(M)$, we can assume that $M$ is $\mathfrak{a}$-torsion-free.

Lemma 3.6. Suppose that $\mathfrak{a}$ is an ideal of a local ring $(R, \mathfrak{m})$ and $t$ be a non-negative integer. If $H^i_\mathfrak{m}(M) \in S$ for all $i \geq t$, then the following are equivalent:

(a) $f^i_\mathfrak{a}(M) \in S$ for all $i \geq t$.

(b) $f^i_\mathfrak{a}\left(\frac{M}{\Gamma_\mathfrak{a}(M)}\right) \in S$ for all $i \geq t$.

Proof. According to the hypothesis $t > \text{cd}_S(\mathfrak{m}, M)$. On the other hand $\text{Supp}_R(\Gamma_\mathfrak{a}(M)) \subseteq \text{Supp}_R(M)$. So by referring to [7, Theorem 3.5], $\text{cd}_S(\mathfrak{m}, \Gamma_\mathfrak{a}(M)) \leq \text{cd}_S(\mathfrak{m}, M)$. Thus, $t > \text{cd}_S(\mathfrak{m}, \Gamma_\mathfrak{a}(M))$ and $H^i_\mathfrak{m}(\Gamma_\mathfrak{a}(M)) \in S$ for all $i \geq t$. Now, consider the following exact sequence:

$$\cdots \rightarrow f^i_\mathfrak{a}(\Gamma_\mathfrak{a}(M)) \rightarrow f^i_\mathfrak{a}(M) \rightarrow f^i_\mathfrak{a}\left(\frac{M}{\Gamma_\mathfrak{a}(M)}\right) \rightarrow f^{i+1}_\mathfrak{a}(\Gamma_\mathfrak{a}(M)) \rightarrow \cdots \ast$$

According to [4, Lemma 2.3] $f^i_\mathfrak{a}(\Gamma_\mathfrak{a}(M)) \cong H^i_\mathfrak{m}(\Gamma_\mathfrak{a}(M))$. By using the hypothesis $f^i_\mathfrak{a}(\Gamma_\mathfrak{a}(M)) \in S$ for all $i \geq t$. So it follows from the exact sequence $(\ast)$ that $f^i_\mathfrak{a}(M) \in S$ if and only if $f^i_\mathfrak{a}\left(\frac{M}{\Gamma_\mathfrak{a}(M)}\right) \in S$ for all $i \geq t$.

Theorem 3.7. Let $(R, \mathfrak{m})$ be a local ring and $M \in S$ be a finitely generated $R$-module of dimension $d$ such that $\text{cd}_S(\mathfrak{m}, M) \leq f \cdot \text{cd}_S(\mathfrak{a}, M)$. Then $f^i(\mathfrak{m}, M) \in S$ where $t = f \cdot \text{cd}_S(\mathfrak{a}, M)$.

Proof. We use induction on $d = \dim(M)$. If $d = 0$, then $\dim\left(\frac{M}{\mathfrak{a}M}\right) = 0$. Accordingly to [3, Theorem 1.1], $f^i_\mathfrak{a}(M) = 0$ for all $i > 0$. 

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Moreover $f_a^t(M) \cong M \in S$. By definition $H^i_m(M) \in S$ for all $i > t$. Therefore from the above lemma we can assume that $M$ is $a$-torsion-free and there is an $M$-regular element $x \in a$. Consider the long exact sequence:

$$\cdots \to f_a^t(M) \xrightarrow{x} f_a^t(M) \xrightarrow{f} \frac{M}{xM} \xrightarrow{g} f_a^{i+1}(M) \xrightarrow{h} \cdots (*)$$

By using the hypothesis $f_a^i(M) \in S$ for all $i > t$ (because $t = f \cdot cd_S(a, M)$). So using the above long exact sequence $f_a^i\left(\frac{M}{xM}\right) \in S$ for all $i > t$. By induction hypothesis, $\frac{f_a^i\left(\frac{M}{xM}\right)}{af_a^i\left(\frac{M}{xM}\right)} \in S$ because $\dim\left(\frac{M}{xM}\right) = \dim(M) - 1$.

Afterwards from the exact sequence $(*)$ we get the following short exact sequence.

$$0 \to \text{Im}(f) \to f_a^t\left(\frac{M}{xM}\right) \to \text{Im}(g) \to 0$$

So we obtain the following long exact sequence.

$$\cdots \to \text{Tor}_1^R\left(\frac{R}{a}, \text{Im}(g)\right) \to \frac{\text{Im}(f)}{a\text{Im}(f)} \to \frac{f_a^t\left(\frac{M}{xM}\right)}{af_a^t\left(\frac{M}{xM}\right)} \to \frac{\text{Im}(g)}{a\text{Im}(g)} \to 0.$$

Since $f_a^t(M) \in S$ and $\text{Im}(g)$ is a submodule of $f_a^{i+1}(M)$, we deduce that $\text{Tor}_1^R\left(\frac{R}{a}, \text{Im}(g)\right) \in S$. On the other hand, $\frac{f_a^i\left(\frac{M}{xM}\right)}{af_a^i\left(\frac{M}{xM}\right)} \in S$. Therefore, $\frac{\text{Im}(f)}{a\text{Im}(f)} \in S$ by the above long exact sequence.

Now, consider the following long exact sequence.

$$\frac{f_a^t(M)}{af_a^t(M)} \xrightarrow{x} \frac{f_a^t(M)}{af_a^t(M)} \xrightarrow{f} \frac{\text{Im}(f)}{a\text{Im}(f)} \to 0.$$ 

So, $\frac{f_a^t(M)}{af_a^t(M)} \cong \frac{\text{Im}(f)}{a\text{Im}(f)}$ because $x \in a$. Consequently, $\frac{f_a^t(M)}{af_a^t(M)} \in S$.

**Proposition 3.8.** For a finitely generated $R$-module $M$,

$$f \cdot cd_S(a, M) = \max \{f \cdot cd_S\left(a, \frac{R}{P}\right) | P \in \text{Ass}_R(M)\}.$$ 

**Proof.** Set $N := \bigoplus_{P \in \text{Ass}_R(M)} \frac{R}{P}$. Then $\text{Supp}_R(M) = \text{Supp}_R(N)$. So, by Theorem 3.2 and Corollary 3.5, $f \cdot cd_S(a, M) = f \cdot cd_S(a, N) = \max \{f \cdot cd_S\left(a, \frac{R}{P}\right) | P \in \text{Ass}_R(M)\}$.

**Proposition 3.9.** Assume that $a$ is an ideal of the local ring $(R, m)$. Then $\text{Hom}_R\left(\frac{R}{m}, f_a^0(M)\right) \in S$ if and only if $\text{Hom}_R\left(\frac{R}{m}, \tilde{M}^a\right) \in S$.

**Proof.** It is enough to consider the following isomorphisms

$$\text{Hom}_R\left(\frac{R}{m}, f_a^0(M)\right) \cong \bigoplus_{i \in \mathbb{N}} \text{Hom}_R\left(\frac{R}{m}, H_0^i(M/\mathfrak{a}^nM)\right) \cong \bigoplus_{i \in \mathbb{N}} \text{Hom}_R\left(\frac{R}{m}, M/\mathfrak{a}^nM\right) \cong \text{Hom}_R\left(\frac{R}{m}, \tilde{M}^a\right).$$
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