

# Formal Local Cohomology Modules and Serre Subcategories

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## Abstract

Let  $(R, \mathfrak{m})$  be a Noetherian local ring,  $\mathfrak{a}$  an ideal of  $R$  and  $M$  a finitely generated  $R$ -module. We investigate some properties of formal local cohomology modules with respect to a Serre subcategory. We provide a common language to indicate some properties of formal local cohomology modules.

## 1. Introduction

Throughout this paper  $(R, \mathfrak{m})$  is a commutative Noetherian local ring,  $\mathfrak{a}$  an ideal of  $R$  and  $M$  is a finitely generated  $R$ -module. For an integer  $i \in \mathbb{N}_0$ ,  $H_{\mathfrak{a}}^i(N)$  denotes the  $i$ -th local cohomology module of  $M$  with respect to  $\mathfrak{a}$  as introduced by Grothendieck (cf. [1], [2]).

We shall consider the family of local cohomology modules  $\{H_{\mathfrak{m}}^i\left(\frac{M}{\mathfrak{a}^n M}\right)\}_{n \in \mathbb{N}}$  for a non-negative integer  $i \in \mathbb{N}_0$ . With natural homomorphisms; this family forms an inverse system. Schenzel introduced the  $i$ -th formal local cohomology of  $M$  with respect to  $\mathfrak{a}$  in the form of  $f_{\mathfrak{a}}^i(M) := \varprojlim_{n \in \mathbb{N}} H_{\mathfrak{m}}^i\left(\frac{M}{\mathfrak{a}^n M}\right)$ , which is the  $i$ -th cohomology module of the  $\mathfrak{a}$ -adic completion of the Čech complex  $\check{c}_{\underline{x}} \otimes_R M$ , where  $\underline{x}$  denotes a system of elements of  $R$  such that  $\text{Rad}(\underline{x}, R) = \mathfrak{m}$  (see [3, Definition 3.1]). He defines the formal grade as  $f.\text{grade}(\mathfrak{a}, M) = \inf \{i \in \mathbb{N}_0 \mid f_{\mathfrak{a}}^i(M) \neq 0\}$ . For any ideal  $\mathfrak{a}$  of  $R$  and finitely generated  $R$ -module  $M$  the following statements hold:

(i) (See [3, Theorem 3.11]). If  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  is a short exact sequence of finitely generated  $R$ -modules, then there is the following long exact sequence:

$$\cdots \rightarrow f_{\mathfrak{a}}^i(M') \rightarrow f_{\mathfrak{a}}^i(M) \rightarrow f_{\mathfrak{a}}^i(M'') \rightarrow \cdots.$$

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(ii) (See [3, Theorem 1.3]).  $f.\text{grade}(\mathfrak{a}, M) \leq \dim(M) - cd(\mathfrak{a}, M)$ ; some properties of formal local cohomology have been presented in [3, 4, 5 and 6].

Throughout this paper  $\mathcal{S}$  denotes a Serre subcategory of the category of  $R$ -modules and  $R$ -homomorphisms (we recall that a class  $\mathcal{S}$  of  $R$ -modules is a Serre subcategory of the category of  $R$ -modules and  $R$ -homomorphisms if  $\mathcal{S}$  is closed under taking submodules, quotients and extensions).

Our paper contains three sections. In Section 2, we shall define the formal grade of  $\mathfrak{a}$  with respect to  $M$  in  $\mathcal{S}$  as the infimum of the integers  $i$  such that  $f_a^i(M) \notin \mathcal{S}$  and is denoted by  $f.\text{grade}_{\mathcal{S}}(\mathfrak{a}, M)$ . (See definition 2.1). Then we shall obtain some properties of this notion. We show that if  $\Gamma_{\mathfrak{a}}(M)$  is a pure submodule of  $M$ , then  $\text{Hom}_R(\frac{R}{\mathfrak{m}}, f_a^t(\Gamma_{\mathfrak{a}}(M)))$  and  $\text{Hom}_R(\frac{R}{\mathfrak{m}}, f_a^{t-1}(\frac{M}{\Gamma_{\mathfrak{a}}(M)}))$  belong to  $\mathcal{S}$ , where  $t = f.\text{grade}_{\mathcal{S}}(\mathfrak{a}, M)$ .

In Section 3, we shall define the formal cohomological dimension of  $\mathfrak{a}$  with respect to  $M$  in  $\mathcal{S}$  as the supremum of the integers  $i$  such that  $f_a^i(M) \notin \mathcal{S}$  and is denoted by  $f.cd_{\mathcal{S}}(\mathfrak{a}, M)$ . (See definition 3.1). The main result of this section is that if  $f_a^i(M) \in \mathcal{S}$  and  $H_m^i(M) \in \mathcal{S}$  for all  $i > t$ , then  $\frac{R}{\mathfrak{a}} \otimes_R f_a^t(M)$  belongs to  $\mathcal{S}$ .

## 2. The formal grade of a module in a Serre subcategory

**Definition 2.1.** The formal grade of  $\mathfrak{a}$  with respect to  $M$  in  $\mathcal{S}$  is the infimum of the integers  $i$  such that  $f_a^i(M) \notin \mathcal{S}$  and is denoted by  $f.\text{grade}_{\mathcal{S}}(\mathfrak{a}, M)$ .

**Proposition 2.2.** Let  $(R, \mathfrak{m})$  be a local ring and  $\mathfrak{a}$  be an ideal of  $R$ . If  $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$  is an exact sequence of finitely generated  $R$ -modules, then the following statements hold.

- (a)  $f.\text{grade}_{\mathcal{S}}(\mathfrak{a}, M) \geq \min\{f.\text{grade}_{\mathcal{S}}(\mathfrak{a}, L), f.\text{grade}_{\mathcal{S}}(\mathfrak{a}, N)\}.$
- (b)  $f.\text{grade}_{\mathcal{S}}(\mathfrak{a}, L) \geq \min\{f.\text{grade}_{\mathcal{S}}(\mathfrak{a}, M), f.\text{grade}_{\mathcal{S}}(\mathfrak{a}, N) + 1\}.$
- (c)  $f.\text{grade}_{\mathcal{S}}(\mathfrak{a}, N) \geq \min\{f.\text{grade}_{\mathcal{S}}(\mathfrak{a}, L) - 1, f.\text{grade}_{\mathcal{S}}(\mathfrak{a}, M)\}.$

**Proof.** According to [3, Theorem 3.11], the above short exact sequence induces the following long exact sequence.

$$\cdots \rightarrow f_a^{i-1}(N) \rightarrow f_a^i(L) \rightarrow f_a^i(M) \rightarrow f_a^i(N) \rightarrow f_a^{i+1}(L) \rightarrow \cdots.$$

So, the result follows.

**Corollary 2.3.** If  $\underline{x} = x_1, \dots, x_n$  is a regular  $M$ -sequence, then  $f.\text{grade}_{\mathcal{S}} \left( \mathbf{a}, \frac{M}{\underline{x}M} \right) \geq f.\text{grade}_{\mathcal{S}} (\mathbf{a}, M) - n$ .

**Proof.** Consider the following exact sequence ( $n \in \mathbb{N}$ )

$$0 \rightarrow \frac{M}{(x_1, \dots, x_{n-1})M} \xrightarrow{x_n} \frac{M}{(x_1, \dots, x_{n-1})M} \xrightarrow{\text{nat.}} \frac{M}{(x_1, \dots, x_n)M} \rightarrow 0$$

whenever  $n = 1$  by  $(x_1, \dots, x_{n-1})M$  we means 0.

**Corollary 2.4.** Let  $\mathbf{a}$  and  $\mathbf{b}$  be ideals of  $R$ . Then

- (a)  $f.\text{grade}_{\mathcal{S}} (\mathbf{a} \cap \mathbf{b}, M) \geq \min\{f.\text{grade}_{\mathcal{S}} (\mathbf{a}, M), f.\text{grade}_{\mathcal{S}} (\mathbf{b}, M), f.\text{grade}_{\mathcal{S}} ((\mathbf{a}, \mathbf{b}), M) + 1\}$ .
- (b)  $f.\text{grade}_{\mathcal{S}} ((\mathbf{a}, \mathbf{b}), M) \geq \min\{f.\text{grade}_{\mathcal{S}} (\mathbf{a} \cap \mathbf{b}, M) - 1, f.\text{grade}_{\mathcal{S}} (\mathbf{a}, M), f.\text{grade}_{\mathcal{S}} (\mathbf{b}, M)\}$ .

**Proof.** For all  $n \in \mathbb{N}$  there is a short exact sequence as follows:

$$0 \rightarrow \frac{M}{\mathbf{a}^n M \cap \mathbf{b}^n M} \rightarrow \frac{M}{\mathbf{a}^n M} \oplus \frac{M}{\mathbf{b}^n M} \rightarrow \frac{M}{(\mathbf{a}^n, \mathbf{b}^n)M} \rightarrow 0.$$

By using [3, Theorem 5.1], the above exact sequence induces the following long exact sequence.

$$\dots \rightarrow \lim_{n \in \mathbb{N}} H_{\mathbf{m}}^i \left( \frac{M}{(\mathbf{a} \cap \mathbf{b})^n M} \right) \rightarrow \lim_{n \in \mathbb{N}} H_{\mathbf{m}}^i \left( \frac{M}{\mathbf{a}^n M} \right) \oplus \lim_{n \in \mathbb{N}} H_{\mathbf{m}}^i \left( \frac{M}{\mathbf{b}^n M} \right) \rightarrow \lim_{n \in \mathbb{N}} H_{\mathbf{m}}^i \left( \frac{M}{(\mathbf{a}, \mathbf{b})^n M} \right) \rightarrow \dots$$

So by using an argument similar to that of Proposition 2.2, the result follows.

**Corollary 2.5.** Assume that  $M$  is a finitely generated  $R$ -module and  $N_1$  and  $N_2$  are submodules of  $M$ . Then considering the exact sequence  $0 \rightarrow \frac{M}{N_1 \cap N_2} \rightarrow$

$$\frac{M}{N_1} \oplus \frac{M}{N_2} \rightarrow \frac{M}{N_1 + N_2} \rightarrow 0$$

- (a)  $f.\text{grade}_{\mathcal{S}} \left( \mathbf{a}, \frac{M}{N_1 \cap N_2} \right) \geq \min\{f.\text{grade}_{\mathcal{S}} \left( \mathbf{a}, \frac{M}{N_1} \right), f.\text{grade}_{\mathcal{S}} \left( \mathbf{a}, \frac{M}{N_2} \right), f.\text{grade}_{\mathcal{S}} \mathbf{a}, MN_1 + N_2 + 1\}$ .
- (b)  $f.\text{grade}_{\mathcal{S}} \left( \mathbf{a}, \frac{M}{N_1 + N_2} \right) \geq \min\left\{f.\text{grade}_{\mathcal{S}} \left( \frac{M}{N_1 \cap N_2} \right) - 1, f.\text{grade}_{\mathcal{S}} \left( \mathbf{a}, \frac{M}{N_1} \right), f.\text{grade}_{\mathcal{S}} \mathbf{a}, MN_2\right\}$ .

**Theorem 2.6.** Let  $\mathbf{a}$  be an ideal of a local ring  $(R, \mathbf{m})$ ,  $M$  be a finitely generated  $R$ -module and  $L$  be a pure submodule of  $M$ . Then  $f.\text{grade}_{\mathcal{S}} (\mathbf{a}, L) \geq f.\text{grade}_{\mathcal{S}} (\mathbf{a}, M)$  where  $\mathcal{S}$  is a Serre subcategory of the category of  $R$ -modules and  $R$ -homomorphisms. In particular,  $\inf \{i | H_{\mathbf{m}}^i(L) \notin \mathcal{S}\} \geq \inf \{i | H_{\mathbf{m}}^i(M) \notin \mathcal{S}\}$ .

**Proof.** Let  $L$  be a pure submodule of  $M$ . So  $\frac{L}{a^n L} \rightarrow \frac{M}{a^n M}$  is pure for each  $n \in \mathbb{N}$ . Now according to [8, Corollary 3.2 (a)] ,  $H_m^i\left(\frac{L}{a^n L}\right) \rightarrow H_m^i\left(\frac{M}{a^n M}\right)$  is injective. Since inverse limit is a left exact functor,  $f_a^i(L)$  is isomorphic to a submodule of  $f_a^i(M)$ . Consequently  $f.\text{grade}_S(\mathfrak{a}, L) \geq f.\text{grade}_S(\mathfrak{a}, M)$ . If  $\mathfrak{a} = 0$  then,  $f.\text{grade}_S(0, M) = \inf \{i | H_m^i(M) \notin \mathcal{S}\}$  and the result follows.

**Corollary 2.7.** If  $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$  is a pure exact sequence of finitely generated  $R$ -modules, then  $\min \{f.\text{grade}_S(\mathfrak{a}, L), f.\text{grade}_S(\mathfrak{a}, N) + 1\} \geq f.\text{grade}_S(\mathfrak{a}, M)$ .

**Proof.** Since  $L$  is a pure submodules of  $M$ , as a result of the previous theorem,  $f.\text{grade}_S(\mathfrak{a}, L) \geq f.\text{grade}_S(\mathfrak{a}, M)$ . Hence we must prove that  $f.\text{grade}_S(\mathfrak{a}, N) + 1 \geq f.\text{grade}_S(\mathfrak{a}, M)$ . We assume that  $i < f.\text{grade}_S(\mathfrak{a}, M)$  and we show that  $i < f.\text{grade}_S(\mathfrak{a}, N) + 1$ . Consider the following long exact sequence.

$$\cdots \rightarrow f_a^{i-1}(M) \rightarrow f_a^{i-1}(N) \rightarrow f_a^i(L) \rightarrow f_a^i(M) \rightarrow f_a^i(N) \rightarrow \cdots (**)$$

If  $i < f.\text{grade}_S(\mathfrak{a}, M)$ , then  $f_a^0(M), f_a^1(M), \dots, f_a^{i-1}(M), f_a^i(M) \in \mathcal{S}$ . On the other hand, since  $i < f.\text{grade}_S(\mathfrak{a}, M) \leq f.\text{grade}_S(\mathfrak{a}, L)$ ,  $f_a^0(L), \dots, f_a^i(L) \in \mathcal{S}$ . Hence, it follows from (\*\*) that  $f_a^0(N), \dots, f_a^{i-1}(N) \in \mathcal{S}$  and so  $i - 1 < f.\text{grade}_S(\mathfrak{a}, N)$ .

**Theorem 2.8.** Let  $(R, \mathfrak{m})$  be a local ring,  $\mathfrak{a}$  be an ideal of  $R$ ,  $\mathcal{S}$  be a Serre subcategory of the category of  $R$ -modules and  $R$ -homomorphisms and  $M \in \mathcal{S}$  be a finitely generated  $R$ -module such that  $\Gamma_{\mathfrak{a}}(M)$  is a pure submodule of  $M$ . Then  $\text{Hom}_R\left(\frac{R}{\mathfrak{a}}, f_a^t(\Gamma_{\mathfrak{a}}(M))\right) \in \mathcal{S}$ , where  $t = f.\text{grade}_S(\mathfrak{a}, M)$ .

**Proof.** Due to the previous theorem,  $f.\text{grade}_S(\mathfrak{a}, \Gamma_{\mathfrak{a}}(M)) \geq f.\text{grade}_S(\mathfrak{a}, M)$ . If  $f.\text{grade}_S(\mathfrak{a}, \Gamma_{\mathfrak{a}}(M)) > f.\text{grade}_S(\mathfrak{a}, M)$ , then the result is obvious. Accordingly, we assume that  $f.\text{grade}_S(\mathfrak{a}, \Gamma_{\mathfrak{a}}(M)) = f.\text{grade}_S(\mathfrak{a}, M)$ . We know that  $\text{Supp}(\Gamma_{\mathfrak{a}}(M)) \subseteq \text{Var}(\mathfrak{a})$ . By using [4, Lemma 2.3],  $f_a^i(\Gamma_{\mathfrak{a}}(M)) \cong H_m^i(\Gamma_{\mathfrak{a}}(M))$  for all  $i \geq 0$ . So, if  $j < f.\text{grade}_S(\mathfrak{a}, M)$ , then  $f_a^j(\Gamma_{\mathfrak{a}}(M)) \cong H_m^j(\Gamma_{\mathfrak{a}}(M)) \in \mathcal{S}$  and  $\text{Ext}_R^k\left(\frac{R}{\mathfrak{m}}, H_m^j(\Gamma_{\mathfrak{a}}(M))\right) \in \mathcal{S}$  for all  $k \geq 0$  and  $j < f.\text{grade}_S(\mathfrak{a}, M)$ . Moreover  $\text{Ext}_R^t\left(\frac{R}{\mathfrak{m}}, \Gamma_{\mathfrak{a}}(M)\right) \in \mathcal{S}$ , because  $\Gamma_{\mathfrak{a}}(M) \in \mathcal{S}$ . Consequently, according to [7, Theorem 2.2],

$$\text{Hom}_R\left(\frac{R}{\mathfrak{m}}, H_m^t(\Gamma_{\mathfrak{a}}(M))\right) \in \mathcal{S}, \text{ where } t = f.\text{grade}_S(\mathfrak{a}, M).$$

**Corollary 2.9** With the same notations as Theorem 2.8, let  $X \in \mathcal{S}$  be a submodule of  $f_a^t(\Gamma_{\mathfrak{a}}(M))$ , where  $t = f.\text{grade}_S(\mathfrak{a}, M)$ . Then  $\text{Hom}_R\left(\frac{R}{\mathfrak{m}}, \frac{f_a^t(\Gamma_{\mathfrak{a}}(M))}{X}\right) \in \mathcal{S}$ .

**Proof.** Consider the long exact sequence:

$$Hom_R\left(\frac{R}{\underline{m}}, f_a^t(\Gamma_a(M))\right) \rightarrow Hom_R\left(\frac{R}{\underline{m}}, \frac{f_a^t(\Gamma_a(M))}{X}\right) \rightarrow Ext_R^1\left(\frac{R}{\underline{m}}, X\right). (*)$$

In accordance with the previous theorem  $Hom_R\left(\frac{R}{\underline{m}}, f_a^t(\Gamma_a(M))\right) \in \mathcal{S}$ . Moreover  $Ext_R^1\left(\frac{R}{\underline{m}}, X\right) \in \mathcal{S}$ . It follows from the exact sequence (\*) that  $Hom_R\left(\frac{R}{\underline{m}}, \frac{f_a^t(\Gamma_a(M))}{X}\right) \in \mathcal{S}$ .

**Theorem 2.10.** Suppose that  $\mathfrak{a}$  is an ideal of  $(R, \underline{m})$  and  $M \in \mathcal{S}$  is a finitely generated  $R$ -module such that  $\Gamma_{\mathfrak{a}}(M)$  is a pure submodule of  $M$ . Then  $Hom_R\left(\frac{R}{\underline{m}}, f_a^{t-1}\left(\frac{M}{\Gamma_a(M)}\right)\right) \in \mathcal{S}$ , where  $t = f.\text{grade}_{\mathcal{S}}(\mathfrak{a}, M)$ .

**Proof.** One has  $f.\text{grade}_{\mathcal{S}}(\mathfrak{a}, \Gamma_a(M)) \geq f.\text{grade}_{\mathcal{S}}(\mathfrak{a}, M)$ , by Theorem 2.6. Now, the exact sequence  $0 \rightarrow \Gamma_a(M) \rightarrow M \rightarrow \frac{M}{\Gamma_a(M)} \rightarrow 0$  induces the following long exact sequence:

$$\dots \xrightarrow{\alpha} f_a^{t-1}(\Gamma_a(M)) \xrightarrow{\beta} f_a^{t-1}(M) \xrightarrow{\gamma} f_a^{t-1}\left(\frac{M}{\Gamma_a(M)}\right) \xrightarrow{\xi} f_a^t(\Gamma_a(M)) \xrightarrow{\varphi} \dots. (*)$$

Using the exact sequence (\*), we obtain the short exact sequence  $0 \rightarrow \text{Im}(\beta) \rightarrow f_a^{t-1}(M) \rightarrow \text{Im}(\gamma) \rightarrow 0$ . Since  $f_a^{t-1}(M) \in \mathcal{S}$ ,  $\text{Im}(\beta) \in \mathcal{S}$  and  $\text{Im}(\gamma) \in \mathcal{S}$ . Furthermore, we have the exact sequence  $0 \rightarrow \text{Im}(\xi) \rightarrow H_m^t(\Gamma_a(M)) \rightarrow \text{Im}(\varphi) \rightarrow 0$  which induces the following long exact sequence:

$$0 \rightarrow Hom_R\left(\frac{R}{\underline{m}}, \text{Im}(\xi)\right) \rightarrow Hom_R\left(\frac{R}{\underline{m}}, H_m^t(\Gamma_a(M))\right) \rightarrow \dots.$$

Thus  $Hom_R\left(\frac{R}{\underline{m}}, \text{Im}(\xi)\right) \in \mathcal{S}$ . Finally, by considering the short exact sequence  $0 \rightarrow \text{Im}(\gamma) \rightarrow f_a^{t-1}\left(\frac{M}{\Gamma_a(M)}\right) \rightarrow \text{Im}(\xi) \rightarrow 0$  we can conclude that  $Hom_R\left(\frac{R}{\underline{m}}, f_a^{t-1}\left(\frac{M}{\Gamma_a(M)}\right)\right) \in \mathcal{S}$ .

**Theorem 2.11.** Suppose that  $R$  is complete with respect to the  $\mathfrak{a}$ -adic topology and  $M \in \mathcal{S}$  be a finitely generated  $R$ -module and  $t$  a positive integer such that  $f_a^i(M) \in \mathcal{S}$  for all  $i < t$ . Then  $Hom_R\left(\frac{R}{\underline{m}}, f_a^t(M)\right) \in \mathcal{S}$ .

**Proof.** We use induction on  $t$ . Let  $t=0$ . Consider the following isomorphisms.

$$\begin{aligned} Hom_R\left(\frac{R}{\underline{m}}, f_a^0(M)\right) &\cong \lim_{\leftarrow n \in \mathbb{N}} Hom_R\left(\frac{R}{\underline{m}}, H_m^0\left(\frac{M}{\underline{a}^n M}\right)\right) \cong \lim_{\leftarrow n \in \mathbb{N}} Hom_R\left(\frac{R}{\underline{m}}, \frac{M}{\underline{a}^n M}\right) \\ &\cong Hom_R\left(\frac{R}{\underline{m}}, \lim_{\leftarrow n \in \mathbb{N}} \frac{M}{\underline{a}^n M}\right) \cong Hom_R\left(\frac{R}{\underline{m}}, \hat{M}^{\mathfrak{a}}\right) \cong Hom_R\left(\frac{R}{\underline{m}}, M\right) \end{aligned}$$

It is clear that  $Hom_R(\frac{R}{\underline{m}}, M) \in \mathcal{S}$ . So by the above isomorphisms, we deduce that

$$Hom_R(\frac{R}{\underline{m}}, f_a^0(M)) \in \mathcal{S}.$$

Suppose that  $t > 0$  and the result is true for all integer  $i$  less than  $t$ . Set  $N := \Gamma_{\underline{m}}(M)$ . Then  $f_a^i(M) \cong f_a^i(\frac{M}{N})$  for all  $i > 0$ , and so we may assume that  $depth_R(M) > 0$ . There is an  $M$ -regular element  $x \in \underline{m}$ . The exact sequence  $0 \rightarrow M \xrightarrow{x} M \rightarrow \frac{M}{xM} \rightarrow 0$  induces the following long exact sequence:

$$\begin{aligned} \cdots \rightarrow f_a^{t-2}(M) &\xrightarrow{x} f_a^{t-2}(\frac{M}{xM}) \xrightarrow{f} f_a^{t-2}(\frac{M}{xM}) \\ &\rightarrow f_a^{t-1}(M) \xrightarrow{x} f_a^{t-1}(\frac{M}{xM}) \xrightarrow{g} f_a^{t-1}(\frac{M}{xM}) \\ &\rightarrow f_a^t(M) \xrightarrow{x} f_a^t(M) \xrightarrow{h} \cdots. \quad (*) \end{aligned}$$

Using the exact sequence  $(*)$  we obtain the short exact sequence

$$0 \rightarrow \frac{f_a^{t-1}(M)}{xf_a^{t-1}(M)} \rightarrow f_a^{t-1}(\frac{M}{xM}) \rightarrow (0 : x)_{f_a^t(M)} \rightarrow 0.$$

Now, this exact sequence induces the following long exact sequence:

$$\begin{aligned} 0 \rightarrow Hom_R\left(\frac{R}{\underline{m}}, \frac{f_a^{t-1}(M)}{xf_a^{t-1}(M)}\right) &\rightarrow Hom_R\left(\frac{R}{\underline{m}}, f_a^{t-1}(\frac{M}{xM})\right) \rightarrow Hom_R\left(\frac{R}{\underline{m}}, (0 : x)_{f_a^t(M)}\right) \rightarrow \\ Ext_R^1\left(\frac{R}{\underline{m}}, \frac{f_a^{t-1}(M)}{xf_a^{t-1}(M)}\right) &\rightarrow \cdots. \quad (**) \end{aligned}$$

By using  $(*)$ ,  $f_a^i(\frac{M}{xM}) \in \mathcal{S}$  for all  $i < t - 1$ . Therefore by the induction hypothesis  $Hom_R(\frac{R}{\underline{m}}, f_a^{t-1}(\frac{M}{xM})) \in \mathcal{S}$ . Furthermore  $Ext_R^1(\frac{R}{\underline{m}}, \frac{f_a^{t-1}(M)}{xf_a^{t-1}(M)}) \in \mathcal{S}$  because  $f_a^{t-1}(M) \in \mathcal{S}$ . Thus in accordance with  $(**)$ ,  $Hom_R(\frac{R}{\underline{m}}, (0 : x)_{f_a^t(M)}) \in \mathcal{S}$ . Since  $x \in \underline{m}$  according to [9,10.86] we have the following isomorphisms.

$$\begin{aligned} Hom_{f_a^t(M)}\left(\frac{R}{\underline{m}}, (0 : x)\right) &\cong Hom_R\left(\frac{R}{\underline{m}}, Hom_R\left(\frac{R}{xR}, f_a^t(M)\right)\right) \cong \\ Hom_R\left(\frac{R}{\underline{m}} \otimes_R \frac{R}{xR}, f_a^t(M)\right) &\cong Hom_R\left(\frac{R}{\underline{m}}, f_a^t(M)\right). \end{aligned}$$

Consequently  $Hom_R(\frac{R}{\underline{m}}, f_a^t(M)) \in \mathcal{S}$ .

### 3. The formal cohomological dimension in a Serre subcategory

We recall from [3, Theorem 1.1] that for a finitely generated  $R$ -module  $M$ ,  $\sup\{i \in \mathbb{N}_0 \mid f_a^i(M) \neq 0\} = \dim(\frac{M}{aM})$ .

**Definition 3.1.** The formal cohomological dimension of  $M$  with respect to  $\underline{a}$  in  $\mathcal{S}$  is The supremum of the integers  $i$  such that  $f_a^i(M) \notin \mathcal{S}$  and is denoted by  $f.cd_{\mathcal{S}}(\underline{a}, M)$ .

**Theorem 3.2.** Suppose that  $\mathcal{S}$  is a Serre subcategory of the category of  $R$ -modules and  $R$ -homomorphisms and  $L$  and  $N$  are two finitely generated  $R$ -modules such that  $Supp_R(L) \subseteq Supp_R(N)$ . Then  $f.cd_{\mathcal{S}}(\underline{a}, L) \leq f.cd_{\mathcal{S}}(\underline{a}, N)$ .

**Proof.** It is enough to prove that  $f_a^i(L) \in \mathcal{S}$  for all  $i > f.cd_{\mathcal{S}}(\underline{a}, N)$  and all finitely generated  $R$ -module  $L$  such that  $Supp_R(L) \subseteq Supp_R(N)$ . We use descending induction on  $i$ . For all  $i > \dim(\frac{L}{aL}) + f.cd_{\mathcal{S}}(\underline{a}, N)$ ,  $f_a^i(L) = 0 \in \mathcal{S}$ . Let  $i > f.cd_{\mathcal{S}}(\underline{a}, N)$  and the result is proved for  $i + 1$ . By Gruson's theorem, there is a chain  $0 = L_0 \subset L_1 \subset \dots \subset L_l = L$  of submodules of  $L$  such that  $\frac{L_i}{L_{i-1}}$  is a homomorphic image of a direct sum of finitely many copies of  $N$ . Consider the exact sequence  $0 \rightarrow L_{i-1} \rightarrow L_i \xrightarrow{\frac{L_i}{L_{i-1}}} \frac{L_i}{L_{i-1}} \rightarrow 0$  ( $i = 0, 1, \dots, l$ ). We may assume that  $l = 1$ . The exact sequence  $0 \rightarrow K \rightarrow \bigoplus_{j=1}^t N \rightarrow L \rightarrow 0$  where  $K$  is a finitely generated  $R$ -module induces the following long exact sequence:

$$\dots \rightarrow f_a^i(\bigoplus_{j=1}^t N) \rightarrow f_a^i(L) \rightarrow f_a^{i+1}(K) \rightarrow \dots. (*)$$

Based on the induction hypothesis  $f_a^{i+1}(K) \in \mathcal{S}$ . Moreover  $f_a^i(\bigoplus_{j=1}^t N) = \bigoplus_{j=1}^t f_a^i(N) \in \mathcal{S}$  for all  $i > f.cd_{\mathcal{S}}(\underline{a}, N)$ . Hence it follows from the exact sequence (\*) that  $f_a^i(L) \in \mathcal{S}$ .

The next example shows that even if  $Supp_R(M) = Supp_R(N)$ , then it may not true that  $f.grade_{\mathcal{S}}(\underline{a}, M) = f.grade_{\mathcal{S}}(\underline{a}, N)$ .

**Example 3.3.** (See [4, Example 4.3 (i)]) Let  $(R, \mathfrak{m})$  be a 2 dimensional complete regular local ring,  $\mathcal{S} = 0$  and  $\underline{a}$  be an ideal of  $R$  with  $\dim(\frac{R}{\underline{a}}) = 1$ . Then by using [5, Theorem 1.1],  $f.grade_{\mathcal{S}}(\underline{a}, R) = 1$  and  $f.grade_{\mathcal{S}}(\underline{a}, \frac{R}{\underline{a}}) = 0$ . Set  $M := R \oplus \frac{R}{\underline{a}}$ .

Then  $Supp_R(M) = Supp_R(R)$ . But

$$f.grade_{\mathcal{S}}(\underline{a}, M) = \inf\{f.grade_{\mathcal{S}}(\underline{a}, R), f.grade_{\mathcal{S}}(\underline{a}, \frac{R}{\underline{a}})\} = 0.$$

**Corollary 3.4.** For all  $x \in \underline{a}$ ,  $f.cd_{\mathcal{S}}(\underline{a}, M) \geq f.cd_{\mathcal{S}}(\underline{a}, \frac{M}{xM})$ .

**Corollary 3.5.** Suppose that  $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$  is an exact sequence of finitely generated  $R$ -modules. Then  $f.cd_{\mathcal{S}}(\underline{a}, M) = \max\{f.cd_{\mathcal{S}}(\underline{a}, L), f.cd_{\mathcal{S}}(\underline{a}, N)\}$ .

**Proof.** Since  $Supp_R(M) = Supp_R(L) \cup Supp_R(N)$  by referring to Theorem 3.2 we deduce that  $f.cd_S(\mathfrak{a}, M) \geq f.cd_S(\mathfrak{a}, L)$  and  $f.cd_S(\mathfrak{a}, M) \geq f.cd_S(\mathfrak{a}, N)$ . Therefore  $f.cd_S(\mathfrak{a}, M) \geq \max \{f.cd_S(\mathfrak{a}, L), f.cd_S(\mathfrak{a}, N)\}$ .

Next we prove that  $\max \{f.cd_S(\mathfrak{a}, L), f.cd_S(\mathfrak{a}, N)\} \geq f.cd_S(\mathfrak{a}, M)$ .

Let  $i > \max \{f.cd_S(\mathfrak{a}, L), f.cd_S(\mathfrak{a}, N)\}$ . Then  $f_a^i(N), f_a^i(L) \in \mathcal{S}$  and from the exact sequence  $f_a^i(L) \rightarrow f_a^i(M) \rightarrow f_a^i(N)$  we conclude that  $f_a^i(M) \in \mathcal{S}$ . Thus,  $\max \{f.cd_S(\mathfrak{a}, L), f.cd_S(\mathfrak{a}, N)\} \geq f.cd_S(\mathfrak{a}, M)$ .

We recall that the cohomological dimension of an  $R$ -module  $M$  with respect to an ideal  $\mathfrak{a}$  of  $R$  in  $\mathcal{S}$  is defined as

$$cd_S(\mathfrak{a}, M) := \sup \{i \in \mathbb{N}_0 | H_a^i(M) \notin \mathcal{S}\}.$$

The following lemma shows that when we considering the Artinianness of  $f_a^i(M)$ , we can assume that  $M$  is  $\mathfrak{a}$ -torsion-free.

**Lemma 3.6.** Suppose that  $\mathfrak{a}$  is an ideal of a local ring  $(R, \mathfrak{m})$  and  $t$  be a non-negative integer. If  $H_{\mathfrak{m}}^i(M) \in \mathcal{S}$  for all  $i \geq t$ , then the following are equivalent:

- (a)  $f_a^i(M) \in \mathcal{S}$  for all  $i \geq t$ .
- (b)  $f_a^i\left(\frac{M}{\Gamma_{\mathfrak{a}}(M)}\right) \in \mathcal{S}$  for all  $i \geq t$ .

**Proof.** According to the hypothesis  $t > cd_S(\mathfrak{m}, M)$ . On the other hand  $Supp_R(\Gamma_{\mathfrak{a}}(M)) \subseteq Supp_R(M)$ . So by referring to [7, Theorem 3.5],  $cd_S(\mathfrak{m}, \Gamma_{\mathfrak{a}}(M)) \leq cd_S(\mathfrak{m}, M)$ . Thus,  $t > cd_S(\mathfrak{m}, \Gamma_{\mathfrak{a}}(M))$  and  $H_{\mathfrak{m}}^i(\Gamma_{\mathfrak{a}}(M)) \in \mathcal{S}$  for all  $i \geq t$ . Now, consider the following long exact sequence:

$$\cdots \rightarrow f_a^i(\Gamma_{\mathfrak{a}}(M)) \rightarrow f_a^i(M) \rightarrow f_a^i\left(\frac{M}{\Gamma_{\mathfrak{a}}(M)}\right) \rightarrow f_a^{i+1}(\Gamma_{\mathfrak{a}}(M)) \rightarrow \cdots (*)$$

According to [4, Lemma 2.3]  $f_a^i(\Gamma_{\mathfrak{a}}(M)) \cong H_{\mathfrak{m}}^i(\Gamma_{\mathfrak{a}}(M))$ . By using the hypothesis  $f_a^i(\Gamma_{\mathfrak{a}}(M)) \in \mathcal{S}$  for all  $i \geq t$ . So it follows from the exact sequence (\*) that  $f_a^i(M) \in \mathcal{S}$  if and only if  $f_a^i\left(\frac{M}{\Gamma_{\mathfrak{a}}(M)}\right) \in \mathcal{S}$  for all  $i \geq t$ .

**Theorem 3.7.** Let  $(R, \mathfrak{m})$  be a local ring and  $M \in \mathcal{S}$  be a finitely generated  $R$ -module of dimension  $d$  such that  $cd_S(\mathfrak{m}, M) \leq f.cd_S(\mathfrak{a}, M)$ . Then  $\frac{f_a^t(M)}{af_a^t(M)} \in \mathcal{S}$  where  $t = f.cd_S(\mathfrak{a}, M)$ .

**Proof.** We use induction on  $d = \dim(M)$ . If  $d = 0$ , then  $\dim\left(\frac{M}{af_a^t(M)}\right) = 0$ . Accordingly to [3, Theorem 1.1],  $f_a^i(M) = 0$  for all  $i > 0$ .



Moreover  $f_a^0(M) \cong M \in \mathcal{S}$ . By definition  $H_m^i(M) \in \mathcal{S}$  for all  $i > t$ . Therefore from the above lemma we can assume that  $M$  is  $\mathbf{a}$ -torsion-free and there is an  $M$ -regular element  $x \in \mathbf{a}$ . Consider the long exact sequence :

$$\cdots \rightarrow f_a^i(M) \xrightarrow{x} f_a^i(M) \xrightarrow{f} f_a^i\left(\frac{M}{xM}\right) \xrightarrow{g} f_a^{i+1}(M) \xrightarrow{h} \cdots (*)$$

By using the hypothesis  $f_a^i(M) \in \mathcal{S}$  for all  $i > t$  (because  $t = f.cd_{\mathcal{S}}(\mathbf{a}, M)$ ). So using the above long exact sequence  $f_a^i\left(\frac{M}{xM}\right) \in \mathcal{S}$  for all  $i > t$ . By induction hypothesis,  $\frac{f_a^t\left(\frac{M}{xM}\right)}{af_a^t\left(\frac{M}{xM}\right)} \in \mathcal{S}$  because  $\dim\left(\frac{M}{xM}\right) = \dim(M) - 1$ .

Afterwards from the exact sequence (\*) we get the following short exact sequence.

$$0 \rightarrow \text{Im}(f) \rightarrow f_a^t\left(\frac{M}{xM}\right) \rightarrow \text{Im}(g) \rightarrow 0$$

So we obtain the following long exact sequence.

$$\cdots \rightarrow \text{Tor}_I^R\left(\frac{R}{\mathbf{a}}, \text{Im}(g)\right) \rightarrow \frac{\text{Im}(f)}{\mathbf{a}\text{Im}(f)} \rightarrow \frac{f_a^t\left(\frac{M}{xM}\right)}{\mathbf{a}f_a^t\left(\frac{M}{xM}\right)} \rightarrow \frac{\text{Im}(g)}{\mathbf{a}\text{Im}(g)} \rightarrow 0.$$

Since  $f_a^t(M) \in \mathcal{S}$  and  $\text{Im}(g)$  is a submodule of  $f_a^{t+1}(M)$ , we deduce that  $\text{Tor}_I^R\left(\frac{R}{\mathbf{a}}, \text{Im}(g)\right) \in \mathcal{S}$ . On the other hand,  $\frac{f_a^t\left(\frac{M}{xM}\right)}{\mathbf{a}f_a^t\left(\frac{M}{xM}\right)} \in \mathcal{S}$ . Therefore,  $\frac{\text{Im}(f)}{\mathbf{a}\text{Im}(f)} \in \mathcal{S}$  by the above long exact sequence.

Now, consider the following long exact sequence.

$$\frac{f_a^t(M)}{\mathbf{a}f_a^t(M)} \xrightarrow{x} \frac{f_a^t(M)}{\mathbf{a}f_a^t(M)} \rightarrow \frac{\text{Im}(f)}{\mathbf{a}\text{Im}(f)} \rightarrow 0.$$

So,  $\frac{f_a^t(M)}{\mathbf{a}f_a^t(M)} \cong \frac{\text{Im}(f)}{\mathbf{a}\text{Im}(f)}$  because  $x \in \mathbf{a}$ . Consequently,  $\frac{f_a^t(M)}{\mathbf{a}f_a^t(M)} \in \mathcal{S}$ .

**Proposition 3.8.** For a finitely generated  $R$ -module  $M$ ,

$$f.cd_{\mathcal{S}}(\mathbf{a}, M) = \max \{f.cd_{\mathcal{S}}\left(\mathbf{a}, \frac{R}{P}\right) | P \in \text{Ass}_R(M)\}.$$

**Proof.** Set  $N := \bigoplus_{P \in \text{Ass}_R(M)} \frac{R}{P}$ . Then  $\text{Supp}_R(M) = \text{Supp}_R(N)$ . So, by Theorem 3.2 and Corollary 3.5,  $f.cd_{\mathcal{S}}(\mathbf{a}, M) = f.cd_{\mathcal{S}}(\mathbf{a}, N) = \max \{f.cd_{\mathcal{S}}\left(\mathbf{a}, \frac{R}{P}\right) | P \in \text{Ass}_R(M)\}.$

**Proposition 3.9.** Assume that  $\mathbf{a}$  is an ideal of the local ring  $(R, \mathbf{m})$ . Then  $\text{Hom}_R\left(\frac{R}{\mathbf{m}}, f_a^0(M)\right) \in \mathcal{S}$  if and only if.  $\text{Hom}_R\left(\frac{R}{\mathbf{m}}, \widehat{M}^{\mathbf{a}}\right) \in \mathcal{S}$ .

**Proof.** It is enough to consider the following isomorphisms

$$\begin{aligned} \text{Hom}_R\left(\frac{R}{\mathbf{m}}, f_a^0(M)\right) &\cong \lim_{n \in \mathbb{N}} \text{Hom}_R\left(\frac{R}{\mathbf{m}}, H_m^0\left(\frac{M}{\mathbf{a}^n M}\right)\right) \cong \lim_{n \in \mathbb{N}} \text{Hom}_R\left(\frac{R}{\mathbf{m}}, \frac{M}{\mathbf{a}^n M}\right) \\ &\cong \text{Hom}_R\left(\frac{R}{\mathbf{m}}, \lim_{n \in \mathbb{N}} \frac{M}{\mathbf{a}^n M}\right) \cong \text{Hom}_R\left(\frac{R}{\mathbf{m}}, \widehat{M}^{\mathbf{a}}\right). \end{aligned}$$

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