On pointwise inner automorphisms of nilpotent groups of class 2

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Abstract
An automorphism \( \theta \) of a group \( G \) is pointwise inner if \( \theta(x) \) is conjugate to \( x \) for any \( x \in G \). The set of all pointwise inner automorphisms of group \( G \), denoted by Aut_{pwi}(G) form a subgroups of Aut(G) containing Inn(G). In this paper, we find a necessary and sufficient condition in certain finitely generated nilpotent groups of class 2 for which Aut_{pwi}(G) = Inn(G). We also prove that in a nilpotent group of class 2 with cyclic commutator subgroup Aut_{pwi}(G) = Inn(G) and the quotient Aut_{pwi}(G)/Inn(G) is torsion. In particular if \( G' \) is a finite cyclic group then Aut_{pwi}(G) = Inn(G).

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Introduction
By definition, a pointwise inner automorphism of a group \( G \) is an automorphism \( \theta: G \rightarrow G \) such that \( t \) and \( \theta(t) \) are conjugate for any \( t \in G \). This notion appears in the famous book of Burnside [1, Note B, p 463]. Denote by Aut_{pwi}(G) the set of all pointwise inner automorphisms of \( G \).

Obviously, Aut_{pwi}(G) contains Inn(G), the group of all inner automorphisms of \( G \). These groups can coincide, for instance when \( G \) is \( S_n, A_n, SL_n(D) \) and \( GL_n(D) \) where \( D \) is an Euclidean domain (see [7], [10], [11]).

By a result of Grossman [5], it turns out that Aut_{pwi}(G) = Inn(G) when \( G \) is a free group. Endimioni in [4] proved that this property remains true in a free nilpotent group.

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Also Yadav in [12] gave a sufficient condition for a finite $p$-group $G$ of nilpotent class 2 to be such that $\text{Aut}_{\text{pwi}}(G) = \text{Inn}(G)$. But the equality does not hold in general.

In fact, in 1911, Burnside posed the following question: Does there exist any finite group $G$ such that $G$ has a non-inner and pointwise inner automorphism? In 1913, Burnside himself gave an affirmative answer to this question [3]. Indeed, there are many examples of groups admitting a pointwise inner automorphism which is not inner (see, for instance [3], [4], [8], [9], [12] where these groups are besides nilpotent).

Segal also gave a subtle example. He constructed a finitely generated torsion-free nilpotent group $G$, in which $\text{Aut}_{\text{pwi}}(G)/\text{Inn}(G)$ contains an element of infinite order (see [9]).

In this paper we study the pointwise inner automorphisms of a finitely generated nilpotent group of class 2 with cyclic commutator subgroup.

We introduce the following definition:

**Definition.** Let $G$ be a finitely generated nilpotent group of class 2. Then $G/Z(G)$ is finitely generated abelian group and thus $G/Z(G) = \langle x_1 Z(G) \rangle \times \cdots \times \langle x_k Z(G) \rangle$ for some $x_1, \ldots, x_k \in G$. The group $G$ is called $d$-group if the following distributive law holds in $G$,

$$[x_1^{\alpha_1} \cdots x_k^{\alpha_k}, G] = [x_1, G]^{\alpha_1} \cdots [x_k, G]^{\alpha_k}$$

where $\alpha_i \in \mathbb{Z}$ and $1 \leq i \leq k$.

Let $G$ be a 2-generator nilpotent group of class 2. It is straightforward to show that $G$ is a $d$-group.

To give an example of an infinite $d$-group, consider the group $G$ with the following presentation

$$G = \langle x_1, x_2, x_3, x_4, x : [x_i, x_j] = x^{m_{ij}}, [x_i, x] = 1; 1 \leq i \leq 4 \text{ and } i < j \rangle,$$

where $m_{ii+1} = 1$ for all $1 \leq i < 4$ and $m_{ij} = 0$ for all $i + 1 < j$. Then $G' = Z(G) = \langle x \rangle \cong \mathbb{Z}$ and $G/Z(G) = \langle x_1, x_2, x_3, x_4 \rangle \cong \mathbb{Z}^4$. A quick calculation shows that

$$[x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} x_4^{\alpha_4}, G] = [x_1, G]^{\alpha_1} [x_2, G]^{\alpha_2} [x_3, G]^{\alpha_3} [x_4, G]^{\alpha_4} = \langle x^{\alpha} \rangle,$$

Where $\alpha_i \in \mathbb{Z}$ for all $1 \leq i \leq 4$ and $\alpha = \gcd(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. Therefore $G$ is an infinite $d$-
Now we give a nilpotent group $G$ of class 2 which is not a d-group. Let $G$ be a free nilpotent group of class 2 on 4 generators $a_1, a_2, a_3$ and $a_4$. If $c_{ij} = [a_i, a_j]$ for $1 \leq i < j \leq 4$, then the relations in $G$ are $[c_{ij}, a_k] = 1$ for $1 \leq i < j \leq 4$ and $1 \leq k \leq 4$, and their consequences. Macdonald in [6] proved that $c_{13}c_{24}$ is not a commutator. Therefore $G$ is not a d-group.

**Theorem 1.** Let $G$ be a finitely generated nilpotent group of class 2 and $G/Z(G) = \langle x_1 \rangle \times \ldots \times \langle x_k \rangle$.

(i) There exists a monomorphism $\text{Aut}_{pwi}(G) \hookrightarrow \prod_{i=1}^{k} \text{Hom}(\langle x_i \rangle, [x_i, G])$.

(ii) If $[x_i, G]$ is cyclic for all $1 \leq i \leq k$, then there exists a monomorphism $\text{Aut}_{pwi}(G) \hookrightarrow \text{Inn}(G)$.

In particular if $G$ is a d-group of class 2 then the monomorphisms in (i) and (ii) are isomorphism. Furthermore $\text{Aut}_{pwi}(G) \cong \text{Inn}(G)$ if and only if $[x_i, G]$ is cyclic for all $1 \leq i \leq k$.

Notice that if $G$ is a finite group then, as consequence of this result, we derive Theorem 3.5 and Corollary 3.6 of Yadav in [12]. In particular, we derive the following consequences of Theorem 1.

**Corollary 1.** Let $G$ be a finitely generated nilpotent group of class 2 in which $G'$ is cyclic, then $\text{Aut}_{pwi}(G) \cong \text{Inn}(G)$. In particular if $G'$ is finite, then $\text{Aut}_{pwi}(G) = \text{Inn}(G)$.

Recall that by Corollary 3.6 in [12], in a finite nilpotent group of class 2, if $G'$ is cyclic then $\text{Aut}_{pwi}(G) = \text{Inn}(G)$. But we cannot hope for a similar conclusion when $G$ is not finite. We will provide an example in the section 2. However, in a finitely generated nilpotent group of class 2, by Corollary 1 we have $\text{Aut}_{pwi}(G) \cong \text{Inn}(G)$. So the structure of $\text{Aut}_{pwi}(G)$ is determined.

**Corollary 2.** Let $G$ be a finitely generated nilpotent group of class 2. If the commutator subgroup of $G$ is cyclic, then $\text{Aut}_{pwi}(G)/\text{Inn}(G)$ is torsion.

Let $G$ be a group and $N$ be a non-trivial proper normal subgroup of $G$. The pair $...$
(G, N) is called a Camina pair if xN \subseteq x^G for all x \in G \setminus N. A group G is called a Camina group if (G, G') is a Camina pair.

Clearly, if G is a Camina group of class 2 then it is a d-group. So, as an immediate consequence of Theorem 1, one readily gets the following corollary.

**Corollary 3.** Let G be a finitely generated nilpotent group of class 2. If G is a Camina group then Aut_{pwi}(G) \cong Inn(G) if and only if G' is cyclic. Particularly, if G/Z(G) is finite, then Aut_{pwi}(G) = Inn(G) if and only if G' is cyclic.

**Preliminary results**

Our notation is standard. Let G be a group, by C_m, G' and Z(G), we denote the cyclic group of order m, the commutator subgroup and the center of G, respectively.

If x, y \in G, then x^y denotes the conjugate element y^{-1}xy \in G. For x \in G, x^G denotes the conjugacy class of x in G. The commutator of two elements x, y \in G is defined by [x, y] = x^{-1}y^{-1}xy and more generally, the left-normed commutator of n elements x_1, \ldots, x_n is defined inductively by

\[ [x_1, \ldots, x_{n-1}, x_n] = [x_1, \ldots, x_{n-1}]^{-1}x_n^{-1}[x_1, \ldots, x_{n-1}]x_n. \]

If H \leq G, [x, H] denotes the set of all [x, h] for h \in H, this is a subgroup of G when G is of class 2. For any group H and abelian group K, Hom(H, K) denotes the group of all homomorphisms from H to K. Also C^* is the set of all central automorphisms of G fixing Z(G) elementwise.

Yadav in [12] shows that in a finite nilpotent group of class 2, there exists a monomorphism from Aut_{pwi}(G) into Hom(G/Z(G), G'). It turns out that this result remains true when G is an infinite nilpotent group of class 2.

For that, let G be a nilpotent group (finite or infinite) of class 2. Let \alpha \in Aut_{pwi}(G). Then the map g \mapsto g^{-1}\alpha(g) is a homomorphism from G into G'. This homomorphism sends Z(G) to 1. So it induces a homomorphism f_\alpha: G/Z(G) \to G', sending \overline{g} = gZ(G) to g^{-1}\alpha(g), for any g \in G. Define

\[ \text{Hom}_{pwi}(G/Z(G), G') = \{ f \in \text{Hom}(G/Z(G), G') : f(\overline{g}) | [g, G] \text{ for all } g \in G \}. \]

To prove Aut_{pwi}(G) \cong \text{Hom}_{pwi}(G/Z(G), G'), we use the following well-known result.
Lemma 1.1 Let $N$ be a normal subgroup of a group $G$. Let $\theta$ be an endomorphism of $G$ such that $\theta(N) \leq N$. Denote by $\bar{\theta}$ and $\theta_0$ the endomorphisms induced by $\theta$ in $G/N$ and $N$, respectively. If $\bar{\theta}$ and $\theta_0$ are surjective (injective), then so is $\theta$.

Proposition 1.2 Let $G$ be a nilpotent group of class 2. Then the above map $\varphi: \alpha \mapsto f_\alpha$ is an isomorphism from $\text{Aut}_{\text{pwi}}(G)$ into $\text{Hom}_{\text{pwi}}(G/Z(G), G')$.

Proof. Since for any $\alpha \in \text{Aut}_{\text{pwi}}(G)$, by the definition $f_\alpha \in \text{Hom}_{\text{pwi}}(G/Z(G), G')$, $\varphi$ is well defined. Let $\alpha_1, \alpha_2 \in \text{Aut}_{\text{pwi}}(G)$ and $g \in G$. We have $\alpha_1(g^{-1}\alpha_2(g)) = g^{-1}\alpha_2(g)$. Since $g^{-1}\alpha_2(g) \in G' \leq Z(G)$. This implies that

\[ f_{\alpha_1\alpha_2}(g) = g^{-1}\alpha_1(g^{-1}\alpha_2(g)) = g^{-1}\alpha_1(gg^{-1}\alpha_2(g)) = g^{-1}\alpha_1(g)g^{-1}\alpha_2(g) = f_{\alpha_1}(g)\cdot f_{\alpha_2}(g). \]

Hence $\varphi$ is a homomorphism. Clearly, $\varphi$ is injective. Now it suffices to show that $\varphi$ is surjective.

Let $f$ be any element of $\text{Hom}_{\text{pwi}}(G/Z(G), G')$. By Lemma 1.1 a quick calculation shows that $\varphi(\alpha) = f$, where $\alpha$ is an element of $\text{Aut}_{\text{pwi}}(G)$, sending $g \in G$ to $gf(gZ(G))$. Then we have $\text{Aut}_{\text{pwi}}(G) \cong \text{Hom}_{\text{pwi}}(G/Z(G), G')$.

* Note that if $G$ is a nilpotent group of class 2 then $\text{Aut}_{\text{pwi}}(G) \cong \text{Hom}_{\text{pwi}}(G/Z(G), G')$.

It is easy to see that in a Camina nilpotent group of class 2, $\text{Hom}_{\text{pwi}}(G/Z(G), G') = \text{Hom}(G/Z(G), G')$. Hence if $G$ is a Camina group of class 2, then $\text{Aut}_{\text{pwi}}(G) \cong \text{Hom}(G/Z(G), G')$.

The following well-known facts will be used repeatedly.

Lemma 1.3 Let $A, B$ and $C$ be abelian groups.

(i) $\text{Hom}(A \times B, C) \cong \text{Hom}(A, C) \times \text{Hom}(B, C)$.

(ii) $\text{Hom}(A, B \times C) \cong \text{Hom}(A, B) \times \text{Hom}(A, C)$.

(iii) $\text{Hom}(C_m, C_n) \cong C_d$ where $d = \gcd(m, n)$.

(iv) $\text{Hom}(\mathbb{Z}, A) \cong A$.

(v) If $A$ is torsion group and $B$ is torsion-free group, then $\text{Hom}(A, B) = 1$.

(vi) If $\gcd(|A|, |B|) \neq 1$, then $\text{Hom}(A, B) \neq 1$. 

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Main Result

Let \( G \) be a finite abelian group. We denote by \( G_p \), the \( p \)-primary component of \( G \). Hence \( G = \prod_{p \in \pi(G)} G_p \) where \( \pi(G) \) denotes the set of all primes \( p \) dividing \( |G| \).

To prove Theorem 1, we need the following Lemma.

Lemma 2.1 ([1, Corollary 1.4]) Let \( A \) and \( B \) be two finite abelian groups and \( \exp(A) \exp(B) \). Then \( \text{Hom}(A,B) \cong A \) if and only if \( B \cong C_m \times H \) in which \( C_m \cong \prod_{p \in \pi(A)} B_p \) and \( H \cong \prod_{p \not\in \pi(A)} B_p \). In particular, if \( \pi(A) = \pi(B) \) then this is equivalent to \( B \) is a cyclic group.

Let \( G \) be a finitely generated nilpotent group of class 2. Then \( G/Z(G) \) is finitely generated abelian group and thus \( G/Z(G) = \langle x_1 Z(G) \rangle \times \ldots \times \langle x_k Z(G) \rangle \) for some \( x_1, \ldots, x_k \in G \).

Let \( f \in \text{Hom}_{pw}(G/Z(G), G) \). So \( f(gZ(G)) \in [g, G] \) for all \( g \in G \). In particular, for all \( 1 \leq i \leq k \) we have \( f(x_i Z(G)) \in [x_i, G] \). Now we prove Theorem 1.

Proof of Theorem 1.

(i) By Proposition 1.2, we have \( \text{Aut}_{pw}(G) \cong \text{Hom}_{pw}(G/Z(G), G') \). It suffices to show that there exists a monomorphism from \( \text{Hom}_{pw}(G/Z(G), G') \) into \( \prod_{i=1}^k \text{Hom}(\langle x_i \rangle, [x_i, G]) \). Let \( f \in \text{Hom}_{pw}(G/Z(G), G') \). Denote by \( f_i \), the homomorphism induced by \( f \) in \( \langle x_i \rangle \), for all \( 1 \leq i \leq k \). Since \( G \) is a nilpotent group of class 2, we have \( [a^m, b] = [a, b]^m = [a, b^m] \) for each \( a, b \in G \) and \( m \in \mathbb{Z} \). Consequently, \( [x_i^m, G] \leq [x_i, G] \) for all \( m \in \mathbb{Z} \) and \( 1 \leq i \leq k \). Therefore \( f_i \in \text{Hom}(\langle x_i \rangle, [x_i, G]) \). Thus the map \( \alpha \) sending any \( f \in \text{Hom}_{pw}(G/Z(G), G') \) to \( \alpha(f) = (f_1, \ldots, f_k) \in \prod_{i=1}^k \text{Hom}(\langle x_i \rangle, [x_i, G]) \) is well defined. Now we prove that this map is a monomorphism. Since \( (fg)_i = f_i g_i \) for each \( f, g \in \text{Hom}_{pw}(G/Z(G), G') \) and \( 1 \leq i \leq k \), \( \alpha \) is homomorphism. Clearly, \( \ker \alpha \) is trivial, this implies that \( \alpha \) is monomorphism. Hence the proof of (i) is complete.

(ii) First we show that \( [x_i, G] \) is finite if and only if \( \langle x_i \rangle \) is finite, and further
exp([x_i, G]) = m. Since G is a nilpotent group of class 2, we have [x_i, G] = [x_i, g]^m = 1 for all g ∈ G and so x_i^m ∈ Z(G).

Hence (x_i) is finite and |x_i| = m. Conversely if |x_i| = m then x_i^m ∈ Z(G) and [x_i, G]^m = [x_i, G] = 1. Consequently [x_i, G] is finite and exp([x_i, G]) = n|m.

Therefore in this case, m = n. Hence by Lemma 2.1, for all 1 ≤ i ≤ k we have
Hom((x_i), [x_i, G]) = (x_i) if and only if [x_i, G] is cyclic.

Now from (i), we have a monomorphism from Aut_pwi(G) into ∏^k i=1 Hom((x_i), [x_i, G]) and therefore we conclude that there exists a monomorphism Aut_pwi(G) ≅ G/Z(G), this completes the proof of (ii).

If G is a d-group, then it is easy to see that the monomorphism defined in (i) is an isomorphism from Aut_pwi(G) into ∏^k i=1 Hom((x_i), [x_i, G]).

Finally to complete the proof, it is sufficient to show that if Aut_pwi(G) ≅ Inn(G), then [x_i, G] is cyclic for all 1 ≤ i ≤ k. Since Aut_pwi(G) ≅ Inn(G), by Proposition 1.2 we have G/Z(G) ≅ Hom_pwi(G/Z(G), G'). On the other hand, G is a d-group and hence
Hom_pwi(G/Z(G), G') ≅ ∏^k i=1 Hom((x_i), [x_i, G])

It follows that
G/Z(G) = (x_1) × ... × (x_k) ≅ ∏^k i=1 Hom((x_i), [x_i, G]).

Now we may assume that (x_1) × ... × (x_n) is the torsion part and (x_{n+1}) × ... × (x_k) is the torsion-free part of G/Z(G). Since for all 1 ≤ i ≤ n, exp([x_i, G]) = exp(x_i) = |x_i| and ∏^n i=1 Hom((x_i), [x_i, G]) ≅ (x_1) × ... × (x_n), Hom((x_i), [x_i, G]) ≅ (x_i) for all 1 ≤ i ≤ n and hence [x_i, G] is cyclic. Furthermore, we have

\[ \prod_{i=n+1}^{k} \text{Hom}((x_i), [x_i, G]) \cong (x_{n+1}) \times \cdots \times (x_k) \cong \mathbb{Z}^{k-n}. \]

Now we have Hom((x_i), [x_i, G]) ≅ [x_i, G], since (x_i) ≅ \mathbb{Z} and hence \[ \prod_{i=n+1}^{m} [x_i, G] \cong \mathbb{Z}^{k-n}. \] That is [x_i, G] ≅ \mathbb{Z} for all n + 1 ≤ i ≤ k. This implies that [x_i, G] is cyclic for all 1 ≤ i ≤ k, as required.

*Notice that if G is a finite group then, as a consequence of this result, we derive Theorem 3.5 and Corollary 3.6 of Yadav in [12].

The following corollary is an easy consequence of the above theorem.

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**Corollary 2.2** Let $G$ be a finitely generated nilpotent group of class 2 with cyclic commutator subgroup. Then there exists a monomorphism from $\text{Aut}_{pwi}(G)$ into $\text{Inn}(G)$ or equivalently $\text{Aut}_{pwi}(G)$ is isomorphic to a subgroup of $G/Z(G)$.

**Remark 2.3** We keep here the notation used in Theorem 1.

(i) By the discussion of (ii) in Theorem 1, if $G'$ is finite cyclic, then $G/Z(G)$ is finite and $|\text{Aut}_{pwi}(G)| \leq |\text{Inn}(G)| = |G/Z(G)|$. On the other hand, $\text{Inn}(G) \leq \text{Aut}_{pwi}(G)$ conclude that $\text{Aut}_{pwi}(G) \cong \text{Inn}(G)$. Note that in this case, $G$ is not necessarily finite.

(ii) If $G'$ is infinite cyclic, it follows from the discussion of (ii) in Theorem 1, that $G/Z(G)$ is a free abelian group of finite rank, say $r(G/Z(G)) = k$. We certainly have $\text{Inn}(G) \leq \text{Aut}_{pwi}(G)$ and thus $r(\text{Inn}(G)) \leq r(\text{Aut}_{pwi}(G))$. Also $r(\text{Aut}_{pwi}(G)) \leq r(\text{Inn}(G))$, since $\text{Aut}_{pwi}(G)$ is isomorphic to a subgroup of $\text{Inn}(G)$. Therefore $\text{Aut}_{pwi}(G)$ and $\text{Inn}(G)$ have the same rank and hence $\text{Aut}_{pwi}(G) \cong \text{Inn}(G)$.

Now it is easy to deduce Corollary 1 from Remark 2.3.

**Remark 2.4** It is known that in a nilpotent groups of class 2, $\text{Inn}(G) \leq \text{Aut}_{pwi}(G) \leq G^*$. So $\text{Inn}(G) = \text{Aut}_{pwi}(G)$ when $\text{Inn}(G) = G^*$. In [1] we characterized all non torsion-free finitely generated groups in which $\text{Inn}(G) = G^*$. We proved that $\text{Inn}(G) = G^*$ if and only if $G$ is an abelian group or nilpotent of class 2 and $Z(G) \cong C_m \times H \times \square^r$ in which $C_m \cong \Pi_{p \in \pi(G/Z(G))} Z(G)_p$, $H \cong \Pi_{p \in \pi(G/Z(G))} Z(G)_p$ and $r \geq 0$ is the torsion-free rank of $Z(G)$ and $G/Z(G)$ has finite exponent.

Hence if $G$ is nilpotent group of class 2, $Z(G) \cong C_m \times H \times \square^r$ and $G/Z(G)$ has finite exponent then we have $\text{Inn}(G) = \text{Aut}_{pwi}(G)$. Notice that in this case, $G'$ is cyclic and the equality $\text{Inn}(G) = \text{Aut}_{pwi}(G)$ also follows from Corollary 1.

Recall that by Corollary 3.6 in [12], in a finite nilpotent group of class 2, if $G'$ is cyclic then $\text{Aut}_{pwi}(G) = \text{Inn}(G)$. But we cannot hope for a similar conclusion when $G$ is not finite.

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For example, consider countably infinitely many copies $H_1, H_2, \ldots$ of a given nilpotent group $H$ of class 2 with cyclic commutator subgroup. Let $G$ (respectively, $\overline{G}$) be the direct product (the cartesian product) of the family $(H_i)_{i>0}$. Clearly, $G$ and $\overline{G}$ are nilpotent of class 2. For each integer $i > 0$, choose an element $a_i \in H_i$ which is not in the center of $H_i$. Then the inner automorphism of $\overline{G}$ defined by $\overline{a}((t_i)_{i>0}) = (a_i^{-1}t_ia_i)_{i>0}$ induces in $G$ a pointwise inner automorphism $\alpha$ which is not inner (see [4]).

However, in a finitely generated nilpotent group of class 2 with cyclic commutator subgroup, we have $\text{Aut}_{\text{pw}}(G) \simeq \text{Inn}(G)$, by Corollary 1. So the structure of $\text{Aut}_{\text{pw}}(G)$ is determined.

Furthermore it is fairly easy to deduce Corollary 2 from Remark 2.3.

We end this part of the paper with some examples of infinite groups $G$ satisfying the conditions of Corollary 1 and therefore $\text{Aut}_{\text{pw}}(G) \simeq \text{Inn}(G)$.

**Example 2.5** Let $G = \langle x_1, x_2, y_1, y_2; x_1^p = x_2^p = y_1^p = y_2^p = 1, [x_1, x_2] = y_1, [y_1, y_2] = [x_1, y_1] = 1; 1 \leq i, j \leq 2 \rangle$. Then $G$ satisfies the condition of Corollary 1. We have $G' = \langle y_1 \rangle \simeq C_p$, $Z(G) = \langle y_1, y_2 \rangle \simeq C_p \times \mathbb{Z}$ and $G/Z(G) = \langle \overline{x_1}, \overline{x_2} \rangle \simeq C_p \times C_p$ and hence $\text{Aut}_{\text{pw}}(G) = \text{Inn}(G)$.

**Example 2.6** Let $G = \langle x_1, x_2, x; [x_1, x_2] = x, [x_1, x] = 1; 1 \leq i \leq 2 \rangle$. Then $G$ satisfies the condition of Corollary 1. We have $G' = Z(G) = \langle x \rangle \simeq \mathbb{Z}$ and $\frac{G}{Z(G)} = \langle \overline{x_1}, \overline{x_2} \rangle \simeq \mathbb{Z} \times \mathbb{Z}$.

Hence $\text{Aut}_{\text{pw}}(G) \simeq \text{Inn}(G)$. It is easy to see that in this case every pointwise inner automorphism is inner and so $\text{Aut}_{\text{pw}}(G) = \text{Inn}(G)$ (see [1, Example 3.4]).

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