On pointwise inner automorphisms of nilpotent groups of class 2

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Abstract

An automorphism \(\theta\) of a group \(G\) is pointwise inner if \(\theta(x)\) is conjugate to \(x\) for any \(x \in G\). The set of all pointwise inner automorphisms of group \(G\), denoted by \(\text{Aut}_{\text{pw}(G)}\) form a subgroups of \(\text{Aut}(G)\) containing \(\text{Inn}(G)\). In this paper, we find a necessary and sufficient condition in certain finitely generated nilpotent groups of class 2 for which \(\text{Aut}_{\text{pw}(G)} = \text{Inn}(G)\). We also prove that in a nilpotent group of class 2 with cyclic commutator subgroup \(\text{Aut}_{\text{pw}(G)} = \text{Inn}(G)\) and the quotient \(\text{Aut}_{\text{pw}(G)}/\text{Inn}(G)\) is torsion. In particular if \(G'\) is a finite cyclic group then \(\text{Aut}_{\text{pw}(G)} = \text{Inn}(G)\).

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Introduction

By definition, a pointwise inner automorphism of a group \(G\) is an automorphism \(\theta: G \rightarrow G\) such that \(t\) and \(\theta(t)\) are conjugate for any \(t \in G\). This notion appears in the famous book of Burnside [1, Note B, p 463]. Denote by \(\text{Aut}_{\text{pw}(G)}\) the set of all pointwise inner automorphisms of \(G\).

Obviously, \(\text{Aut}_{\text{pw}(G)}\) contains \(\text{Inn}(G)\), the group of all inner automorphisms of \(G\). These groups can coincide, for instance when \(G\) is \(S_n, A_n, SL_n(D)\) and \(GL_n(D)\) where \(D\) is an Euclidean domain (see [7], [10], [11]).

By a result of Grossman [5], it turns out that \(\text{Aut}_{\text{pw}(G)} = \text{Inn}(G)\) when \(G\) is a free group. Endimioni in [4] proved that this property remains true in a free nilpotent group.

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Also Yadav in [12] gave a sufficient condition for a finite p-group G of nilpotent class 2 to be such that Aut_{pwi}(G) = Inn(G). But the equality does not hold in general.

In fact, in 1911, Burnside posed the following question: Does there exist any finite group G such that G has a non-inner and pointwise inner automorphism? In 1913, Burnside himself gave an affirmative answer to this question [3]. Indeed, there are many examples of groups admitting a pointwise inner automorphism which is not inner (see, for instance [3], [4], [8], [9], [12] where these groups are besides nilpotent).

Segal also gave a subtle example. He constructed a finitely generated torsion-free nilpotent group G, in which Aut_{pwi}(G)/Inn(G) contains an element of infinite order (see [9]).

In this paper we study the pointwise inner automorphisms of a finitely generated nilpotent group of class 2 with cyclic commutator subgroup.

We introduce the following definition:

**Definition.** Let G be a finitely generated nilpotent group of class 2. Then G/Z(G) is finitely generated abelian group and thus G/Z(G) = \langle x_1 Z(G) \rangle \times \cdots \times \langle x_k Z(G) \rangle \text{ for some } x_1, \ldots, x_k \in G. The group G is called d-group if the following distributive law holds in G,

\[ [x_1^{\alpha_1} \cdots x_k^{\alpha_k}, G] = [x_1, G]^{\alpha_1} \cdots [x_k, G]^{\alpha_k} \]

where \( \alpha_i \in \mathbb{Z} \) and 1 ≤ i ≤ k.

Let G be a 2-generator nilpotent group of class 2. It is straightforward to show that G is a d-group.

To give an example of an infinite d-group, consider the group G with the following presentation

\[ G = \langle x_1, x_2, x_3, x_4, x : [x_i, x_j] = x^{m_{ij}}, [x_i, x] = 1; 1 \leq i \leq 4 \text{ and } i < j \rangle, \]

where \( m_{i+1} = 1 \) for all 1 ≤ i < 4 and \( m_{ij} = 0 \) for all \( i + 1 < j \). Then \( G' = Z(G) = \langle x \rangle \cong \mathbb{Z} \) and \( G/Z(G) = \langle x_1, x_2, x_3, x_4 \rangle \cong \mathbb{Z}^4 \). A quick calculation shows that

\[ [x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} x_4^{\alpha_4}, G] = [x_1, G]^{\alpha_1} [x_2, G]^{\alpha_2} [x_3, G]^{\alpha_3} [x_4, G]^{\alpha_4} = \langle x^\alpha \rangle, \]

Where \( \alpha_i \in \mathbb{Z} \) for all 1 ≤ i ≤ 4 and \( \alpha = \gcd(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \). Therefore G is an infinite d-
group.

Now we give a nilpotent group $G$ of class 2 which is not a $d$-group.

Let $G$ be a free nilpotent group of class 2 on 4 generators $a_1, a_2, a_3$ and $a_4$. If $c_{ij} = [a_i, a_j]$ for $1 \leq i < j \leq 4$, then the relations in $G$ are $[c_{ij}, a_k] = 1$ for $1 \leq i < j \leq 4$ and $1 \leq k \leq 4$, and their consequences. Macdonald in [6] proved that $c_{13}c_{24}$ is not a commutator. Therefore $G$ is not a $d$-group.

**Theorem 1.** Let $G$ be a finitely generated nilpotent group of class 2 and $G/Z(G) = \langle x_1 \rangle \times \cdots \times \langle x_k \rangle$.

(i) There exists a monomorphism $\text{Aut}_{\text{pwi}}(G) \hookrightarrow \prod_{i=1}^{k} \text{Hom}((\langle x_i \rangle), [x_i, G])$.

(ii) If $[x_i, G]$ is cyclic for all $1 \leq i \leq k$, then there exists a monomorphism $\text{Aut}_{\text{pwi}}(G) \hookrightarrow \text{Inn}(G)$.

In particular if $G$ is a $d$-group of class 2 then the monomorphisms in (i) and (ii) are isomorphism. Furthermore $\text{Aut}_{\text{pwi}}(G) \cong \text{Inn}(G)$ if and only if $[x_i, G]$ is cyclic for all $1 \leq i \leq k$.

Notice that if $G$ is a finite group then, as consequence of this result, we derive Theorem 3.5 and Corollary 3.6 of Yadav in [12].

In particular, we derive the following consequences of Theorem 1.

**Corollary 1.** Let $G$ be a finitely generated nilpotent group of class 2 in which $G'$ is cyclic, then $\text{Aut}_{\text{pwi}}(G) \cong \text{Inn}(G)$. In particular if $G'$ is finite, then $\text{Aut}_{\text{pwi}}(G) = \text{Inn}(G)$.

Recall that by Corollary 3.6 in [12], in a finite nilpotent group of class 2, if $G'$ is cyclic then $\text{Aut}_{\text{pwi}}(G) = \text{Inn}(G)$. But we cannot hope for a similar conclusion when $G$ is not finite. We will provide an example in the section 2. However, in a finitely generated nilpotent group of class 2, by Corollary 1 we have $\text{Aut}_{\text{pwi}}(G) \cong \text{Inn}(G)$. So the structure of $\text{Aut}_{\text{pwi}}(G)$ is determined.

**Corollary 2.** Let $G$ be a finitely generated nilpotent group of class 2. If the commutator subgroup of $G$ is cyclic, then $\text{Aut}_{\text{pwi}}(G)/\text{Inn}(G)$ is torsion.

Let $G$ be a group and $N$ be a non-trivial proper normal subgroup of $G$. The pair
(G,N) is called a Camina pair if xN ⊆ x^G for all x ∈ G\N. A group G is called a Camina group if (G,G') is a Camina pair.

Clearly, if G is a Camina group of class 2 then it is a d-group. So, as an immediate consequence of Theorem 1, one readily gets the following corollary.

**Corollary 3.** Let G be a finitely generated nilpotent group of class 2. If G is a Camina group then Aut_{pwi}(G) ∼= Inn(G) if and only if G' is cyclic. Particularly, if G/Z(G) is finite, then Aut_{pwi}(G) = Inn(G) if and only if G' is cyclic.

**Preliminary results**

Our notation is standard. Let G be a group, by $C_m, G'$ and Z(G), we denote the cyclic group of order m, the commutator subgroup and the center of G, respectively.

If $x, y \in G$, then $x^y$ denotes the conjugate element $y^{-1}xy \in G$. For $x \in G$, $x^G$ denotes the conjugacy class of x in G. The commutator of two elements $x, y \in G$ is defined by $[x, y] = x^{-1}y^{-1}xy$ and more generally, the left-normed commutator of n elements $x_1, \ldots, x_n$ is defined inductively by

$$[x_1, \ldots, x_{n-1}, x_n] = [x_1, \ldots, x_{n-1}]^{-1}x_n^{-1}[x_1, \ldots, x_{n-1}]x_n.$$  

If $H \leq G$, $[x, H]$ denotes the set of all $[x, h]$ for $h \in H$, this is a subgroup of G when G is of class 2. For any group H and abelian group K, Hom(H,K) denotes the group of all homomorphisms from H to K. Also $G^*$ is the set of all central automorphisms of G fixing Z(G) elementwise.

Yadav in [12] shows that in a finite nilpotent group of class 2, there exists a monomorphism from Aut_{pwi}(G) into Hom(G/Z(G),G'). It turns out that this result remains true when G is an infinite nilpotent group of class 2.

For that, let G be a nilpotent group (finite or infinite) of class 2. Let $a \in$ Aut_{pwi}(G).

Then the map $g \mapsto g^{-1}a(g)$ is a homomorphism from G into G'. This homomorphism sends Z(G) to 1. So it induces a homomorphism $f_a: G/Z(G) \to G'$, sending $\bar{g} = gZ(G)$ to $g^{-1}a(g)$, for any $g \in G$. Define

$$\text{Hom}_{pwi}(G/Z(G), G') = \{f \in \text{Hom}(\frac{G}{Z(G)}, G'): f(\bar{g}) \in [g, G] \text{ for all } g \in G\}.$$  

To prove Aut_{pwi}(G) ∼= Hom_{pwi}(G/Z(G), G'), we use the following well-known result.
Lemma 1.1 Let $N$ be a normal subgroup of a group $G$. Let $\theta$ be an endomorphism of $G$ such that $\theta(N) \leq N$. Denote by $\tilde{\theta}$ and $\theta_0$ the endomorphisms induced by $\theta$ in $G/N$ and $N$, respectively. If $\tilde{\theta}$ and $\theta_0$ are surjective (injective), then so is $\theta$.

Proposition 1.2 Let $G$ be a nilpotent group of class 2. Then the above map $\varphi: \alpha \mapsto f_{\alpha}$ is an isomorphism from $\text{Aut}_{\text{pwi}}(G)$ into $\text{Hom}_{\text{pwi}}(G/Z(G), G')$.

Proof. Since for any $\alpha \in \text{Aut}_{\text{pwi}}(G)$, by the definition $f_{\alpha} \in \text{Hom}_{\text{pwi}}(G/Z(G), G')$, $\varphi$ is well defined. Let $\alpha_1, \alpha_2 \in \text{Aut}_{\text{pwi}}(G)$ and $g \in G$. We have $\alpha_1(g^{-1}\alpha_2(g)) = g^{-1}\alpha_2(g)$, since $g^{-1}\alpha_2(g) \in G' \leq Z(G)$. This implies that

$$f_{\alpha_1\alpha_2}(g) = g^{-1}\alpha_1(\alpha_2(g)) = g^{-1}\alpha_1(g)g^{-1}\alpha_2(g)$$

$$= g^{-1}\alpha_1(g)g^{-1}\alpha_2(g) = f_{\alpha_1}(g), f_{\alpha_2}(g).$$

Hence $\varphi$ is a homomorphism. Clearly, $\varphi$ is injective. Now it suffices to show that $\varphi$ is surjective.

Let $f$ be any element of $\text{Hom}_{\text{pwi}}(G/Z(G), G')$. By Lemma 1.1 a quick calculation shows that $\varphi(\alpha) = f$, where $\alpha$ is an element of $\text{Aut}_{\text{pwi}}(G)$, sending $g \in G$ to $gf(gZ(G))$. Then we have $\text{Aut}_{\text{pwi}}(G) \cong \text{Hom}_{\text{pwi}}(G/Z(G), G')$.

* Note that if $G$ is a nilpotent group of class 2 then $\text{Aut}_{\text{pwi}}(G) \cong \text{Hom}_{\text{pwi}}(G/Z(G), G')$.

It is easy to see that in a Camina nilpotent group of class 2, $\text{Hom}_{\text{pwi}}(G/Z(G), G') = \text{Hom}(G/Z(G), G')$. Hence if $G$ is a Camina group of class 2, then $\text{Aut}_{\text{pwi}}(G) \cong \text{Hom}(G/Z(G), G')$.

The following well-known facts will be used repeatedly.

Lemma 1.3 Let $A, B$ and $C$ be abelian groups.

(i) $\text{Hom}(A \times B, C) \cong \text{Hom}(A, C) \times \text{Hom}(B, C)$.

(ii) $\text{Hom}(A, B \times C) \cong \text{Hom}(A, B) \times \text{Hom}(A, C)$.

(iii) $\text{Hom}(C_m, C_n) \cong C_d$ where $d = \gcd(m, n)$.

(iv) $\text{Hom}(\mathbb{Z}, A) \cong A$.

(v) If $A$ is torsion group and $B$ is torsion-free group, then $\text{Hom}(A, B) = 1$.

(vi) If $\gcd(|A|, |B|) \neq 1$, then $\text{Hom}(A, B) \neq 1$. 289
Main Result

Let $G$ be a finite abelian group. We denote by $G_p$, the $p$-primary component of $G$. Hence $G = \prod_{p \in \pi(G)} G_p$ where $\pi(G)$ denotes the set of all primes $p$ dividing $|G|$.

To prove Theorem 1, we need the following Lemma.

Lemma 2.1 ([1, Corollary 1.4]) Let $A$ and $B$ be two finite abelian groups and $\exp(A)|\exp(B)$. Then $\text{Hom}(A, B) \simeq A$ if and only if $B \simeq C_m \times H$ in which $C_m \simeq \prod_{p \in \pi(A)} B_p$ and $H \simeq \prod_{p \notin \pi(A)} B_p$. In particular, if $\pi(A) = \pi(B)$ then this is equivalent to $B$ is a cyclic group.

Let $G$ be a finitely generated nilpotent group of class 2. Then $G/Z(G)$ is finitely generated abelian group and thus $G/Z(G) = \langle x_1Z(G) \rangle \times \cdots \times \langle x_kZ(G) \rangle$ for some $x_1, \ldots, x_k \in G$.

Let $f \in \text{Hom}_{\text{pwf}}(G/Z(G), G')$. So $f(gZ(G)) \in [g, G]$ for all $g \in G$. In particular, for all $1 \leq i \leq k$ we have $f(x_iZ(G)) \in [x_i, G]$. Now we prove Theorem 1.

Proof of Theorem 1.

(i) By Proposition 1.2, we have $\text{Aut}_{\text{pwf}}(G) \simeq \text{Hom}_{\text{pwf}}(G/Z(G), G')$. It suffices to show that there exists a monomorphism from $\text{Hom}_{\text{pwf}}(G/Z(G), G')$ into $\prod_{i=1}^k \text{Hom}((\langle x_i \rangle), [x_i, G])$. Let $f \in \text{Hom}_{\text{pwf}}(G/Z(G), G')$. Denote by $f_i$, the homomorphism induced by $f$ in $\langle x_i \rangle$, for all $1 \leq i \leq k$. Since $G$ is a nilpotent group of class 2, we have $[a^m, b] = [a, b]^m = [a, b^m]$ for each $a, b \in G$ and $m \in \mathbb{Z}$. Consequently, $[x_i^m, G] \leq [x_i, G]$ for all $m \in \mathbb{Z}$ and $1 \leq i \leq k$. Therefore $f_i \in \text{Hom}((\langle x_i \rangle), [x_i, G])$. Thus the map $\alpha$ sending any $f \in \text{Hom}_{\text{pwf}}(G/Z(G), G')$ to $\alpha(f) = (f_1, \ldots, f_k) \in \prod_{i=1}^k \text{Hom}((\langle x_i \rangle), [x_i, G])$ is well defined. Now we prove that this map is a monomorphism. Since $(fg)_i = f_i g_i$ for each $f, g \in \text{Hom}_{\text{pwf}}(G/Z(G), G')$ and $1 \leq i \leq k$, $\alpha$ is homomorphism. Clearly, $\text{ker} \alpha$ is trivial, this implies that $\alpha$ is monomorphism. Hence the proof of (i) is complete.

(ii) First we show that $[x_i, G]$ is finite if and only if $\langle x_i \rangle$ is finite, and further
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For this, let 
\([x_i, G] = n\). Since \(G\) is a nilpotent group of class 2, we have 
\([x_i^n, g] = [x_i, g]^n = 1\) for all \(g \in G\) and so \(x_i^n \in Z(G)\).

Hence \((\langle x_i \rangle)\) is finite and \(|\langle x_i \rangle| = n\). Conversely if \(|\langle x_i \rangle| = m\) then \(x_i^m \in Z(G)\) and 
\([x_i, G]^m = [x_i^n, G] = 1\). Consequently \([x_i, G]\) is finite and \(\exp([x_i, G]) = n|m\).

Therefore in this case, \(m = n\). Hence by Lemma 2.1, for all \(1 \leq i \leq k\) we have 
\(\text{Hom}((\langle x_i \rangle), [x_i, G]) \cong (\langle x_i \rangle)\) if and only if \([x_i, G]\) is cyclic.

Now from (i), we have a monomorphism from \(\text{Aut}_{\text{pwli}}(G)\) into \(\prod_{i=1}^{k} \text{Hom}((\langle x_i \rangle), [x_i, G])\) and therefore we conclude that there exists a monomorphism \(\text{Aut}_{\text{pwli}}(G) \hookrightarrow G/Z(G)\), this completes the proof of (ii).

If \(G\) is a d-group, then it is easy to see that the monomorphism defined in (i) is an isomorphism from \(\text{Aut}_{\text{pwli}}(G)\) into \(\prod_{i=1}^{k} \text{Hom}((\langle x_i \rangle), [x_i, G])\).

Finally to complete the proof, it is sufficient to show that if \(\text{Aut}_{\text{pwli}}(G) \cong \text{Inn}(G)\), then 
\([x_i, G]\) is cyclic for all \(1 \leq i \leq k\). Since \(\text{Aut}_{\text{pwli}}(G) \cong \text{Inn}(G)\), by Proposition 1.2 we have 
\(G/Z(G) \cong \text{Hom}_{\text{pwli}}(G/Z(G), G')\). On the other hand, \(G\) is a d-group and hence 
\(\text{Hom}_{\text{pwli}}(G/Z(G), G') \cong \prod_{i=1}^{k} \text{Hom}((\langle x_i \rangle), [x_i, G]).\)

It follows that 
\(G/Z(G) = \langle x_1 \rangle \times \ldots \times \langle x_k \rangle \cong \prod_{i=1}^{k} \text{Hom}((\langle x_i \rangle), [x_i, G]).\)

Now we may assume that \(\langle x_1 \rangle \times \ldots \times \langle x_n \rangle\) is the torsion part and \(\langle x_{n+1} \rangle \times \ldots \times \langle x_k \rangle\) is the torsion-free part of \(G/Z(G)\). Since for all \(1 \leq i \leq n\), \(\exp([x_i, G]) = \exp(\langle x_i \rangle) = |\langle x_i \rangle|\) and 
\(\prod_{i=1}^{n} \text{Hom}((\langle x_i \rangle), [x_i, G]) \cong \langle x_1 \rangle \times \ldots \times \langle x_n \rangle\), \(\text{Hom}((\langle x_i \rangle), [x_i, G]) \cong (\langle x_i \rangle)\) for all \(1 \leq i \leq n\) and hence \([x_i, G]\) is cyclic. Furthermore, we have 
\(\prod_{i=n+1}^{k} \text{Hom}((\langle x_i \rangle), [x_i, G]) \cong \langle x_{n+1} \rangle \times \ldots \times \langle x_k \rangle \cong \mathbb{Z}^{k-n}.

Now we have \(\text{Hom}((\langle x_i \rangle), [x_i, G]) \cong [x_i, G]\), since \(\langle x_i \rangle \cong \mathbb{Z}\) and hence \(\prod_{i=n+1}^{m} [x_i, G] \cong \mathbb{Z}^{k-n}\). That is \([x_i, G] \cong \mathbb{Z}\) for all \(n + 1 \leq i \leq k\). This implies that \([x_i, G]\) is cyclic for all \(1 \leq i \leq k\), as required.

*Notice that if \(G\) is a finite group then, as a consequence of this result, we derive
Theorem 3.5 and Corollary 3.6 of Yadav in [12].

The following corollary is an easy consequence of the above theorem.

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Corollary 2.2 Let $G$ be a finitely generated nilpotent group of class 2 with cyclic commutator subgroup. Then there exists a monomorphism from $\text{Aut}_{pwi}(G)$ into $\text{Inn}(G)$ or equivalently $\text{Aut}_{pwi}(G)$ is isomorphic to a subgroup of $G/Z(G)$.

Remark 2.3 We keep here the notation used in Theorem 1.

(i) By the discussion of (ii) in Theorem 1, if $G'$ is finite cyclic, then $G/Z(G)$ is finite and $|\text{Aut}_{pwi}(G)| \leq |\text{Inn}(G)| = |G/Z(G)|$. On the other hand, $\text{Inn}(G) \leq \text{Aut}_{pwi}(G)$, conclude that $\text{Aut}_{pwi}(G) = \text{Inn}(G)$. Note that in this case, $G$ is not necessarily finite.

(ii) If $G'$ is infinite cyclic, it follows from the discussion of (ii) in Theorem 1, that $G/Z(G)$ is a free abelian group of finite rank, say $r(G/Z(G)) = k$. We certainly have $\text{Inn}(G) \leq \text{Aut}_{pwi}(G)$ and thus $r(\text{Inn}(G)) \leq r(\text{Aut}_{pwi}(G))$. Also $r(\text{Aut}_{pwi}(G)) \leq r(\text{Inn}(G))$, since $\text{Aut}_{pwi}(G)$ is isomorphic to a subgroup of $\text{Inn}(G)$. Therefore $\text{Aut}_{pwi}(G)$ and $\text{Inn}(G)$ have the same rank and hence $\text{Aut}_{pwi}(G) \cong \text{Inn}(G)$.

Now it is easy to deduce Corollary 1 from Remark 2.3.

Remark 2.4 It is known that in a nilpotent groups of class 2, $\text{Inn}(G) \leq \text{Aut}_{pwi}(G) \leq C^*$. So $\text{Inn}(G) = \text{Aut}_{pwi}(G)$ when $\text{Inn}(G) = C^*$. In [1] we characterized all non torsion-free finitely generated groups in which $\text{Inn}(G) = C^*$. We proved that $\text{Inn}(G) = C^*$ if and only if $G$ is an abelian group or nilpotent of class 2 and $Z(G) \cong C_m \times H \times \square^r$ in which $C_m \cong \Pi_{p \in \pi(G/Z(G))} Z(G)_p$, $H \cong \Pi_{p \notin \pi(G/Z(G))} Z(G)_p$ and $r \geq 0$ is the torsion-free rank of $Z(G)$ and $G/Z(G)$ has finite exponent.

Hence if $G$ is nilpotent group of class 2, $Z(G) \cong C_m \times H \times \square^r$ and $G/Z(G)$ has finite exponent then we have $\text{Inn}(G) = \text{Aut}_{pwi}(G)$. Notice that in this case, $G'$ is cyclic and the equality $\text{Inn}(G) = \text{Aut}_{pwi}(G)$ also follows from Corollary 1.

Recall that by Corollary 3.6 in [12], in a finite nilpotent group of class 2, if $G'$ is cyclic then $\text{Aut}_{pwi}(G) = \text{Inn}(G)$. But we cannot hope for a similar conclusion when $G$ is not finite.

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For example, consider countably infinitely many copies $H_1, H_2, \ldots$ of a given nilpotent group $H$ of class 2 with cyclic commutator subgroup. Let $G$ (respectively, $\overline{G}$) be the direct product (the cartesian product) of the family $(H_i)_{i>0}$. Clearly, $G$ and $\overline{G}$ are nilpotent of class 2. For each integer $i > 0$, choose an element $a_i \in H_i$ which is not in the center of $H_i$. Then the inner automorphism of $\overline{G}$ defined by $\overline{a}(\overline{(t_i)_{i>0}}) = (a_i^{-1}t_i a_i)_{i>0}$ induces in $G$ a pointwise inner automorphism $\alpha$ which is not inner (see [4]).

However, in a finitely generated nilpotent group of class 2 with cyclic commutator subgroup, we have $\text{Aut}_{pwi}(G) \cong \text{Inn}(G)$, by Corollary 1. So the structure of $\text{Aut}_{pwi}(G)$ is determined.

Furthermore it is fairly easy to deduce Corollary 2 from Remark 2.3.

We end this part of the paper with some examples of infinite groups $G$ satisfying the conditions of Corollary 1 and therefore $\text{Aut}_{pwi}(G) \cong \text{Inn}(G)$.

**Example 2.5** Let $G = \langle x_1, x_2, y_1, y_2; x_1^p = x_2^p = y_1^p = 1, [x_1, x_2] = y_1, [y_1, y_2] = [x_1, y_2] = 1; 1 \leq i, j \leq 2 \rangle$. Then $G$ satisfies the condition of Corollary 1. We have $G' = \langle y_1 \rangle \cong C_p$, $Z(G) = \langle y_1, y_2 \rangle \cong C_p \times \mathbb{Z}$ and $G/Z(G) = \langle \overline{x_1}, \overline{x_2} \rangle \cong C_p \times C_p$ and hence $\text{Aut}_{pwi}(G) = \text{Inn}(G)$.

**Example 2.6** Let $G = \langle x_1, x_2; [x_1, x_2] = x_1 [x_1, x] = 1; 1 \leq i \leq 2 \rangle$. Then $G$ satisfies the condition of Corollary 1. We have $G' = Z(G) = \langle x \rangle \cong \mathbb{Z}$ and $\frac{G}{Z(G)} = \langle \overline{x_1}, \overline{x_2} \rangle \cong \mathbb{Z} \times \mathbb{Z}$. Hence $\text{Aut}_{pwi}(G) \cong \text{Inn}(G)$. It is easy to see that in this case every pointwise inner automorphism is inner and so $\text{Aut}_{pwi}(G) = \text{Inn}(G)$ (see [1, Example 3.4]).

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