On pointwise inner automorphisms of nilpotent groups of class $2$

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Abstract
An automorphism $\theta$ of a group $G$ is pointwise inner if $\theta(x)$ is conjugate to $x$ for any $x \in G$. The set of all pointwise inner automorphisms of group $G$, denoted by $\text{Aut}_{pwi}(G)$, form a subgroups of $\text{Aut}(G)$ containing $\text{Inn}(G)$. In this paper, we find a necessary and sufficient condition in certain finitely generated nilpotent groups of class $2$ for which $\text{Aut}_{pwi}(G) \cong \text{Inn}(G)$. We also prove that in a nilpotent group of class $2$ with cyclic commutator subgroup $\text{Aut}_{pwi}(G) \cong \text{Inn}(G)$ and the quotient $\text{Aut}_{pwi}(G)/\text{Inn}(G)$ is torsion. In particular if $G'$ is a finite cyclic group then $\text{Aut}_{pwi}(G) = \text{Inn}(G)$.

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Introduction
By definition, a pointwise inner automorphism of a group $G$ is an automorphism $\theta: G \rightarrow G$ such that $t$ and $\theta(t)$ are conjugate for any $t \in G$. This notion appears in the famous book of Burnside [1, Note B, p 463]. Denote by $\text{Aut}_{pwi}(G)$ the set of all pointwise inner automorphisms of $G$.

Obviously, $\text{Aut}_{pwi}(G)$ contains $\text{Inn}(G)$, the group of all inner automorphisms of $G$. These groups can coincide, for instance when $G$ is $S_n$, $A_n$, $SL_n(D)$ and $GL_n(D)$ where $D$ is an Euclidean domain (see [7], [10], [11]).

By a result of Grossman [5], it turns out that $\text{Aut}_{pwi}(G) = \text{Inn}(G)$ when $G$ is a free group. Endimioni in [4] proved that this property remains true in a free nilpotent group.

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Also Yadav in [12] gave a sufficient condition for a finite $p$-group $G$ of nilpotent class 2 to be such that $\text{Aut}_{\text{pwi}}(G) = \text{Inn}(G)$. But the equality does not hold in general.

In fact, in 1911, Burnside posed the following question: Does there exist any finite group $G$ such that $G$ has a non-inner and pointwise inner automorphism? In 1913, Burnside himself gave an affirmative answer to this question [3]. Indeed, there are many examples of groups admitting a pointwise inner automorphism which is not inner (see, for instance [3], [4], [8], [9], [12] where these groups are besides nilpotent).

Segal also gave a subtle example. He constructed a finitely generated torsion-free nilpotent group $G$, in which $\text{Aut}_{\text{pwi}}(G)/\text{Inn}(G)$ contains an element of infinite order (see [9]).

In this paper we study the pointwise inner automorphisms of a finitely generated nilpotent group of class 2 with cyclic commutator subgroup.

We introduce the following definition:

**Definition.** Let $G$ be a finitely generated nilpotent group of class 2. Then $G/Z(G)$ is finitely generated abelian group and thus $G/Z(G) = \langle x_1 Z(G) \rangle \times \ldots \times \langle x_k Z(G) \rangle$ for some $x_1, \ldots, x_k \in G$. The group $G$ is called $d$-group if the following distributive law holds in $G$,

$$[x_1^{a_1}, \ldots, x_k^{a_k}, G] = [x_1, G]^{a_1} \ldots [x_k, G]^{a_k}$$

where $a_i \in \mathbb{Z}$ and $1 \leq i \leq k$.

Let $G$ be a 2-generator nilpotent group of class 2. It is straightforward to show that $G$ is a $d$-group.

To give an example of an infinite $d$-group, consider the group $G$ with the following presentation

$$G = \langle x_1, x_2, x_3, x_4, x : [x_i, x_j] = x^{m_{ij}}, [x_i, x] = 1; 1 \leq i \leq 4 \text{ and } i < j \rangle,$$

where $m_{ii+1} = 1$ for all $1 \leq i < 4$ and $m_{ij} = 0$ for all $i + 1 < j$. Then $G' = Z(G) = \langle x \rangle \cong \mathbb{Z}$ and $G/Z(G) = \langle x_1, x_2, x_3, x_4 \rangle \cong \mathbb{Z}^4$. A quick calculation shows that

$$[x_1^{a_1}, x_2^{a_2}, x_3^{a_3}, x_4^{a_4}, G] = [x_1, G]^{a_1} [x_2, G]^{a_2} [x_3, G]^{a_3} [x_4, G]^{a_4} = \langle x^a \rangle,$$

where $a_i \in \mathbb{Z}$ for all $1 \leq i \leq 4$ and $a = \gcd(a_1, a_2, a_3, a_4)$. Therefore $G$ is an infinite $d$-
group.

Now we give a nilpotent group $G$ of class 2 which is not a d-group.

Let $G$ be a free nilpotent group of class 2 on 4 generators $a_1$, $a_2$, $a_3$ and $a_4$. If $c_{ij} = [a_i, a_j]$ for $1 \leq i < j \leq 4$, then the relations in $G$ are $[c_{ij}, a_k] = 1$ for $1 \leq i < j \leq 4$ and $1 \leq k \leq 4$, and their consequences. Macdonald in [6] proved that $c_{13}c_{24}$ is not a commutator. Therefore $G$ is not a d-group.

**Theorem 1.** Let $G$ be a finitely generated nilpotent group of class 2 and $G/Z(G) = \langle x_1 \rangle \times ... \times \langle x_k \rangle$.

(i) There exists a monomorphism $\text{Aut}_{pwi}(G) \hookrightarrow \prod_{i=1}^{k} \text{Hom}(\langle x_i \rangle, [x_i, G])$.

(ii) If $[x_i, G]$ is cyclic for all $1 \leq i \leq k$, then there exists a monomorphism $\text{Aut}_{pwi}(G) \hookrightarrow \text{Inn}(G)$.

In particular if $G$ is a d-group of class 2 then the monomorphisms in (i) and (ii) are isomorphism. Furthermore $\text{Aut}_{pwi}(G) \simeq \text{Inn}(G)$ if and only if $[x_i, G]$ is cyclic for all $1 \leq i \leq k$.

Notice that if $G$ is a finite group then, as consequence of this result, we derive Theorem 3.5 and Corollary 3.6 of Yadav in [12].

In particular, we derive the following consequences of Theorem 1.

**Corollary 1.** Let $G$ be a finitely generated nilpotent group of class 2 in which $G'$ is cyclic, then $\text{Aut}_{pwi}(G) \simeq \text{Inn}(G)$. In particular if $G'$ is finite, then $\text{Aut}_{pwi}(G) = \text{Inn}(G)$.

Recall that by Corollary 3.6 in [12], in a finite nilpotent group of class 2, if $G'$ is cyclic then $\text{Aut}_{pwi}(G) = \text{Inn}(G)$. But we cannot hope for a similar conclusion when $G$ is not finite. We will provide an example in the section 2. However, in a finitely generated nilpotent group of class 2, by Corollary 1 we have $\text{Aut}_{pwi}(G) \simeq \text{Inn}(G)$. So the structure of $\text{Aut}_{pwi}(G)$ is determined.

**Corollary 2.** Let $G$ be a finitely generated nilpotent group of class 2. If the commutator subgroup of $G$ is cyclic, then $\text{Aut}_{pwi}(G)/\text{Inn}(G)$ is torsion.

Let $G$ be a group and $N$ be a non-trivial proper normal subgroup of $G$. The pair
(G, N) is called a Camina pair if xN ≤ xG for all x ∈ G\N. A group G is called a Camina group if (G, G') is a Camina pair. Clearly, if G is a Camina group of class 2 then it is a d-group. So, as an immediate consequence of Theorem 1, one readily gets the following corollary.

**Corollary 3.** Let G be a finitely generated nilpotent group of class 2. If G is a Camina group then Aut_pwi(G) ≃ Inn(G) if and only if G' is cyclic. Particularly, if G/Z(G) is finite, then Aut_pwi(G) = Inn(G) if and only if G' is cyclic.

**Preliminary results**

Our notation is standard. Let G be a group, by C_m, G' and Z(G), we denote the cyclic group of order m, the commutator subgroup and the center of G, respectively.

If x, y ∈ G, then x^y denotes the conjugate element y^{-1}xy ∈ G. For x ∈ G, x^G denotes the conjugacy class of x in G. The commutator of two elements x, y ∈ G is defined by [x, y] = x^{-1}y^{-1}xy and more generally, the left-normed commutator of n elements x_1, ..., x_n is defined inductively by

\[ [x_1, ..., x_{n-1}, x_n] = [x_1, ..., x_{n-1}]^{-1}x_n^{-1}[x_1, ..., x_{n-1}]x_n. \]

If H ≤ G, [x, H] denotes the set of all [x, h] for h ∈ H, this is a subgroup of G when G is of class 2. For any group H and abelian group K, Hom(H, K) denotes the group of all homomorphisms from H to K. Also C^* is the set of all central automorphisms of G fixing Z(G) elementwise.

Yadav in [12] shows that in a finite nilpotent group of class 2, there exists a monomorphism from Aut_pwi(G) into Hom(G/Z(G), G'). It turns out that this result remains true when G is an infinite nilpotent group of class 2.

For that, let G be a nilpotent group (finite or infinite) of class 2. Let α ∈ Aut_pwi(G). Then the map g ↦ g^{-1}α(g) is a homomorphism from G into G'. This homomorphism sends Z(G) to 1. So it induces a homomorphism f_α: G/Z(G) → G', sending \overline{g} = gZ(G) to g^{-1}α(g), for any g ∈ G. Define

\[ \text{Hom}_{pwi}(G/Z(G), G') = \{ f ∈ \text{Hom}(\overline{G/Z(G)}, G') : f(\overline{g}) ∈ [g, G] \text{ for all } g ∈ G \}. \]

To prove Aut_pwi(G) ≃ Hom_pwi(G/Z(G), G'), we use the following well-known result.
**Lemma 1.1** Let $N$ be a normal subgroup of a group $G$. Let $\theta$ be an endomorphism of $G$ such that $\theta(N) \leq N$. Denote by $\overline{\theta}$ and $\theta_0$ the endomorphisms induced by $\theta$ in $G/N$ and $N$, respectively. If $\overline{\theta}$ and $\theta_0$ are surjective (injective), then so is $\theta$.

**Proposition 1.2** Let $G$ be a nilpotent group of class 2. Then the above map $\varphi: \alpha \mapsto f_\alpha$ is an isomorphism from $\text{Aut}_{\text{pwi}}(G)$ into $\text{Hom}_{\text{pwi}}(\z(G), G')$.

**Proof.** Since for any $\alpha \in \text{Aut}_{\text{pwi}}(G)$, by the definition $f_\alpha \in \text{Hom}_{\text{pwi}}(\z(G), G')$, $\varphi$ is well defined. Let $\alpha_1, \alpha_2 \in \text{Aut}_{\text{pwi}}(G)$ and $g \in G$. We have $\alpha_1(g^{-1}\alpha_2(g)) = g^{-1}\alpha_2(g)$, since $g^{-1}\alpha_2(g) \in G' \leq \z(G)$. This implies that

$$f_{\alpha_1 g^{-1} \alpha_2}(g) = g^{-1}\alpha_1(g^{-1}\alpha_2(g)) = g^{-1}\alpha_1(gg^{-1}\alpha_2(g)) = g^{-1}\alpha_1(g). \alpha_1(g^{-1}\alpha_2(g)) = f_{\alpha_1}(g). f_{\alpha_2}(g).$$

Hence $\varphi$ is a homomorphism. Clearly, $\varphi$ is injective. Now it suffices to show that $\varphi$ is surjective.

Let $f$ be any element of $\text{Hom}_{\text{pwi}}(\z(G), G')$. By Lemma 1.1 a quick calculation shows that $\varphi(\alpha) = f$, where $\alpha$ is an element of $\text{Aut}_{\text{pwi}}(G)$, sending $g \in G$ to $gf(g\z(G))$. Then we have $\text{Aut}_{\text{pwi}}(G) \cong \text{Hom}_{\text{pwi}}(\z(G), G')$.

* Note that if $G$ is a nilpotent group of class 2 then $\text{Aut}_{\text{pwi}}(G) \cong \text{Hom}_{\text{pwi}}(\z(G), G')$.

It is easy to see that in a Camina nilpotent group of class 2, $\text{Hom}_{\text{pwi}}(\z(G), G') = \text{Hom}(\z(G), G')$. Hence if $G$ is a Camina group of class 2, then $\text{Aut}_{\text{pwi}}(G) \cong \text{Hom}(\z(G), G')$.

The following well-known facts will be used repeatedly.

**Lemma 1.3** Let $A, B$ and $C$ be abelian groups.

(i) $\text{Hom}(A \times B, C) \cong \text{Hom}(A, C) \times \text{Hom}(B, C)$.

(ii) $\text{Hom}(A, B \times C) \cong \text{Hom}(A, B) \times \text{Hom}(A, C)$.

(iii) $\text{Hom}(C_m, C_n) \cong C_d$ where $d = \gcd(m, n)$.

(iv) $\text{Hom}(\mathbb{Z}, A) \cong A$.

(v) If $A$ is torsion group and $B$ is torsion-free group, then $\text{Hom}(A, B) = 1$.

(vi) If $\gcd(|A|, |B|) \neq 1$, then $\text{Hom}(A, B) \neq 1$. 

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Main Result

Let $G$ be a finite abelian group. We denote by $G_p$, the $p$-primary component of $G$. Hence $G = \prod_{p \in \pi(G)} G_p$ where $\pi(G)$ denotes the set of all primes $p$ dividing $|G|$. To prove Theorem 1, we need the following Lemma.

Lemma 2.1 ([1, Corollary 1.4]) Let $A$ and $B$ be two finite abelian groups and $\exp(A) \exp(B)$. Then $\text{Hom}(A, B) \simeq A$ if and only if $B \simeq C_m \times H$ in which $C_m \simeq \prod_{p \in \pi(A)} B_p$ and $H \simeq \prod_{p \notin \pi(A)} B_p$. In particular, if $\pi(A) = \pi(B)$ then this is equivalent to $B$ is a cyclic group.

Let $G$ be a finitely generated nilpotent group of class 2. Then $G/Z(G)$ is finitely generated abelian group and thus $G/Z(G) = \langle x_1 Z(G) \rangle \times \cdots \times \langle x_k Z(G) \rangle$ for some $x_1, \ldots, x_k \in G$.

Let $f \in \text{Hom}_{pwi}(G/Z(G), G')$. So $f(gZ(G)) \in [g, G]$ for all $g \in G$. In particular, for all $1 \leq i \leq k$ we have $f(x_i Z(G)) \in [x_i, G]$. Now we prove Theorem 1.

Proof of Theorem 1.

(i) By Proposition 1.2, we have $\text{Aut}_{pwi}(G) \simeq \text{Hom}_{pwi}(G/Z(G), G')$. It suffices to show that there exists a monomorphism from $\text{Hom}_{pwi}(G/Z(G), G')$ into $\prod_{i=1}^k \text{Hom}((x_i), [x_i, G])$. Let $f \in \text{Hom}_{pwi}(G/Z(G), G')$. Denote by $f_i$, the homomorphism induced by $f$ in $(x_i)$, for all $1 \leq i \leq k$. Since $G$ is a nilpotent group of class 2, we have $[a^m, b] = [a, b]^m = [a, b]^{m}$ for each $a, b \in G$ and $m \in \mathbb{Z}$. Consequently, $[x_i^m, G] \leq [x_i, G]$ for all $m \in \mathbb{Z}$ and $1 \leq i \leq k$. Therefore $f_i \in \text{Hom}((x_i), [x_i, G])$. Thus the map $\alpha$ sending any $f \in \text{Hom}_{pwi}(G/Z(G), G')$ to $\alpha(f) = (f_1, \ldots, f_k) \in \prod_{i=1}^k \text{Hom}((x_i), [x_i, G])$ is well defined. Now we prove that this map is a monomorphism. Since $(fg)_i = f_i g_i$ for each $f, g \in \text{Hom}_{pwi}(G/Z(G), G')$ and $1 \leq i \leq k$, $\alpha$ is homomorphism. Clearly, $\ker \alpha$ is trivial, this implies that $\alpha$ is monomorphism. Hence the proof of (i) is complete.

(ii) First we show that $[x_i, G]$ is finite if and only if $(x_i)$ is finite, and further
exp([x_i, G]) = exp(⟨x_i⟩) = |⟨x_i⟩|. For this, let |[x_i, G]| = n. Since G is a nilpotent group of class 2, we have [x_i^n, g] = [x_i, g]^n = 1 for all g ∈ G and so x_i^n ∈ Z(G).

Hence ⟨x_i⟩ is finite and |⟨x_i⟩| = n. Conversely if |⟨x_i⟩| = m then x_i^m ∈ Z(G) and [x_i, G]^m = [x_i^m, G] = 1. Consequently [x_i, G] is finite and \( \exp([x_i, G]) = n|m \).

Therefore in this case, m = n. Hence by Lemma 2.1, for all 1 ≤ i ≤ k we have \( \text{Hom}(⟨x_i⟩, [x_i, G]) ≅ ⟨x_i⟩ \) if and only if [x_i, G] is cyclic.

Now from (i), we have a monomorphism from Aut_{pw}(G) into \( \prod_{i=1}^k \text{Hom}(⟨x_i⟩, [x_i, G]) \) and therefore we conclude that there exists a monomorphism Aut_{pw}(G) ↪ G/Z(G), this completes the proof of (ii).

If G is a d-group, then it is easy to see that the monomorphism defined in (i) is an isomorphism from Aut_{pw}(G) into \( \prod_{i=1}^k \text{Hom}(⟨x_i⟩, [x_i, G]) \).

Finally to complete the proof, it is sufficient to show that if Aut_{pw}(G) ≅ Inn(G), then [x_i, G] is cyclic for all 1 ≤ i ≤ k. Since Aut_{pw}(G) ≅ Inn(G), by Proposition 1.2 we have G/Z(G) ≅ Hom_{pw}(G/Z(G), G'). On the other hand, G is a d-group and hence

\[
\text{Hom}_{pw}(G/Z(G), G') ≅ \prod_{i=1}^k \text{Hom}(⟨x_i⟩, [x_i, G]).
\]

It follows that

\[
G/Z(G) = ⟨x_1⟩ × ... × ⟨x_k⟩ ≅ \prod_{i=1}^k \text{Hom}(⟨x_i⟩, [x_i, G]).
\]

Now we may assume that \( ⟨x_1⟩ × ... × ⟨x_n⟩ \) is the torsion part and \( ⟨x_{n+1}, ... × ⟨x_k⟩ \) is the torsion-free part of G/Z(G). Since for all 1 ≤ i ≤ n, \( \exp([x_i, G]) = \exp(⟨x_i⟩) = |⟨x_i⟩| \) and \( \prod_{i=1}^n \text{Hom}(⟨x_i⟩, [x_i, G]) ≅ ⟨x_1⟩ × ... × ⟨x_n⟩ \) and \( \text{Hom}(⟨x_i⟩, [x_i, G]) ≅ ⟨x_i⟩ \) for all 1 ≤ i ≤ n and hence [x_i, G] is cyclic. Furthermore, we have

\[
\prod_{i=n+1}^k \text{Hom}(⟨x_i⟩, [x_i, G]) ≅ ⟨x_{n+1}, ... × ⟨x_k⟩ ≅ Z^{k-n}.
\]

Now we have Hom(⟨x_i⟩, [x_i, G]) ≅ [x_i, G], since ⟨x_i⟩ ≅ Z and hence \( \prod_{i=n+1}^m [x_i, G] ≅ Z^{k-n} \). That is \([x_i, G] ≅ Z\) for all \( n + 1 ≤ i ≤ k \). This implies that \([x_i, G] \) is cyclic for all \( 1 ≤ i ≤ k \), as required.

*Notice that if G is a finite group then, as a consequence of this result, we derive Theorem 3.5 and Corollary 3.6 of Yadav in [12].

The following corollary is an easy consequence of the above theorem.
Corollary 2.2 Let $G$ be a finitely generated nilpotent group of class 2 with cyclic commutator subgroup. Then there exists a monomorphism from $\text{Aut}_{\text{pwi}}(G)$ into $\text{Inn}(G)$ or equivalently $\text{Aut}_{\text{pwi}}(G)$ is isomorphic to a subgroup of $G/Z(G)$.

Remark 2.3 We keep here the notation used in Theorem 1.

(i) By the discussion of (ii) in Theorem 1, if $G'$ is finite cyclic, then $G/Z(G)$ is finite and $|\text{Aut}_{\text{pwi}}(G)| \leq |\text{Inn}(G)| = |G/Z(G)|$. On the other hand, $\text{Inn}(G) \leq \text{Aut}_{\text{pwi}}(G)$ conclude that $\text{Aut}_{\text{pwi}}(G) = \text{Inn}(G)$. Note that in this case, $G$ is not necessarily finite.

(ii) If $G'$ is infinite cyclic, it follows from the discussion of (ii) in Theorem 1, that $G/Z(G)$ is a free abelian group of finite rank, say $r(G/Z(G)) = k$. We certainly have $\text{Inn}(G) \leq \text{Aut}_{\text{pwi}}(G)$ and thus $r(\text{Inn}(G)) \leq r(\text{Aut}_{\text{pwi}}(G))$. Also $r(\text{Aut}_{\text{pwi}}(G)) \leq r(\text{Inn}(G))$, since $\text{Aut}_{\text{pwi}}(G)$ is isomorphic to a subgroup of $\text{Inn}(G)$. Therefore $\text{Aut}_{\text{pwi}}(G)$ and $\text{Inn}(G)$ have the same rank and hence $\text{Aut}_{\text{pwi}}(G) \simeq \text{Inn}(G)$.

Now it is easy to deduce Corollary 1 from Remark 2.3.

Remark 2.4 It is known that in a nilpotent groups of class 2, $\text{Inn}(G) \leq \text{Aut}_{\text{pwi}}(G) \leq C^\ast$. So $\text{Inn}(G) = \text{Aut}_{\text{pwi}}(G)$ when $\text{Inn}(G) = C^\ast$. In [1] we characterized all non torsion-free finitely generated groups in which $\text{Inn}(G) = C^\ast$. We proved that $\text{Inn}(G) = C^\ast$ if and only if $G$ is an abelian group or nilpotent of class 2 and $Z(G) = C_m \times H \times \square^r$ in which $C_m \simeq \Pi_{p \in \pi(G/Z(G))} Z(G)_p$, $H \simeq \Pi_{p \in \pi(G/Z(G))} Z(G)_p$ and $r \geq 0$ is the torsion-free rank of $Z(G)$ and $G/Z(G)$ has finite exponent.

Hence if $G$ is nilpotent group of class 2, $Z(G) \simeq C_m \times H \times \square^r$ and $G/Z(G)$ has finite exponent then we have $\text{Inn}(G) = \text{Aut}_{\text{pwi}}(G)$. Notice that in this case, $G'$ is cyclic and the equality $\text{Inn}(G) = \text{Aut}_{\text{pwi}}(G)$ also follows from Corollary 1.

Recall that by Corollary 3.6 in [12], in a finite nilpotent group of class 2, if $G'$ is cyclic then $\text{Aut}_{\text{pwi}}(G) = \text{Inn}(G)$. But we cannot hope for a similar conclusion when $G$ is not finite.
For example, consider countably infinitely many copies $H_1, H_2, \ldots$ of a given nilpotent group $H$ of class 2 with cyclic commutator subgroup. Let $G$ (respectively, $\overline{G}$) be the direct product (the cartesian product) of the family $(H_i)_{i>0}$. Clearly, $G$ and $\overline{G}$ are nilpotent of class 2. For each integer $i > 0$, choose an element $a_i \in H_i$ which is not in the center of $H_i$. Then the inner automorphism of $\overline{G}$ defined by $\overline{a}((t_i)_{i>0}) = (a_i^{-1}t_ia_i)_{i>0}$ induces in $G$ a pointwise inner automorphism $\alpha$ which is not inner (see [4]).

However, in a finitely generated nilpotent group of class 2 with cyclic commutator subgroup, we have $\text{Aut}_{\text{pwi}}(G) \approx \text{Inn}(G)$, by Corollary 1. So the structure of $\text{Aut}_{\text{pwi}}(G)$ is determined.

Furthermore it is fairly easy to deduce Corollary 2 from Remark 2.3.

We end this part of the paper with some examples of infinite groups $G$ satisfying the conditions of Corollary 1 and therefore $\text{Aut}_{\text{pwi}}(G) \approx \text{Inn}(G)$.

**Example 2.5** Let $G = \langle x_1, x_2, y_1, y_2 ; x_1^p = x_2^p = y_1^p = y_2^p = 1, [x_1, x_2] = y_1, [y_1, y_2] = [x_1, y_1] = 1 ; 1 \leq i, j \leq 2 \rangle$. Then $G$ satisfies the condition of Corollary 1. We have $G' = \langle y_1 \rangle \cong C_p$, $Z(G) = \langle y_1, y_2 \rangle \cong C_p \times \mathbb{Z}$ and $G/Z(G) = \langle x_1, x_2 \rangle \cong C_p \times C_p$ and hence $\text{Aut}_{\text{pwi}}(G) = \text{Inn}(G)$.

**Example 2.6** Let $G = \langle x_1, x_2 ; [x_1, x_2] = x_1, [x_1, x] = 1 ; 1 \leq i \leq 2 \rangle$. Then $G$ satisfies the condition of Corollary 1. We have $G' = Z(G) = \langle x \rangle \cong \mathbb{Z}$ and $\frac{G}{Z(G)} = \langle x_1, x_2 \rangle \cong \mathbb{Z} \times \mathbb{Z}$ . Hence $\text{Aut}_{\text{pwi}}(G) \approx \text{Inn}(G)$. It is easy to see that in this case every pointwise inner automorphism is inner and so $\text{Aut}_{\text{pwi}}(G) = \text{Inn}(G)$ (see [1, Example 3.4]).

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