Formal Local Cohomology Modules and Serre Subcategories

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Abstract

Let $(R, m)$ be a Noetherian local ring, $a$ an ideal of $R$ and $M$ a finitely generated $R$-module. We investigate some properties of formal local cohomology modules with respect to a Serre subcategory. We provide a common language to indicate some properties of formal local cohomology modules.

1. Introduction

Throughout this paper $(R, m)$ is a commutative Noetherian local ring, $a$ an ideal of $R$ and $M$ is a finitely generated $R$-module. For an integer $i \in \mathbb{N}_0$, $H^i_a(N)$ denotes the $i$-th local cohomology module of $M$ with respect to $a$ as introduced by Grothendieck (cf. [1], [2]).

We shall consider the family of local cohomology modules $\{H^i_m\left(\frac{M}{a^iM}\right)_{i\in \mathbb{N}}\}$ for a non-negative integer $i \in \mathbb{N}_0$. With natural homomorphisms; this family forms an inverse system. Schenzel introduced the $i$-th formal local cohomology of $M$ with respect to $a$ in the form of $f^i_a(M) := \varprojlim_{n\in \mathbb{N}} H^i_m\left(\frac{M}{a^iM}\right)$, which is the $i$-th cohomology module of the $a$-adic completion of the Čech complex $\check{c}_x \otimes_R M$, where $x$ denotes a system of elements of $R$ such that $\text{Rad}(x, R) = m$ (see [3, Definition 3.1]). He defines the formal grade as $f.\text{grade}(a, M) = \inf \{i \in \mathbb{N}_0 | f^i_a(M) \neq 0\}$. For any ideal $a$ of $R$ and finitely generated $R$-module $M$ the following statements hold:

(i) (See [3, Theorem 3.11]). If $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is a short exact sequence of finitely generated $R$-modules, then there is the following long exact sequence:

$$\cdots \rightarrow f^i_a(M') \rightarrow f^i_a(M) \rightarrow f^i_a(M'') \rightarrow \cdots$$

**Keywords:** Local cohomology, Formal local cohomology, Serre subcategory, Formal grade, Formal cohomological dimension.


Received: 26 Feb 2012 Revised 18 Dec 2013

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(ii) (See [3, Theorem 1.3]). \( f.\ grade(\ a, M) \leq \dim(M) - cd(\ a, M) \); some properties of formal local cohomology have been presented in [3, 4, 5 and 6].

Throughout this paper \( S \) denotes a Serre subcategory of the category of \( R \)-modules and \( R \)-homomorphisms (we recall that a class \( S \) of \( R \)-modules is a Serre subcategory of the category of \( R \)-modules and \( R \)-homomorphisms if \( S \) is closed under taking submodules, quotients and extensions).

Our paper contains three sections. In Section 2, we shall define the formal grade of \( a \) with respect to \( M \) in \( S \) as the infimum of the integers \( i \) such that \( f_a^i(M) \notin S \) and is denoted by \( f.\ grade_S(\ a, M) \). (See definition 2.1). Then we shall obtain some properties of this notion. We show that if \( \Gamma_a(M) \) is a pure submodule of \( M \), then \( \Hom_R(\frac{R}{m}, f_a^t(\Gamma_a(M))) \) and \( \Hom_R(\frac{R}{m}, f_a^{t-1}(\frac{M}{f_a(M)})) \) belong to \( S \), where \( t = f.\ grade_S(\ a, M) \).

In Section 3, we shall define the formal cohomological dimension of \( a \) with respect to \( M \) in \( S \) as the supremum of the integers \( i \) such that \( f_a^i(M) \notin S \) and is denoted by \( f.\ cd_S(\ a, M) \). (See definition 3.1). The main result of this section is that if \( f_a^i(M) \in S \) and \( H_a^i(M) \in S \) for all \( i > t \), then \( \frac{R}{a} \otimes_R f_a^t(M) \) belongs to \( S \).

## 2. The formal grade of a module in a Serre subcategory

**Definition 2.1.** The formal grade of \( a \) with respect to \( M \) in \( S \) is the infimum of the integers \( i \) such that \( f_a^i(M) \notin S \) and is denoted by \( f.\ grade_S(\ a, M) \).

**Proposition 2.2.** Let \( (R, m) \) be a local ring and \( a \) be an ideal of \( R \). If \( 0 \to L \to M \to N \to 0 \) is an exact sequence of finitely generated \( R \)-modules, then the following statements hold.

\[
\begin{align*}
(a) & \quad f.\ grade_S(\ a, M) \geq \min\{f.\ grade_S(\ a, L), f.\ grade_S(\ a, N)\}.
(b) & \quad f.\ grade_S(\ a, L) \geq \min\{f.\ grade_S(\ a, M), f.\ grade_S(\ a, N) + 1\}.
(c) & \quad f.\ grade_S(\ a, N) \geq \min\{f.\ grade_S(\ a, L) - 1, f.\ grade_S(\ a, M)\}.
\end{align*}
\]

**Proof.** According to [3, Theorem 3.11], the above short exact sequence induces the following long exact sequence.

\[
\ldots \to f_a^{i-1}(N) \to f_a^i(L) \to f_a^i(M) \to f_a^i(N) \to f_a^{i+1}(L) \to \ldots
\]

So, the result follows.
Corollary 2.3. If $\underline{x} = x_1, \ldots, x_n$ is a regular $M$-sequence, then $f.\text{grade}_S \left( \frac{a}{\sum_{i=1}^{n} x_i M} \right) \geq f.\text{grade}_S (a, M) - n$.

Proof. Consider the following exact sequence $(n \in \mathbb{N})$

$$0 \to \frac{M}{(x_1, \ldots, x_{n-1})M} \xrightarrow{x_n} \frac{M}{(x_1, \ldots, x_n)M} \xrightarrow{\text{nat.}} \frac{M}{(x_1, \ldots, x_n)M} \to 0$$

whenever $n = 1$ by $(x_1, \ldots, x_{n-1})M$ we means $0$.

Corollary 2.4. Let $a$ and $b$ be ideals of $R$. Then

(a) $f.\text{grade}_S (a \cap b, M) \geq \min \{ f.\text{grade}_S (a, M), f.\text{grade}_S (b, M), f.\text{grade}_S ((a, b), M) + 1 \}.$

(b) $f.\text{grade}_S ((a, b), M) \geq \min \{ f.\text{grade}_S (a \cap b, M) - 1, f.\text{grade}_S (a, M), f.\text{grade}_S (b, M) \}.$

Proof. For all $n \in \mathbb{N}$ there is a short exact sequence as follows:

$$0 \to \frac{M}{a^n M \cap b^n M} \xrightarrow{\text{nat.}} \frac{M}{a^n M \oplus b^n M} \xrightarrow{\text{nat.}} \frac{M}{a^n b^n M} \to 0.$$

By using [3, Theorem 5.1], the above exact sequence induces the following long exact sequence.

$$\ldots \to \lim_{\to n \in \mathbb{N}} \text{H}^i_m \left( \frac{M}{(a \cap b)^n M} \right) \to \lim_{\to n \in \mathbb{N}} \text{H}^i_m \left( \frac{M}{a^n M} \right) \oplus \lim_{\to n \in \mathbb{N}} \text{H}^i_m \left( \frac{M}{b^n M} \right) \to \lim_{\to n \in \mathbb{N}} \text{H}^i_m \left( \frac{M}{(a \cap b)^n M} \right) \to \ldots.$$

So by using an argument similar to that of Proposition 2.2, the result follows.

Corollary 2.5. Assume that $M$ is a finitely generated $R$-module and $N_1$ and $N_2$ are submodules of $M$. Then considering the exact sequence $0 \to \frac{M}{N_1 \cap N_2} \xrightarrow{\text{nat.}} \frac{M}{N_1} \oplus \frac{M}{N_2} \to \frac{M}{N_1 + N_2} \to 0$ we shall have

(a) $f.\text{grade}_S \left( \frac{M}{N_1 \cap N_2} , \frac{M}{N_1} , \frac{M}{N_2} \right) \geq \min \{ f.\text{grade}_S (a, M), f.\text{grade}_S (a, M) , MN2 , f.\text{grade}_S (a, MN1+ N2 + 1) \}.$

(b) $f.\text{grade}_S \left( \frac{M}{N_1 \cap N_2} , \frac{M}{N_1} , \frac{M}{N_2} \right) \geq \min \{ f.\text{grade}_S (a, M) - 1, f.\text{grade}_S (a, MN1) , f.\text{grade}_S (a, MN2) \}.$

Theorem 2.6. Let $a$ be an ideal of a local ring $(R, m)$, $M$ be a finitely generated $R$-module and $L$ be a pure submodule of $M$. Then $f.\text{grade}_S (a, L) \geq f.\text{grade}_S (a, M)$ where $S$ is a Serre subcategory of the category of $R$-modules and $R$-homomorphisms. In particular, $\inf \{ \text{inf} \{ \text{H}^i_m (L) \notin S \} \geq \inf \{ \text{inf} \{ \text{H}^i_m (M) \notin S \} \}.$
Proof. Let $L$ be a pure submodule of $M$. So $\frac{L}{a^n L} \to \frac{M}{a^n M}$ is pure for each $n \in \mathbb{N}$. Now according to [8, Corollary 3.2 (a)], $H^i_m \left( \frac{L}{a^n L} \right) \to H^i_m \left( \frac{M}{a^n M} \right)$ is injective. Since inverse limit is a left exact functor, $f^i_a(L)$ is isomorphic to a submodule of $f^i_a(M)$. Consequently, $f.\text{grade}_S(a, L) \geq f.\text{grade}_S(a, M)$. If $a = 0$ then, $f.\text{grade}_S(0, M) = \inf \{ l | H^i_m(M) \notin S \}$ and the result follows.

Corollary 2.7. If $0 \to L \to M \to N \to 0$ is a pure exact sequence of finitely generated $R$-modules, then $\min \{ f.\text{grade}_S(a, L), f.\text{grade}_S(a, N) + 1 \} \geq f.\text{grade}_S(a, M)$.

Proof. Since $L$ is a pure submodule of $M$, as a result of the previous theorem, $f.\text{grade}_S(a, L) \geq f.\text{grade}_S(a, M)$. Hence we must prove that $f.\text{grade}_S(a, N) + 1 \geq f.\text{grade}_S(a, M)$. We assume that $i < f.\text{grade}_S(a, M)$ and we show that $i < f.\text{grade}_S(a, N) + 1$. Consider the following long exact sequence.

$$\cdots \to f^i_a(N) \to f^{i-1}_a(L) \to f^i_a(M) \to f^{i}_a(N) \to \cdots \quad \text{(**)}$$

If $i < f.\text{grade}_S(a, M)$, then $f^0_a(M), f^1_a(M), \ldots, f^{i-1}_a(M), f^i_a(M) \in S$. On the other hand, since $i < f.\text{grade}_S(a, M) \leq f.\text{grade}_S(a, L), f^0_a(L), \ldots, f^i_a(L) \in S$. Hence, it follows from (**), $f^0_a(N), \ldots, f^{i-1}_a(N) \in S$ and so $i - 1 < f.\text{grade}_S(a, N)$.

Theorem 2.8. Let $(R, m)$ be a local ring, $a$ be an ideal of $R$, $S$ be a Serre subcategory of the category of $R$-modules and $R$-homomorphisms and $M \in S$ be a finitely generated $R$-module such that $\Gamma_a(M)$ is a pure submodule of $M$. Then $\text{Hom}_R \left( \frac{R}{a}, f^i_a(\Gamma_a(M)) \right) \in S$, where $t = f.\text{grade}_S(a, M)$.

Proof. Due to the previous theorem, $f.\text{grade}_S(a, \Gamma_a(M)) \geq f.\text{grade}_S(a, M)$. If $f.\text{grade}_S(a, \Gamma_a(M)) > f.\text{grade}_S(a, M)$, then the result is obvious. Accordingly, we assume that $f.\text{grade}_S(a, \Gamma_a(M)) = f.\text{grade}_S(a, M)$. We know that $\text{Supp}(\Gamma_a(M)) \subseteq \text{Var}(a)$. By using [4, Lemma 2.3], $f^i_a(\Gamma_a(M)) \cong H^i_m(\Gamma_a(M))$ for all $i \geq 0$. So, if $i < f.\text{grade}_S(a, M)$, then $f^i_a(\Gamma_a(M)) \cong H^i_m(\Gamma_a(M)) \in S$ and $\text{Ext}_R^k \left( \frac{R}{m}, f^i_a(\Gamma_a(M)) \right) \in S$ for all $k \geq 0$ and $j < f.\text{grade}_S(a, M)$. Moreover $\text{Ext}_R^i \left( \frac{R}{m}, \Gamma_a(M) \right) \in S$, because $\Gamma_a(M) \in S$. Consequently, according to [7, Theorem 2.2],

$$\text{Hom}_R \left( \frac{R}{m}, H^i_m(\Gamma_a(M)) \right) \in S,$$

where $t = f.\text{grade}_S(a, M)$.

Corollary 2.9 With the same notations as Theorem 2.8, let $X \in S$ be a submodule of $f^i_a(\Gamma_a(M))$, where $t = f.\text{grade}_S(a, M)$. Then $\text{Hom}_R \left( \frac{R}{m}, \frac{f^i_a(\Gamma_a(M))}{X} \right) \in S$.

Proof. Consider the long exact sequence:
In accordance with the previous theorem $\text{Hom}_R\left(\frac{R}{m}, f_\alpha^t(\Gamma_a(M))\right) \rightarrow \text{Hom}_R\left(\frac{R}{m}, \frac{f_\alpha^t(\Gamma_a(M))}{x}\right) \rightarrow \text{Ext}_R^1\left(\frac{R}{m}, X\right). (*)$

**Theorem 2.10.** Suppose that $\mathfrak{a}$ is an ideal of $(R, m)$ and $M \in S$ is a finitely generated $R$-module such that $\Gamma_a(M)$ is a pure submodule of $M$. Then $\text{Hom}_R\left(\frac{R}{m}, f_\alpha^{t-1}\left(\frac{M}{\Gamma_a(M)}\right)\right) \in S$, where $t = \text{f.grade}_S(\mathfrak{a}, M)$.

**Proof.** One has $f.\text{grade}_S(\mathfrak{a}, \Gamma_a(M)) \geq f.\text{grade}_S(\mathfrak{a}, M)$, by Theorem 2.6. Now, the exact sequence $0 \rightarrow \Gamma_a(M) \rightarrow M \rightarrow \frac{M}{\Gamma_a(M)} \rightarrow 0$ induces the following long exact sequence:

$$\cdots \rightarrow f_\alpha^{t-1}(\Gamma_a(M)) \rightarrow f_\alpha^{t-1}(M) \rightarrow \frac{M}{\Gamma_a(M)} \rightarrow \cdots. (*)$$

Using the exact sequence $(*)$, we obtain the short exact sequence $0 \rightarrow \text{Im}(\beta) \rightarrow f_\alpha^{t-1}(M) \rightarrow \text{Im}(\gamma) \rightarrow 0$. Since $f_\alpha^{t-1}(M) \in S$, $\text{Im}(\beta) \in S$ and $\text{Im}(\gamma) \in S$. Furthermore, we have the exact sequence $0 \rightarrow \text{Im}(\xi) \rightarrow H^t_m(\Gamma_a(M)) \rightarrow \text{Im}(\varphi) \rightarrow 0$ which induces the following long exact sequence:

$$0 \rightarrow \text{Hom}_R\left(\frac{R}{m}, \text{Im}(\xi)\right) \rightarrow \text{Hom}_R\left(\frac{R}{m}, H^t_m(\Gamma_a(M))\right) \rightarrow \cdots.$$ 

Thus $\text{Hom}_R\left(\frac{R}{m}, \text{Im}(\xi)\right) \in S$. Finally, by considering the short exact sequence $0 \rightarrow \text{Im}(\gamma) \rightarrow f_\alpha^{t-1}\left(\frac{M}{\Gamma_a(M)}\right) \rightarrow \text{Im}(\xi) \rightarrow 0$ we can conclude that $\text{Hom}_R\left(\frac{R}{m}, f_\alpha^{t-1}\left(\frac{M}{\Gamma_a(M)}\right)\right) \in S$.

**Theorem 2.11.** Suppose that $R$ is complete with respect to the $\mathfrak{a}$–adic topology and $M \in S$ be a finitely generated $R$-module and $t$ a positive integer such that $f_\alpha^i(M) \in S$ for all $i < t$. Then $\text{Hom}_R\left(\frac{R}{m}, f_\alpha^t(M)\right) \in S$.

**Proof.** We use induction on $t$. Let $t=0$. Consider the following isomorphisms.

$$\text{Hom}_R\left(\frac{R}{m}, f_\alpha^0(M)\right) \cong \text{lim}_{\rightarrow} \text{Hom}_R\left(\frac{R}{m}, H^0_x\left(\frac{M}{a^iM}\right)\right) \cong \text{lim}_{\rightarrow} \text{Hom}_R\left(\frac{R}{m}, \frac{M}{a^iM}\right) \cong \text{Hom}_R\left(\frac{R}{m}, \text{Im}\right)$$
It is clear that $\text{Hom}_R\left(\frac{R}{m}, M\right) \in S$. So by the above isomorphisms, we deduce that

$\text{Hom}_R\left(\frac{R}{m}, f_{a}^{i}(M)\right) \in S$.

Suppose that $t > 0$ and the result is true for all integer $i$ less than $t$. Set $N := \cap_{m}(M)$.

Then $f_{a}^{i}(M) \cong f_{a}^{i}\left(\frac{M}{N}\right)$ for all $i > 0$, and so we may assume that $\text{depth}_R(M) > 0$. There is an $M -$regular element $x \in m$. The exact sequence $0 \to M \to M \to \frac{M}{xM} \to 0$ induces the following long exact sequence:

$$\ldots \to f_{a}^{t-2}(M) \xrightarrow{x} f_{a}^{t-2}(M) \xrightarrow{f} f_{a}^{t-2}\left(\frac{M}{xM}\right) \to f_{a}^{t-1}(M) \xrightarrow{x} f_{a}^{t-1}(M) \xrightarrow{g} f_{a}^{t-1}\left(\frac{M}{xM}\right) \to f_{a}^{t}(M) \xrightarrow{x} f_{a}^{t}(M) \xrightarrow{h} \ldots.$$

Using the exact sequence $(\ast)$ we obtain the short exact sequence

$$0 \to f_{a}^{t-1}(M) \to f_{a}^{t-1}\left(\frac{M}{xM}\right) \to \left(0 : x \right) \to 0.$$

Now, this exact sequence induces the following long exact sequence:

$$0 \to \text{Hom}_R\left(\frac{R}{m}, f_{a}^{t-1}(M)\right) \to \text{Hom}_R\left(\frac{R}{m}, f_{a}^{t-1}\left(\frac{M}{xM}\right)\right) \to \text{Hom}_R\left(\frac{R}{m}, (0 : x)\right) \to \text{Ext}_R^{1}\left(\frac{R}{m}, f_{a}^{t-1}\left(\frac{M}{xM}\right)\right) \to \ldots. (\ast\ast)$$

By using $(\ast)$, $f_{a}^{i}\left(\frac{M}{xM}\right) \in S$ for all $i < t - 1$. Therefore by the induction hypothesis $\text{Hom}_R\left(\frac{R}{m}, f_{a}^{t-1}\left(\frac{M}{xM}\right)\right) \in S$. Furthermore $\text{Ext}_R^{1}\left(\frac{R}{m}, f_{a}^{t-1}\left(\frac{M}{xM}\right)\right) \in S$ because $f_{a}^{t-1}(M) \in S$. Thus in accordance with $(\ast\ast)$, $\text{Hom}_R\left(\frac{R}{m}, (0 : x)\right) \in S$. Since $x \in m$ according to [9,10.86] we have the following isomorphisms.

$$\text{Hom}_R\left(\frac{R}{m}, (0 : x)\right) \cong \text{Hom}_R\left(\frac{R}{xR}, \text{Hom}_R\left(\frac{R}{xR}, f_{a}^{i}(M)\right)\right) \cong \text{Hom}_R\left(\frac{R}{xR} \otimes_R \frac{R}{xR}, f_{a}^{i}(M)\right) \cong \text{Hom}_R\left(\frac{R}{m}, f_{a}^{i}(M)\right).$$

Consequently $\text{Hom}_R\left(\frac{R}{m}, f_{a}^{i}(M)\right) \in S$. 

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3. The formal cohomological dimension in a Serre subcategory

We recall from [3,Theorem 1.1] that for a finitely generated \( R \)-module \( M \),
\[
\sup \{ i \in \mathbb{N} \mid f^i_a(M) \neq 0 \} = \dim \left( \frac{M}{aM} \right).
\]

**Definition 3.1.** The formal cohomological dimension of \( M \) with respect to \( a \) in \( S \) is the supremum of the integers \( i \) such that \( f^i_a(M) \notin S \) and is denoted by \( f \cdot \text{cd}_S(a, M) \).

**Theorem 3.2.** Suppose that \( S \) is a Serre subcategory of the category of \( R \)-modules and \( R \)-homomorphisms and \( L \) and \( N \) are two finitely generated \( R \)-modules such that \( \text{Supp}_R(L) \subseteq \text{Supp}_R(N) \). Then \( f \cdot \text{cd}_S(a, L) \leq f \cdot \text{cd}_S(a, N) \).

**Proof.** It is enough to prove that \( f^i_a(L) \in S \) for all \( i > f \cdot \text{cd}_S(a, N) \) and all finitely generated \( R \)-module \( L \) such that \( \text{Supp}_R(L) \subseteq \text{Supp}_R(N) \). We use descending induction on \( i \). For all \( i > \dim \left( \frac{L}{aL} \right) + f \cdot \text{cd}_S(a, N) \), \( f^i_a(L) = 0 \in S \). Let \( i > f \cdot \text{cd}_S(a, N) \) and the result is proved for \( i + 1 \). By Gruson’s theorem, there is a chain \( 0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_l = L \) of submodules of \( L \) such that \( \frac{L_i}{L_{i-1}} \) is a homomorphic image of a direct sum of finitely many copies of \( N \). Consider the exact sequence \( 0 \to L_{i-1} \to L_i \to \frac{L_i}{L_{i-1}} \to 0 \) \((i = 0, l, \ldots, l)\). We may assume that \( l = l \). The exact sequence \( 0 \to K \to \bigoplus_{j=1}^l N \to L \to 0 \) where \( K \) is a finitely generated \( R \)-module induces the following long exact sequence:

\[
\cdots \to f^i_a\left( \bigoplus_{j=1}^l N \right) \to f^i_a(L) \to f^{i+1}_a(K) \to \cdots \tag{*}
\]

Based on the induction hypothesis \( f^{i+1}_a(K) \in S \). Moreover \( f^i_a\left( \bigoplus_{j=1}^l N \right) = \bigoplus_{j=1}^l f^i_a(N) \in S \) for all \( i > f \cdot \text{cd}_S(a, N) \). Hence it follows from the exact sequence \((*)\) that \( f^i_a(L) \in S \).

The next example shows that even if \( \text{Supp}_R(M) = \text{Supp}_R(N) \), then it may not true that \( f \cdot \text{grade}_S(a, M) = f \cdot \text{grade}_S(a, N) \).

Example 3.3. (See [4, Example 4.3 (i)]) Let \( (R, m) \) be a 2 dimensional complete regular local ring, \( S = 0 \) and \( a \) be an ideal of \( R \) with \( \dim \left( \frac{R}{a} \right) = 1 \). Then by using [5,Theorem 1.1], \( f \cdot \text{grade}_S(a, R) = 1 \) and \( f \cdot \text{grade}_S\left( a, \frac{R}{m} \right) = 0 \). Set \( M := R + \frac{R}{m} \).

Then \( \text{Supp}_R(M) = \text{Supp}_R(R) \). But

\[
f \cdot \text{grade}_S(a, M) = \inf \left\{ f \cdot \text{grade}_S(a, R), f \cdot \text{grade}_S\left( a, \frac{R}{m} \right) \right\} = 0.
\]

**Corollary 3.4.** For all \( x \in a \cdot f \cdot \text{cd}_S(a, M) \geq f \cdot \text{cd}_S\left( a, \frac{M}{xM} \right) \).

**Corollary 3.5.** Suppose that \( 0 \to L \to M \to N \to 0 \) is an exact sequence of finitely generated \( R \)-modules. Then \( f \cdot \text{cd}_S(a, M) = \max \{ f \cdot \text{cd}_S(a, L), f \cdot \text{cd}_S(a, N) \} \).
Proof. Since $\text{Supp}_R(M) = \text{Supp}_R(L) \cup \text{Supp}_R(N)$ by referring to Theorem 3.2 we deduce that $f \cdot \text{cd}_S(a, M) \geq f \cdot \text{cd}_S(a, L)$ and $f \cdot \text{cd}_S(a, M) \geq f \cdot \text{cd}_S(a, N)$. Therefore $f \cdot \text{cd}_S(a, M) \geq \max \{f \cdot \text{cd}_S(a, L), f \cdot \text{cd}_S(a, N)\}$. Next we prove that $\max \{f \cdot \text{cd}_S(a, L), f \cdot \text{cd}_S(a, N)\} \geq f \cdot \text{cd}_S(a, M)$.

Let $i > \max \{f \cdot \text{cd}_S(a, L), f \cdot \text{cd}_S(a, N)\}$. Then $f^i_a(N), f^i_a(L) \in S$ and from the exact sequence $f^i_a(L) \rightarrow f^i_a(M) \rightarrow f^i_a(N)$ we conclude that $f^i_a(M) \in S$. Thus, $\max \{f \cdot \text{cd}_S(a, L), f \cdot \text{cd}_S(a, N)\} \geq f \cdot \text{cd}_S(a, M)$.

We recall that the cohomological dimension of an $R$-module $M$ with respect to an ideal $a$ of $R$ in $S$ is defined as:

$$\text{cd}_S(a, M) := \sup \{i \in \mathbb{N}_0 | H^i_a(M) \notin S\}.$$ 

The following lemma shows that when we considering the Artinianness of $f^i_a(M)$, we can assume that $M$ is $a$-torsion-free.

**Lemma 3.6.** Suppose that $a$ is an ideal of a local ring $(R, m)$ and $t$ be a non-negative integer. If $H^i_m(M) \in S$ for all $i \geq t$, then the following are equivalent:

(a) $f^i_a(M) \in S$ for all $i \geq t$.

(b) $f^i_a\left(\frac{M}{\Gamma_a(M)}\right) \in S$ for all $i \geq t$.

**Proof.** According to the hypothesis $t > \text{cd}_S(m, M)$. On the other hand $\text{Supp}_R(\Gamma_a(M)) \subseteq \text{Supp}_R(M)$. So by referring to [7, Theorem 3.5], $\text{cd}_S(m, \Gamma_a(M)) \leq \text{cd}_S(m, M)$. Thus, $t > \text{cd}_S(m, \Gamma_a(M))$ and $H^i_m(\Gamma_a(M)) \in S$ for all $i \geq t$. Now, consider the following long exact sequence:

$$\cdots \rightarrow f^i_a(\Gamma_a(M)) \rightarrow f^i_a(M) \rightarrow f^i_a\left(\frac{M}{\Gamma_a(M)}\right) \rightarrow f^{i+1}_a(\Gamma_a(M)) \rightarrow \cdots \ (*)$$

According to [4, Lemma 2.3] $f^i_a(\Gamma_a(M)) \cong H^i_m(\Gamma_a(M))$. By using the hypothesis $f^i_a(\Gamma_a(M)) \in S$ for all $i \geq t$. So it follows from the exact sequence $(*)$ that $f^i_a(M) \in S$ if and only if $f^i_a\left(\frac{M}{\Gamma_a(M)}\right) \in S$ for all $i \geq t$.

**Theorem 3.7.** Let $(R, m)$ be a local ring and $M \in S$ be a finitely generated $R$-module of dimension $d$ such that $\text{cd}_S(m, M) \leq f \cdot \text{cd}_S(a, M)$. Then $\frac{f^d_a(M)}{a^{f^d_a(M)}} \in S$ where $t = f \cdot \text{cd}_S(a, M)$.

**Proof.** We use induction on $d = \dim (M)$. If $d = 0$, then $\dim \left(\frac{M}{aM}\right) = 0$. Accordingly to [3, Theorem 1.1], $f^i_a(M) = 0$ for all $i > 0$. 344
Moreover $f_a^i(M) \cong M \in \mathcal{S}$. By definition $H^i_{\mathfrak{m}}(M) \in \mathcal{S}$ for all $i > t$. Therefore from the above lemma we can assume that $M$ is $\mathfrak{a}$-torsion-free and there is an $M$-regular element $x \in \mathfrak{a}$. Consider the long exact sequence:

$$
\cdots \rightarrow f_a^i(M) \xrightarrow{x} f_a^i(M) \xrightarrow{f} f_a^i\left(\frac{M}{xM}\right) \xrightarrow{g} f_a^{i+1}(M) \xrightarrow{h} \cdots. (*)
$$

By using the hypothesis $f_a^i(M) \in \mathcal{S}$ for all $i > t$ (because $t = f.cds(\mathfrak{a}, M)$). So using the above long exact sequence $f_a^i\left(\frac{M}{xM}\right) \in \mathcal{S}$ for all $i > t$. By induction hypothesis, $\frac{f_a^i\left(M\right)}{af_a^i\left(M\right)} \in \mathcal{S}$ because $\dim\left(\frac{M}{xM}\right) = \dim(M) - 1$.

Afterwards from the exact sequence $(*)$ we get the following short exact sequence:

$$
0 \rightarrow \text{Im}(f) \rightarrow f_a^i\left(\frac{M}{xM}\right) \rightarrow \text{Im}(g) \rightarrow 0.
$$

So we obtain the following long exact sequence.

$$
\cdots \rightarrow \text{Tor}_1^{\mathcal{R}}\left(\frac{R}{\mathfrak{a}}, \text{Im}(g)\right) \rightarrow \frac{\text{Im}(f)}{a\text{Im}(f)} \rightarrow f_a^i\left(\frac{M}{xM}\right) \rightarrow \frac{\text{Im}(g)}{a\text{Im}(g)} \rightarrow 0.
$$

Since $f_a^i(M) \in \mathcal{S}$ and $\text{Im}(g)$ is a submodule of $f_a^{i+1}(M)$, we deduce that $\text{Tor}_1^{\mathcal{R}}\left(\frac{R}{\mathfrak{a}}, \text{Im}(g)\right) \in \mathcal{S}$. On the other hand, $\frac{f_a^i\left(M\right)}{af_a^i\left(M\right)} \in \mathcal{S}$. Therefore, $\frac{\text{Im}(f)}{a\text{Im}(f)} \in \mathcal{S}$ by the above long exact sequence.

Now, consider the following long exact sequence.

$$
\frac{f_a^i(M)}{af_a^i(M)} \xrightarrow{x} \frac{f_a^i(M)}{af_a^i(M)} \xrightarrow{y} \frac{\text{Im}(f)}{a\text{Im}(f)} \rightarrow 0.
$$

So, $\frac{f_a^i(M)}{af_a^i(M)} \cong \frac{\text{Im}(f)}{a\text{Im}(f)}$ because $x \in \mathfrak{a}$. Consequently, $\frac{f_a^i(M)}{af_a^i(M)} \in \mathcal{S}$.

**Proposition 3.8.** For a finitely generated $R$-module $M$,

$$f.cds(\mathfrak{a}, M) = \max\{f.cds\left(\mathfrak{a}, \frac{R}{P}\right) | P \in \text{Ass}_R(M)\}.$$

**Proof.** Set $N := \bigoplus_{P \in \text{Ass}_R(M)} \frac{R}{P}$. Then $\text{Supp}_R(M) = \text{Supp}_R(N)$. So, by Theorem 3.2 and Corollary 3.5, $f.cds(\mathfrak{a}, M) = f.cds(\mathfrak{a}, N) = \max\{f.cds\left(\mathfrak{a}, \frac{R}{P}\right) | P \in \text{Ass}_R(M)\}$.

**Proposition 3.9.** Assume that $\mathfrak{a}$ is an ideal of the local ring $(R, \mathfrak{m})$. Then $\text{Hom}_R\left(\frac{R}{\mathfrak{m}}, f_a^0(M)\right) \in \mathcal{S}$ if and only if $\text{Hom}_R\left(\frac{R}{\mathfrak{m}}, \mathfrak{m}\mathfrak{a}\right) \in \mathcal{S}$.

**Proof.** It is enough to consider the following isomorphisms

$$
\text{Hom}_R\left(\frac{R}{\mathfrak{m}}, f_a^0(M)\right) \cong \lim_{\mathfrak{n} \in \mathcal{N}} \text{Hom}_R\left(\frac{R}{\mathfrak{m}}, H^0_{\mathfrak{a}^\mathfrak{n}} M\right) \cong \lim_{\mathfrak{n} \in \mathcal{N}} \text{Hom}_R\left(\frac{R}{\mathfrak{m}}, \mathfrak{a}^\mathfrak{n} M\right) \cong \text{Hom}_R\left(\frac{R}{\mathfrak{m}}, \mathfrak{m}\mathfrak{a}\right).
$$
Acknowledgements

The authors would like to thank the referees for their helpful comments.

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