Formal Local Cohomology Modules and Serre Subcategories

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Abstract

Let \(( R, \mathfrak{m} )\) be a Noetherian local ring, \( \mathfrak{a}\) an ideal of \( R \) and \( M \) a finitely generated \( R\)-module. We investigate some properties of formal local cohomology modules with respect to a Serre subcategory. We provide a common language to indicate some properties of formal local cohomology modules.

1. Introduction

Throughout this paper \(( R, \mathfrak{m} )\) is a commutative Noetherian local ring, \( \mathfrak{a}\) an ideal of \( R \) and \( M \) is a finitely generated \( R\)-module. For an integer \( i \in \mathbb{N}_0 \), \( H^i_{\mathfrak{a}}(N) \) denotes the \( i\)-th local cohomology module of \( M \) with respect to \( \mathfrak{a}\) as introduced by Grothendieck (cf. [1], [2]).

We shall consider the family of local cohomology modules \( \{H^i_{\mathfrak{m}}(\frac{M}{a^n M})\}_{n \in \mathbb{N}} \) for a non-negative integer \( i \in \mathbb{N}_0 \). With natural homomorphisms; this family forms an inverse system. Schenzel introduced the \( i\)-th formal local cohomology of \( M \) with respect to \( \mathfrak{a}\) in the form of \( f^i_{\mathfrak{a}}(M) := \lim_{n \in \mathbb{N}} H^i_{\mathfrak{m}}(\frac{M}{a^n M}) \), which is the \( i\)-th cohomology module of the \( \mathfrak{a}\)-adic completion of the Čech complex \( \check{\mathcal{C}}_x \otimes_R M \), where \( x \) denotes a system of elements of \( R \) such that \( \text{Rad}(x, R) = \mathfrak{m} \) (see [3, Definition 3.1]). He defines the formal grade as \( f.\text{grade}(\mathfrak{a}, M) = \inf \{ i \in \mathbb{N}_0 | f^i_{\mathfrak{a}}(M) \neq 0 \} \). For any ideal \( \mathfrak{a}\) of \( R \) and finitely generated \( R\)-module \( M \) the following statements hold:

(i) (See [3, Theorem 3.11]). If \( 0 \to M' \to M \to M'' \to 0 \) is a short exact sequence of finitely generated \( R\)-modules, then there is the following long exact sequence:

\[
\cdots \to f^i_{\mathfrak{a}}(M') \to f^i_{\mathfrak{a}}(M) \to f^i_{\mathfrak{a}}(M'') \to \cdots
\]

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(ii) (See [3, Theorem 1.3]). \( f\cdot \text{grade} (\mathfrak{a}, M) \leq \dim(M) - \text{cd}(\mathfrak{a}, M); \) some properties of formal local cohomology have been presented in [3, 4, 5 and 6].

Throughout this paper \( S \) denotes a Serre subcategory of the category of \( R \)-modules and \( R \)-homomorphisms (we recall that a class \( S \) of \( R \)-modules is a Serre subcategory of the category of \( R \)-modules and \( R \)-homomorphisms if \( S \) is closed under taking submodules, quotients and extensions).

Our paper contains three sections. In Section 2, we shall define the formal grade of \( \mathfrak{a} \) with respect to \( M \) in \( S \) as the infimum of the integers \( i \) such that \( f^i_{\mathfrak{a}}(M) \notin S \) and is denoted by \( f\cdot \text{grades}_S (\mathfrak{a}, M) \). (See definition 2.1). Then we shall obtain some properties of this notion. We show that if \( \Gamma_{\mathfrak{a}}(M) \) is a pure submodule of \( M \), then \( \text{Hom}_R(\frac{R}{m}, f^i_{\mathfrak{a}}(\Gamma_{\mathfrak{a}}(M))) \) and \( \text{Hom}_R(\frac{R}{m}, f^{i-1}_{\mathfrak{a}}((\frac{M}{\Gamma_{\mathfrak{a}}(M)}))) \) belong to \( S \), where \( t = f\cdot \text{grade}_S (\mathfrak{a}, M) \).

In Section 3, we shall define the formal cohomological dimension of \( \mathfrak{a} \) with respect to \( M \) in \( S \) as the supremum of the integers \( i \) such that \( f^i_{\mathfrak{a}}(M) \notin S \) and is denoted by \( f\cdot \text{cd}_S (\mathfrak{a}, M) \). (See definition 3.1). The main result of this section is that if \( f^i_{\mathfrak{a}}(M) \in S \) and \( H^i_m(M) \in S \) for all \( i > t \), then \( \frac{R}{\mathfrak{a}} \otimes_R f^t_{\mathfrak{a}}(M) \) belongs to \( S \).

2. The formal grade of a module in a Serre subcategory

**Definition 2.1.** The formal grade of \( \mathfrak{a} \) with respect to \( M \) in \( S \) is the infimum of the integers \( i \) such that \( f^i_{\mathfrak{a}}(M) \notin S \) and is denoted by \( f\cdot \text{grade}_S (\mathfrak{a}, M) \).

**Proposition 2.2.** Let \( (R, m) \) be a local ring and \( \mathfrak{a} \) be an ideal of \( R \). If \( 0 \to L \to M \to N \to 0 \) is an exact sequence of finitely generated \( R \)-modules, then the following statements hold.

\( (a) \quad f\cdot \text{grade}_S (\mathfrak{a}, M) \geq \min\{f\cdot \text{grade}_S (\mathfrak{a}, L), f\cdot \text{grade}_S (\mathfrak{a}, N)\}. \\
(b) \quad f\cdot \text{grade}_S (\mathfrak{a}, L) \geq \min\{f\cdot \text{grade}_S (\mathfrak{a}, M), f\cdot \text{grade}_S (\mathfrak{a}, N) + 1\}. \\
(c) \quad f\cdot \text{grade}_S (\mathfrak{a}, N) \geq \min\{f\cdot \text{grade}_S (\mathfrak{a}, L) - 1, f\cdot \text{grade}_S (\mathfrak{a}, M)\}.

**Proof.** According to [3, Theorem 3.11], the above short exact sequence induces the following long exact sequence.

\[ ... \to f^{i-1}_{\mathfrak{a}}(N) \to f^i_{\mathfrak{a}}(L) \to f^i_{\mathfrak{a}}(M) \to f^i_{\mathfrak{a}}(N) \to f^{i+1}_{\mathfrak{a}}(L) \to ... \]

So, the result follows.
Corollary 2.3. If \( x = x_1, ..., x_n \) is a regular \( M \)-sequence, then \( f.\text{grade}_S \left( \frac{a}{\sum M} \right) \geq f.\text{grade}_S (a,M) - n \).

Proof. Consider the following exact sequence \((n \in \mathbb{N})\)
\[
0 \rightarrow \frac{M}{(x_1, ..., x_{n-1})M} \xrightarrow{x_n} \frac{M}{(x_1, ..., x_n, M)} \xrightarrow{\text{nat.}} \frac{M}{(x_1, ..., x_n)M} \rightarrow 0
\]
whenever \( n = 1 \) by \((x_1, ..., x_{n-1})M\) we mean 0.

Corollary 2.4. Let \( a \) and \( b \) be ideals of \( R \). Then
(a) \( f.\text{grade}_S (a \cap b, M) \geq \min \{ f.\text{grade}_S (a, M), f.\text{grade}_S (b, M), f.\text{grade}_S ((a,b), M) + 1 \} \).
(b) \( f.\text{grade}_S ((a,b), M) \geq \min \{ f.\text{grade}_S (a \cap b, M) - 1, f.\text{grade}_S (a, M), f.\text{grade}_S (b, M) \} \).

Proof. For all \( n \in \mathbb{N} \) there is a short exact sequence as follows:
\[
0 \rightarrow \frac{M}{a^n M \cap b^n M} \rightarrow \frac{M}{a^n M} \oplus \frac{M}{b^n M} \rightarrow \frac{M}{(a^n, b^n)M} \rightarrow 0.
\]
By using [3,Theorem 5.1], the above exact sequence induces the following long exact sequence.
\[
\cdots \rightarrow \lim_{n \in \mathbb{N}} \text{lim}_{n \in \mathbb{N}} H^i_m \left( \frac{M}{(a^n, b^n)M} \right) \rightarrow \lim_{n \in \mathbb{N}} \text{lim}_{n \in \mathbb{N}} H^i_m \left( \frac{M}{a^n M} \right) \oplus \lim_{n \in \mathbb{N}} \text{lim}_{n \in \mathbb{N}} H^i_m \left( \frac{M}{b^n M} \right) \rightarrow \lim_{n \in \mathbb{N}} \text{lim}_{n \in \mathbb{N}} H^i_m \left( \frac{M}{(a^n, b^n)M} \right) \rightarrow \cdots.
\]
So by using an argument similar to that of Proposition 2.2, the result follows.

Corollary 2.5. Assume that \( M \) is a finitely generated \( R \)-module and \( N_1 \) and \( N_2 \) are submodules of \( M \). Then considering the exact sequence \( 0 \rightarrow \frac{M}{N_1 \cap N_2} \rightarrow \frac{M}{N_1} \oplus \frac{M}{N_2} \rightarrow \frac{M}{N_1 + N_2} \rightarrow 0 \) we shall have
(a) \( f.\text{grade}_S (a, \frac{M}{N_1 + N_2}) \geq \min \{ f.\text{grade}_S (a, \frac{M}{N_1}), f.\text{grade}_S (a, \frac{M}{N_2}), f.\text{grade}_S (a, \frac{M}{N_1 N_2}), f.\text{grade}_S (a, \frac{M}{N_2 N_1}), f.\text{grade}_S (a, \frac{M}{N_1 N_2 + N_2}), f.\text{grade}_S (a, \frac{M}{N_1 N_2 + N_1}), f.\text{grade}_S (a, \frac{M}{N_1 N_2 + N_1 + N_2}) \} \).
(b) \( f.\text{grade}_S (a, \frac{M}{N_1 + N_2}) \geq \min \{ f.\text{grade}_S (a, \frac{M}{N_1 N_2}), f.\text{grade}_S (a, \frac{M}{N_1 N_2 + N_2}), f.\text{grade}_S (a, \frac{M}{N_1 N_2 + N_1}), f.\text{grade}_S (a, \frac{M}{N_1 N_2 + N_1 + N_2}) \} \).

Theorem 2.6. Let \( a \) be an ideal of a local ring \((R, m)\), \( M \) be a finitely generated \( R \)-module and \( L \) be a pure submodule of \( M \). Then \( f.\text{grade}_S (a, L) \geq f.\text{grade}_S (a, M) \) where \( S \) is a Serre subcategory of the category of \( R \)-modules and \( R \)-homomorphisms.
In particular, \( \inf \{ i | H^i_m(L) \notin S \} \geq \inf \{ i | H^i_m(M) \notin S \} \).

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**Proof.** Let $L$ be a pure submodule of $M$. So \( \frac{L}{a^nL} \to \frac{M}{a^nM} \) is pure for each $n \in \mathbb{N}$. Now according to [8, Corollary 3.2 (a)], \( \hat{H}^*_m \left( \frac{L}{a^nL} \right) \to \hat{H}^*_m \left( \frac{M}{a^nM} \right) \) is injective. Since inverse limit is a left exact functor, $f_a^i(L)$ is isomorphic to a submodule of $f_a^i(M)$. Consequently, $f.\text{grade}_S(a, L) \geq f.\text{grade}_S(a, M)$. If $a = 0$, then $f.\text{grade}_S(0, M) = \inf \{ |H^*_m(M) \notin S \}$ and the result follows.

**Corollary 2.7.** If $0 \to L \to M \to N \to 0$ is a pure exact sequence of finitely generated $R$-modules, then $\min \{ f.\text{grade}_S(a, L), f.\text{grade}_S(a, N) + 1 \} \geq f.\text{grade}_S(a, M)$.

**Proof.** Since $L$ is a pure submodule of $M$, as a result of the previous theorem, $f.\text{grade}_S(a, L) \geq f.\text{grade}_S(a, M)$. Hence we must prove that $f.\text{grade}_S(a, N) + 1 \geq f.\text{grade}_S(a, M)$. We assume that $i < f.\text{grade}_S(a, M)$ and we show that $i < f.\text{grade}_S(a, N) + 1$. Consider the following long exact sequence.

\[
\cdots \to f_a^{i+1}(M) \to f_a^i(N) \to f_a^i(L) \to f_a^i(M) \to f_a^i(N) \to \cdots \quad (**) 
\]

If $i < f.\text{grade}_S(a, M)$, then $f_a^i(M), f_a^i(M), \ldots, f_a^{i-1}(M), f_a^0(M) \in S$. On the other hand, since $i < f.\text{grade}_S(a, M)$, $f_a^i(M), f_a^0(L), \ldots, f_a^i(L) \in S$. Hence, it follows from (**) that $f_a^i(N), \ldots, f_a^{i-1}(N) \in S$ and so $i - 1 < f.\text{grade}_S(a, N)$.

**Theorem 2.8.** Let $(R, m)$ be a local ring, $a$ be an ideal of $R$, $S$ be a Serre subcategory of the category of $R$-modules and $R$-homomorphisms and $M \in S$ be a finitely generated $R$-module such that $\Gamma_a(M)$ is a pure submodule of $M$. Then $\text{Hom}_R \left( \frac{R}{a}, f_a^t(\Gamma_a(M)) \right) \in S$, where $t = f.\text{grade}_S(a, M)$.

**Proof.** Due to the previous theorem, $f.\text{grade}_S(a, \Gamma_a(M)) \geq f.\text{grade}_S(a, M)$. If $f.\text{grade}_S(a, \Gamma_a(M)) > f.\text{grade}_S(a, M)$, then the result is obvious. Accordingly, we assume that $f.\text{grade}_S(a, \Gamma_a(M)) = f.\text{grade}_S(a, M)$. We know that $\text{Supp}(\Gamma_a(M)) \subseteq \text{Var}(a)$. By using [4, Lemma 2.3], $f_a^i(\Gamma_a(M)) \cong H^i_m(\Gamma_a(M))$ for all $i \geq 0$. So, if $j < f.\text{grade}_S(a, M)$, then $f_a^j(\Gamma_a(M)) \cong H^j_m(\Gamma_a(M)) \in S$ and $\text{Ext}_R^k \left( \frac{R}{m}, H^j_m(\Gamma_a(M)) \right) \in S$ for all $k \geq 0$ and $j < f.\text{grade}_S(a, M)$. Moreover, $\text{Ext}_R^k \left( \frac{R}{m}, \Gamma_a(M) \right) \in S$, because $\Gamma_a(M) \in S$. Consequently, according to [7, Theorem 2.2],

\[
\text{Hom}_R \left( \frac{R}{m}, H^i_m(\Gamma_a(M)) \right) \in S, \text{ where } t = f.\text{grade}_S(a, M).
\]

**Corollary 2.9.** With the same notations as Theorem 2.8, let $X \in S$ be a submodule of $f_a^t(\Gamma_a(M))$, where $t = f.\text{grade}_S(a, M)$. Then $\text{Hom}_R \left( \frac{R}{m}, \frac{f_a^t(\Gamma_a(M))}{X} \right) \in S$.

**Proof.** Consider the long exact sequence:
In accordance with the previous theorem $\text{Hom}_R\left(\frac{R}{m}, f_\alpha^t(\Gamma_a(M))\right) \to \text{Hom}_R\left(\frac{R}{m}, \frac{f_\alpha^t(\Gamma_a(M))}{X}\right) \to \text{Ext}_R^1\left(\frac{R}{m}, X\right). (*)$

Theorem 2.10. Suppose that $a$ is an ideal of $(R, m)$ and $M \in S$ is a finitely generated $R$-module such that $\Gamma_a(M)$ is a pure submodule of $M$. Then $\text{Hom}_R\left(\frac{R}{m}, f_\alpha^{t-1}\left(\frac{M}{\Gamma_a(M)}\right)\right) \in S$, where $t = f. \text{grade}_S(a, M)$.

Proof. One has $f. \text{grade}_S(a, \Gamma_a(M)) \geq f. \text{grade}_S(a, M)$, by Theorem 2.6. Now, the exact sequence $0 \to \Gamma_a(M) \to M \to \frac{M}{\Gamma_a(M)} \to 0$ induces the following long exact sequence:

$$
\cdots \to f_\alpha^{t-1}(\Gamma_a(M)) \xrightarrow{\alpha} f_\alpha^{t-1}(M) \xrightarrow{\beta} f_\alpha^{t-1}\left(\frac{M}{\Gamma_a(M)}\right) \xrightarrow{\gamma} f_\alpha^{t-1}(\Gamma_a(M)) \xrightarrow{\delta} \cdots. (*)
$$

Using the exact sequence $(*)$, we obtain the short exact sequence $0 \to \text{Im}(\beta) \to f_\alpha^{t-1}(M) \to \text{Im}(\gamma) \to 0$. Since $f_\alpha^{t-1}(M) \in S$, $\text{Im}(\beta), \text{Im}(\gamma) \in S$. Furthermore, we have the exact sequence $0 \to \text{Im}(\xi) \to H_\alpha^t(\Gamma_a(M)) \to \text{Im}(\varphi) \to 0$ which induces the following long exact sequence:

$$
0 \to \text{Hom}_R\left(\frac{R}{m}, \text{Im}(\xi)\right) \to \text{Hom}_R\left(\frac{R}{m}, H_\alpha^t(\Gamma_a(M))\right) \to \cdots.
$$

Thus $\text{Hom}_R\left(\frac{R}{m}, \text{Im}(\xi)\right) \in S$. Finally, by considering the short exact sequence $0 \to \text{Im}(\gamma) \to f_\alpha^{t-1}\left(\frac{M}{\Gamma_a(M)}\right) \to \text{Im}(\xi) \to 0$ we can conclude that $\text{Hom}_R\left(\frac{R}{m}, f_\alpha^{t-1}\left(\frac{M}{\Gamma_a(M)}\right)\right) \in S$.

Theorem 2.11. Suppose that $R$ is complete with respect to the $a$-adic topology and $M \in S$ be a finitely generated $R$-module and $t$ a positive integer such that $f_\alpha^i(M) \in S$ for all $i < t$. Then $\text{Hom}_R\left(\frac{R}{m}, f_\alpha^t(M)\right) \in S$.

Proof. We use induction on $t$. Let $t=0$. Consider the following isomorphisms.

$$
\text{Hom}_R\left(\frac{R}{m}, f_\alpha^0(M)\right) \cong \lim_{\leftarrow n} \text{Hom}_R\left(\frac{R}{m}, H_\alpha^n(\frac{M}{a^nM})\right) \cong \lim_{\leftarrow n} \text{Hom}_R\left(\frac{R}{m}, \frac{M}{a^nM}\right) \cong \text{Hom}_R\left(\frac{R}{m}, \frac{M}{a^0M}\right) \cong \text{Hom}_R\left(\frac{R}{m}, M\right)
$$
It is clear that $\text{Hom}_R\left(\frac{R}{m}, M\right) \in S$. So by the above isomorphisms, we deduce that

$\text{Hom}_R\left(\frac{R}{m}, f^a_i(M)\right) \in S$.

Suppose that $t>0$ and the result is true for all integer $i$ less than $t$. Set $N := f^a_m(M)$. Then $f^a_i(M) \cong \frac{f^a_i(M)}{N}$ for all $i > 0$, and so we may assume that $\text{depth}_R(M) > 0$. There is an $M$-regular element $x \in m$. The exact sequence $0 \to M \to M \to M/xM \to 0$ induces the following long exact sequence:

$$
\cdots \to f^{t-2}_a(M) \xrightarrow{x} f^{t-2}_a(M) \xrightarrow{f} f^{t-2}_a\left(\frac{M}{xM}\right) \\
\to f^{t-1}_a(M) \xrightarrow{x} f^{t-1}_a(M) \xrightarrow{g} f^{t-1}_a\left(\frac{M}{xM}\right) \\
\to f^t_a(M) \xrightarrow{x} f^t_a(M) \xrightarrow{h} \cdots. \text{(*)}
$$

Using the exact sequence (⋆) we obtain the short exact sequence

$$0 \to \frac{f^{t-1}_a(M)}{xf^{t-1}_a(M)} \to f^{t-1}_a\left(\frac{M}{xM}\right) \to (0 : x) \to 0.$$

Now, this exact sequence induces the following long exact sequence:

$$0 \to \text{Hom}_R\left(\frac{R}{m}, \frac{f^{t-1}_a(M)}{xf^{t-1}_a(M)}\right) \to \text{Hom}_R\left(\frac{R}{m}, f^{t-1}_a\left(\frac{M}{xM}\right)\right) \to \text{Hom}_R\left(\frac{R}{m}, (0 : x)\right) \to \text{Ext}^1_R\left(\frac{R}{m}, \frac{f^{t-1}_a(M)}{xf^{t-1}_a(M)}\right) \to \cdots. \text{(**)}$$

By using (⋆), $f^i_a\left(\frac{M}{xM}\right) \in S$ for all $i < t - 1$. Therefore by the induction hypothesis $\text{Hom}_R\left(\frac{R}{m}, f^{t-1}_a\left(\frac{M}{xM}\right)\right) \in S$. Furthermore $\text{Ext}^1_R\left(\frac{R}{m}, \frac{f^{t-1}_a(M)}{xf^{t-1}_a(M)}\right) \in S$ because $f^{t-1}_a(M) \in S$. Thus in accordance with (⋆⋆), $\text{Hom}_R\left(\frac{R}{m}, (0 : x)\right) \in S$. Since $x \in m$ according to [9,10.86] we have the following isomorphisms.

$$\text{Hom}_R\left(\frac{R}{m}, (0 : x)\right) \cong \text{Hom}_R\left(\frac{R}{m}, \text{Hom}_R\left(\frac{R}{xR}, f^t_a(M)\right)\right) \cong$$

$$\text{Hom}_R\left(\frac{R}{m} \otimes_R \frac{R}{xR}, f^t_a(M)\right) \cong \text{Hom}_R\left(\frac{R}{m}, f^t_a(M)\right).$$

Consequently $\text{Hom}_R\left(\frac{R}{m}, f^t_a(M)\right) \in S$. 

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3. The formal cohomological dimension in a Serre subcategory

We recall from [3,Theorem 1.1] that for a finitely generated $R$-module $M$, $\operatorname{sup}\{i \in \mathbb{N}_0 | f^i_a(M) \neq 0\} = \dim \left( \frac{M}{aM} \right)$.

**Definition 3.1.** The formal cohomological dimension of $M$ with respect to $a$ in $S$ is the supremum of the integers $i$ such that $f^i_a(M) \notin S$ and is denoted by $f.c.d_S(a, M)$.

**Theorem 3.2.** Suppose that $S$ is a Serre subcategory of the category of $R$-modules and $R$-homomorphisms and $L$ and $N$ are two finitely generated $R$-modules such that $\operatorname{Supp}_R(L) \subseteq \operatorname{Supp}_R(N)$. Then $f.c.d_S(a, L) \leq f.c.d_S(a, N)$.

**Proof.** It is enough to prove that $f^i_a(L) \in S$ for all $i > f.c.d_S(a, N)$ and all finitely generated $R$-module $L$ such that $\operatorname{Supp}_R(L) \subseteq \operatorname{Supp}_R(N)$. We use descending induction on $i$. For all $i > \dim \left( \frac{L}{aL} \right) + f.c.d_S(a, N)$, $f^i_a(L) = 0 \in S$. Let $i > f.c.d_S(a, N)$ and the relation is proved for $i + 1$. By Gruson’s theorem, there is a chain $0 = L_0 \subset L_1 \subset \cdots \subset L_i = L$ of submodules of $L$ such that $\frac{L}{L_{i-1}}$ is a homomorphic image of a direct sum of finitely many copies of $N$. Consider the exact sequence $0 \to L_{i-1} \to L_i \to \frac{L}{L_{i-1}} \to 0$ $(i = 0, 1, \ldots, l)$. We may assume that $l = 1$. The exact sequence $0 \to K \to \bigoplus_{j=1}^t N \to L \to 0$ where $K$ is a finitely generated $R$-module induces the following long exact sequence:

$$\cdots \to f^{i}_a\left(\bigoplus_{j=1}^t N\right) \to f^{i}_a(L) \to f^{i+1}_a(K) \to \cdots.$$ 

Based on the induction hypothesis $f^{i+1}_a(K) \in S$. Moreover $f^i_a\left(\bigoplus_{j=1}^t N\right) = \bigoplus_{j=1}^t f^i_a(N) \in S$ for all $i > f.c.d_S(a, N)$. Hence it follows from the exact sequence (*) that $f^i_a(L) \in S$.

The next example shows that even if $\operatorname{Supp}_R(M) = \operatorname{Supp}_R(N)$, then it may not true that $f.\operatorname{grade}_S(a, M) = f.\operatorname{grade}_S(a, N)$.

**Example 3.3.** (See [4, Example 4.3 (i)]) Let $(R, m)$ be a 2 dimensional complete regular local ring, $S = 0$ and $a$ be an ideal of $R$ with $\dim \left( \frac{R}{a} \right) = 1$. Then by using [5,Theorem 1.1], $f.\operatorname{grade}_S(a, R) = 1$ and $f.\operatorname{grade}_S\left( a, \frac{R}{m} \right) = 0$. Set $M := R \oplus \frac{R}{m}$. Then $\operatorname{Supp}_R(M) = \operatorname{Supp}_R(R)$. But $f.\operatorname{grade}_S(a, M) = \inf \left\{ f.\operatorname{grade}_S(a, R), f.\operatorname{grade}_S\left( a, \frac{R}{m} \right) \right\} = 0$.

**Corollary 3.4.** For all $x \in a \cdot f.c.d_S(a, M) \geq f.c.d_S\left( a, \frac{M}{xM} \right)$.

**Corollary 3.5.** Suppose that $0 \to L \to M \to N \to 0$ is an exact sequence of finitely generated $R$-modules. Then $f.c.d_S(a, M) = \max \{ f.c.d_S(a, L), f.c.d_S(a, N) \}$. 

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Proof. Since $\text{Supp}_R(M) = \text{Supp}_R(L) \cup \text{Supp}_R(N)$ by referring to Theorem 3.2 we deduce that $f \cdot \text{cd}_S(\mathfrak{a}, M) \geq f \cdot \text{cd}_S(\mathfrak{a}, L)$ and $f \cdot \text{cd}_S(\mathfrak{a}, M) \geq f \cdot \text{cd}_S(\mathfrak{a}, N)$. Therefore $f \cdot \text{cd}_S(\mathfrak{a}, M) \geq \max \{f \cdot \text{cd}_S(\mathfrak{a}, L), f \cdot \text{cd}_S(\mathfrak{a}, N)\}$.

Next we prove that $\max \{f \cdot \text{cd}_S(\mathfrak{a}, L), f \cdot \text{cd}_S(\mathfrak{a}, N)\} \geq f \cdot \text{cd}_S(\mathfrak{a}, M)$.

Let $i > \max \{f \cdot \text{cd}_S(\mathfrak{a}, L), f \cdot \text{cd}_S(\mathfrak{a}, N)\}$. Then $f_{i}^{\mathfrak{a}}(N), f_{i}^{\mathfrak{a}}(L) \in S$ and from the exact sequence $f_{i}^{\mathfrak{a}}(L) \to f_{i}^{\mathfrak{a}}(M) \to f_{i}^{\mathfrak{a}}(N)$ we conclude that $f_{i}^{\mathfrak{a}}(M) \in S$. Thus, $\max \{f \cdot \text{cd}_S(\mathfrak{a}, L), f \cdot \text{cd}_S(\mathfrak{a}, N)\} \geq f \cdot \text{cd}_S(\mathfrak{a}, M)$.

We recall that the cohomological dimension of an $R$-module $M$ with respect to an ideal $\mathfrak{a}$ of $R$ in $S$ is defined as

$$\text{cd}_S(\mathfrak{a}, M) := \sup \{i \in \mathbb{N} \mid H^i_{\mathfrak{a}}(M) \notin S\}.$$ 

The following lemma shows that when we considering the Artinianness of $f_{i}^{\mathfrak{a}}(M)$, we can assume that $M$ is $\mathfrak{a}$-torsion-free.

Lemma 3.6. Suppose that $\mathfrak{a}$ is an ideal of a local ring $(R, \mathfrak{m})$ and $t$ be a non-negative integer. If $H_{\mathfrak{m}}^i(M) \in S$ for all $i \geq t$, then the following are equivalent:

(a) $f_{i}^{\mathfrak{a}}(M) \in S$ for all $i \geq t$.
(b) $f_{i}^{\mathfrak{a}} \left(\frac{M}{\Gamma_{\mathfrak{m}}(M)}\right) \in S$ for all $i \geq t$.

Proof. According to the hypothesis $t > \text{cd}_S(\mathfrak{m}, M)$. On the other hand $\text{Supp}_R(\Gamma_{\mathfrak{a}}(M)) \subseteq \text{Supp}_R(M)$. So by referring to [7, Theorem 3.5], $\text{cd}_S(\mathfrak{m}, \Gamma_{\mathfrak{a}}(M)) \leq \text{cd}_S(\mathfrak{m}, M)$. Thus, $t > \text{cd}_S(\mathfrak{m}, \Gamma_{\mathfrak{a}}(M))$ and $H_{\mathfrak{m}}^i(\Gamma_{\mathfrak{a}}(M)) \in S$ for all $i \geq t$. Now, consider the following long exact sequence:

$$\cdots \to f_{i}^{\mathfrak{a}}(\Gamma_{\mathfrak{a}}(M)) \to f_{i}^{\mathfrak{a}}(M) \to f_{i}^{\mathfrak{a}} \left(\frac{M}{\Gamma_{\mathfrak{a}}(M)}\right) \to f_{i}^{\mathfrak{a}+1}(\Gamma_{\mathfrak{a}}(M)) \to \cdots \tag{*}$$

According to [4, Lemma 2.3] $f_{i}^{\mathfrak{a}}(\Gamma_{\mathfrak{a}}(M)) \equiv H_{\mathfrak{m}}^i(\Gamma_{\mathfrak{a}}(M))$. By using the hypothesis $f_{i}^{\mathfrak{a}}(\Gamma_{\mathfrak{a}}(M)) \in S$ for all $i \geq t$. So it follows from the exact sequence (*) that $f_{i}^{\mathfrak{a}}(M) \in S$ if and only if $f_{i}^{\mathfrak{a}} \left(\frac{M}{\Gamma_{\mathfrak{a}}(M)}\right) \in S$ for all $i \geq t$.

Theorem 3.7. Let $(R, \mathfrak{m})$ be a local ring and $M \in S$ be a finitely generated $R$-module of dimension $d$ such that $\text{cd}_S(\mathfrak{m}, M) \leq f \cdot \text{cd}_S(\mathfrak{a}, M)$. Then $\frac{f_{i}^{\mathfrak{a}}(M)}{a_{f_{i}^{\mathfrak{a}}}(M)} \in S$ where $t = f \cdot \text{cd}_S(\mathfrak{a}, M)$.

Proof. We use induction on $d = \dim(M)$. If $d = 0$, then $\dim \left(\frac{M}{a_{f_{i}^{\mathfrak{a}}}(M)}\right) = 0$. Accordingly to [3, Theorem 1.1], $f_{i}^{\mathfrak{a}}(M) = 0$ for all $i > 0$. 

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Moreover \( f_a^0(M) \cong M \in S \). By definition \( H_i^a(M) \in S \) for all \( i > t \). Therefore from the above lemma we can assume that \( M \) is \( a \)-torsion-free and there is an \( M \)-regular element \( x \in a \). Consider the long exact sequence:
\[
\cdots \to f_a^t(M) \xrightarrow{x} f_a^t(M) \xrightarrow{f_a^t} f_a^t\left( \frac{M}{xM} \right) \xrightarrow{g} f_a^{t+1}(M) \xrightarrow{h} \cdots. (*)
\]

By using the hypothesis \( f_a^i(M) \in S \) for all \( i > t \) (because \( t = f.cd_S(a,M) \)). So using the above long exact sequence \( f_a^i\left( \frac{M}{xM} \right) \in S \) for all \( i > t \). By induction hypothesis, \( \frac{f_a^t(M)}{af_a^t(M)} \in S \) because \( \dim\left( \frac{M}{xM} \right) = \dim(M) - 1 \).

Afterwards from the exact sequence \((*)\) we get the following short exact sequence.
\[
0 \to \text{Im}(f) \to f_a^t\left( \frac{M}{xM} \right) \to \text{Im}(g) \to 0
\]
So we obtain the following long exact sequence.
\[
\cdots \to \text{Tor}^R_1\left( \frac{R}{a}, \text{Im}(g) \right) \to \frac{\text{Im}(f)}{a\text{Im}(f)} \to \frac{f_a^t\left( \frac{M}{xM} \right)}{af_a^t(M)} \to \frac{\text{Im}(g)}{a\text{Im}(g)} \to 0.
\]

Since \( f_a^t(M) \in S \) and \( \text{Im}(g) \) is a submodule of \( f_a^{t+1}(M) \), we deduce that \( \text{Tor}^R_1\left( \frac{R}{a}, \text{Im}(g) \right) \in S \). On the other hand, \( \frac{f_a^t\left( \frac{M}{xM} \right)}{af_a^t(M)} \in S \). Therefore, \( \frac{\text{Im}(f)}{a\text{Im}(f)} \in S \) by the above long exact sequence.

Now, consider the following long exact sequence.
\[
\frac{f_a^t(M)}{af_a^t(M)} \xrightarrow{x} \frac{f_a^t(M)}{af_a^t(M)} \xrightarrow{I} \frac{\text{Im}(f)}{a\text{Im}(f)} \to 0.
\]
So, \( \frac{f_a^t(M)}{af_a^t(M)} \cong \frac{\text{Im}(f)}{a\text{Im}(f)} \) because \( x \in a \). Consequently, \( \frac{f_a^t(M)}{af_a^t(M)} \in S \).

**Proposition 3.8.** For a finitely generated \( R \)-module \( M \),
\[
f.cd_S(a,M) = \max\{ f.cd_S(a,\frac{R}{P}) | P \in \text{Ass}_R(M) \}.
\]

**Proof.** Set \( N := \bigoplus_{P \in \text{Ass}_R(M) \setminus \{0\}} \frac{R}{P} \). Then \( \text{Supp}_R(M) = \text{Supp}_R(N) \). So, by Theorem 3.2 and Corollary 3.5, \( f.cd_S(a,M) = f.cd_S(a,N) = \max\{ f.cd_S(a,\frac{R}{P}) | P \in \text{Ass}_R(M) \} \).

**Proposition 3.9.** Assume that \( a \) is an ideal of the local ring \((R, m)\). Then \( \text{Hom}_R(\frac{R}{m}, f_a^0(M)) \in S \) if and only if \( \text{Hom}_R(\frac{R}{m}, a^a) \in S \).

**Proof.** It is enough to consider the following isomorphisms
\[
\text{Hom}_R\left( \frac{R}{m}, f_a^0(M) \right) \cong \text{lim}_{n \in \mathbb{N}} \text{Hom}_R\left( \frac{R}{m}, H^0_m\left( \frac{M}{a^nM} \right) \right) \cong \text{lim}_{n \in \mathbb{N}} \text{Hom}_R\left( \frac{R}{m}, a^nM \right) \cong \text{Hom}_R\left( \frac{R}{m}, \hat{M}^a \right).
\]
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