On pointwise inner automorphisms of nilpotent groups of class 2

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Abstract

An automorphism $\theta$ of a group $G$ is pointwise inner if $\theta(x)$ is conjugate to $x$ for any $x \in G$. The set of all pointwise inner automorphisms of group $G$, denoted by $\text{Aut}_{\text{pwi}}(G)$, form a subgroups of $\text{Aut}(G)$ containing $\text{Inn}(G)$. In this paper, we find a necessary and sufficient condition in certain finitely generated nilpotent groups of class 2 for which $\text{Aut}_{\text{pwi}}(G) = \text{Inn}(G)$. We also prove that in a nilpotent group of class 2 with cyclic commutator subgroup $\text{Aut}_{\text{pwi}}(G) = \text{Inn}(G)$ and the quotient $\text{Aut}_{\text{pwi}}(G)/\text{Inn}(G)$ is torsion. In particular if $G'$ is a finite cyclic group then $\text{Aut}_{\text{pwi}}(G) = \text{Inn}(G)$.

MSC: Primary 20D45; Secondary 20E36

Introduction

By definition, a pointwise inner automorphism of a group $G$ is an automorphism $\theta: G \to G$ such that $t$ and $\theta(t)$ are conjugate for any $t \in G$. This notion appears in the famous book of Burnside [1, Note B, p 463]. Denote by $\text{Aut}_{\text{pwi}}(G)$ the set of all pointwise inner automorphisms of $G$.

Obviously, $\text{Aut}_{\text{pwi}}(G)$ contains $\text{Inn}(G)$, the group of all inner automorphisms of $G$. These groups can coincide, for instance when $G$ is $S_n, A_n, SL_n(D)$ and $GL_n(D)$ where $D$ is an Euclidean domain (see [7], [10], [11]).

By a result of Grossman [5], it turns out that $\text{Aut}_{\text{pwi}}(G) = \text{Inn}(G)$ when $G$ is a free group. Endimioni in [4] proved that this property remains true in a free nilpotent group.

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Also Yadav in [12] gave a sufficient condition for a finite p-group G of nilpotent class 2 to be such that $\text{Aut}_{\text{pwi}}(G) = \text{Inn}(G)$. But the equality does not hold in general.

In fact, in 1911, Burnside posed the following question: Does there exist any finite group G such that G has a non-inner and pointwise inner automorphism? In 1913, Burnside himself gave an affirmative answer to this question [3]. Indeed, there are many examples of groups admitting a pointwise inner automorphism which is not inner (see, for instance [3], [4], [8], [9], [12] where these groups are besides nilpotent).

Segal also gave a subtle example. He constructed a finitely generated torsion-free nilpotent group G, in which $\text{Aut}_{\text{pwi}}(G) / \text{Inn}(G)$ contains an element of infinite order (see [9]).

In this paper we study the pointwise inner automorphisms of a finitely generated nilpotent group of class 2 with cyclic commutator subgroup.

We introduce the following definition:

**Definition.** Let G be a finitely generated nilpotent group of class 2. Then $G/Z(G)$ is finitely generated abelian group and thus $G/Z(G) = \langle x_1 Z(G) \rangle \times \cdots \times \langle x_k Z(G) \rangle$ for some $x_1, \ldots, x_k \in G$. The group G is called **d-group** if the following distributive law holds in G,

$$[x_1^{\alpha_1} \cdots x_k^{\alpha_k}, G] = [x_1, G]^{\alpha_1} \cdots [x_k, G]^{\alpha_k}$$

where $\alpha_i \in \mathbb{Z}$ and $1 \leq i \leq k$.

Let G be a 2-generator nilpotent group of class 2. It is straightforward to show that G is a d-group.

To give an example of an infinite d-group, consider the group G with the following presentation

$$G = \langle x_1, x_2, x_3, x_4, x : [x_i, x_j] = x^{m_{ij}}, [x_i, x] = 1 ; 1 \leq i \leq 4 \text{ and } i < j \rangle,$$

where $m_{i+1} = 1$ for all $1 \leq i < 4$ and $m_{ij} = 0$ for all $i + 1 < j$. Then $G' = Z(G) = \langle x \rangle \cong \mathbb{Z}$ and $G/Z(G) = \langle x_1, x_2, x_3, x_4 \rangle \cong \mathbb{Z}^4$. A quick calculation shows that

$$[x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} x_4^{\alpha_4}, G] = [x_1, G]^{\alpha_1} [x_2, G]^{\alpha_2} [x_3, G]^{\alpha_3} [x_4, G]^{\alpha_4} = \langle x^{\alpha} \rangle,$$

Where $\alpha_i \in \mathbb{Z}$ for all $1 \leq i \leq 4$ and $\alpha = \gcd(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. Therefore G is an infinite d-
group.

Now we give a nilpotent group $G$ of class 2 which is not a d-group.

Let $G$ be a free nilpotent group of class 2 on 4 generators $a_1, a_2, a_3$ and $a_4$. If $c_{ij} = [a_i, a_j]$ for $1 \leq i < j \leq 4$, then the relations in $G$ are $[c_{ij}, a_k] = 1$ for $1 \leq i < j \leq 4$ and $1 \leq k \leq 4$, and their consequences. Macdonald in [6] proved that $c_{13}c_{24}$ is not a commutator. Therefore $G$ is not a d-group.

**Theorem 1.** Let $G$ be a finitely generated nilpotent group of class 2 and $G/Z(G) = \langle x_1 \rangle \times \ldots \times \langle x_k \rangle$.

(i) There exists a monomorphism $\text{Aut}_{pwi}(G) \hookrightarrow \prod_{i=1}^{k} \text{Hom}(\langle x_i \rangle, [x_i, G])$.

(ii) If $[x_i, G]$ is cyclic for all $1 \leq i \leq k$, then there exists a monomorphism $\text{Aut}_{pwi}(G) \hookrightarrow \text{Inn}(G)$.

In particular if $G$ is a d-group of class 2 then the monomorphisms in (i) and (ii) are isomorphism. Furthermore $\text{Aut}_{pwi}(G) \cong \text{Inn}(G)$ if and only if $[x_i, G]$ is cyclic for all $1 \leq i \leq k$.

Notice that if $G$ is a finite group then, as consequence of this result, we derive Theorem 3.5 and Corollary 3.6 of Yadav in [12].

In particular, we derive the following consequences of Theorem 1.

**Corollary 1.** Let $G$ be a finitely generated nilpotent group of class 2 in which $G'$ is cyclic, then $\text{Aut}_{pwi}(G) \cong \text{Inn}(G)$. In particular if $G'$ is finite, then $\text{Aut}_{pwi}(G) = \text{Inn}(G)$.

Recall that by Corollary 3.6 in [12], in a finite nilpotent group of class 2, if $G'$ is cyclic then $\text{Aut}_{pwi}(G) = \text{Inn}(G)$. But we cannot hope for a similar conclusion when $G$ is not finite. We will provide an example in the section 2. However, in a finitely generated nilpotent group of class 2, by Corollary 1 we have $\text{Aut}_{pwi}(G) \cong \text{Inn}(G)$. So the structure of $\text{Aut}_{pwi}(G)$ is determined.

**Corollary 2.** Let $G$ be a finitely generated nilpotent group of class 2. If the commutator subgroup of $G$ is cyclic, then $\text{Aut}_{pwi}(G)/\text{Inn}(G)$ is torsion.

Let $G$ be a group and $N$ be a non-trivial proper normal subgroup of $G$. The pair
On pointwise inner automorphisms of nilpotent groups of class 2

zahedeh azhdari & et al.

$G_N$ is called a Camina pair if $xN \leq x^G$ for all $x \in G \setminus N$. A group $G$ is called a Camina group if $(G, G')$ is a Camina pair.

Clearly, if $G$ is a Camina group of class 2 then it is a $d$-group. So, as an immediate consequence of Theorem 1, one readily gets the following corollary.

**Corollary 3.** Let $G$ be a finitely generated nilpotent group of class 2. If $G$ is a Camina group then $\text{Aut}_{\text{pwi}}(G) \cong \text{Inn}(G)$ if and only if $G'$ is cyclic. Particularly, if $G/Z(G)$ is finite, then $\text{Aut}_{\text{pwi}}(G) = \text{Inn}(G)$ if and only if $G'$ is cyclic.

**Preliminary results**

Our notation is standard. Let $G$ be a group, by $C_m, G'$ and $Z(G)$, we denote the cyclic group of order $m$, the commutator subgroup and the center of $G$, respectively.

If $x, y \in G$, then $x^y$ denotes the conjugate element $y^{-1}xy \in G$. For $x \in G$, $x^G$ denotes the conjugacy class of $x$ in $G$. The commutator of two elements $x, y \in G$ is defined by $[x, y] = x^{-1}y^{-1}xy$ and more generally, the left-normed commutator of $n$ elements $x_1, \ldots, x_n$ is defined inductively by

$$[x_1, \ldots, x_{n-1}, x_n] = [x_1, \ldots, x_{n-1}]^{-1}x_n^{-1}[x_1, \ldots, x_{n-1}]x_n.$$  

If $H \leq G$, $[x, H]$ denotes the set of all $[x, h]$ for $h \in H$, this is a subgroup of $G$ when $G$ is of class 2. For any group $H$ and abelian group $K$, $\text{Hom}(H, K)$ denotes the group of all homomorphisms from $H$ to $K$. Also $C^*$ is the set of all central automorphisms of $G$ fixing $Z(G)$ elementwise.

Yadav in [12] shows that in a finite nilpotent group of class 2, there exists a monomorphism from $\text{Aut}_{\text{pwi}}(G)$ into $\text{Hom}(G/Z(G), G')$. It turns out that this result remains true when $G$ is an infinite nilpotent group of class 2.

For that, let $G$ be a nilpotent group (finite or infinite) of class 2. Let $\alpha \in \text{Aut}_{\text{pwi}}(G)$. Then the map $g \mapsto g^{-1}\alpha(g)$ is a homomorphism from $G$ into $G'$. This homomorphism sends $Z(G)$ to 1. So it induces a homomorphism $f_\alpha: G/Z(G) \to G'$, sending $\overline{g} = gZ(G)$ to $g^{-1}\alpha(g)$, for any $g \in G$. Define

$$\text{Hom}_{\text{pwi}}(G/Z(G), G') = \{f \in \text{Hom}\left(\frac{G}{Z(G)}, G'\right) : f(\overline{g}) \in [g, G] \text{ for all } g \in G\}.$$  

To prove $\text{Aut}_{\text{pwi}}(G) \cong \text{Hom}_{\text{pwi}}(G/Z(G), G')$, we use the following well-known result.
Lemma 1.1 Let $N$ be a normal subgroup of a group $G$. Let $\theta$ be an endomorphism of $G$ such that $\theta(N) \leq N$. Denote by $\bar{\theta}$ and $\theta_0$ the endomorphisms induced by $\theta$ in $G/N$ and $N$, respectively. If $\bar{\theta}$ and $\theta_0$ are surjective (injective), then so is $\theta$.

Proposition 1.2 Let $G$ be a nilpotent group of class 2. Then the above map $\phi: \alpha \mapsto f_\alpha$ is an isomorphism from $\text{Aut}_{pwi}(G)$ into $\text{Hom}_{pwi}(G/Z(G), G')$. 

Proof. Since for any $\alpha \in \text{Aut}_{pwi}(G)$, by the definition $f_\alpha \in \text{Hom}_{pwi}(G/Z(G), G')$, $\phi$ is well defined. Let $\alpha_1, \alpha_2 \in \text{Aut}_{pwi}(G)$ and $g \in G$. We have $\alpha_1(g^{-1} \alpha_2(g)) = g^{-1} \alpha_2(g)$, since $g^{-1} \alpha_2(g) \in G' \leq Z(G)$. This implies that

\[ f_{\alpha_1 \cdot \alpha_2}(g) = g^{-1} \alpha_1(\alpha_2(g)) = g^{-1} \alpha_1(gg^{-1} \alpha_2(g)) = g^{-1} \alpha_1(g) \cdot g^{-1} \alpha_2(g) = f_{\alpha_1}(g) \cdot f_{\alpha_2}(g). \]

Hence $\phi$ is a homomorphism. Clearly, $\phi$ is injective. Now it suffices to show that $\phi$ is surjective.

Let $f$ be any element of $\text{Hom}_{pwi}(G/Z(G), G')$. By Lemma 1.1 a quick calculation shows that $\phi(\alpha) = f$, where $\alpha$ is an element of $\text{Aut}_{pwi}(G)$, sending $g \in G$ to $gf(gZ(G))$. Then we have $\text{Aut}_{pwi}(G) \cong \text{Hom}_{pwi}(G/Z(G), G')$.

* Note that if $G$ is a nilpotent group of class 2 then $\text{Aut}_{pwi}(G) \cong \text{Hom}_{pwi}(G/Z(G), G')$.

It is easy to see that in a Camina nilpotent group of class 2, $\text{Hom}_{pwi}(G/Z(G), G') = \text{Hom}(G/Z(G), G')$. Hence if $G$ is a Camina group of class 2, then $\text{Aut}_{pwi}(G) \cong \text{Hom}(G/Z(G), G')$.

The following well-known facts will be used repeatedly.

Lemma 1.3 Let $A, B$ and $C$ be abelian groups.

(i) $\text{Hom}(A \times B, C) \cong \text{Hom}(A, C) \times \text{Hom}(B, C)$.

(ii) $\text{Hom}(A, B \times C) \cong \text{Hom}(A, B) \times \text{Hom}(A, C)$.

(iii) $\text{Hom}(C_m, C_n) \cong C_d$ where $d = \gcd(m, n)$.

(iv) $\text{Hom}(\mathbb{Z}, A) \cong A$.

(v) If $A$ is torsion group and $B$ is torsion-free group, then $\text{Hom}(A, B) = 1$.

(vi) If $\gcd(|A|, |B|) \neq 1$, then $\text{Hom}(A, B) \neq 1$. 

289
Main Result

Let $G$ be a finite abelian group. We denote by $G_p$, the $p$-primary component of $G$. Hence $G = \prod_{p \in \pi(G)} G_p$ where $\pi(G)$ denotes the set of all primes $p$ dividing $|G|$.

To prove Theorem 1, we need the following Lemma.

**Lemma 2.1** ([1, Corollary 1.4]) Let $A$ and $B$ be two finite abelian groups and $\exp(A)\exp(B)$. Then $\text{Hom}(A,B) \simeq A$ if and only if $B \simeq C_m \times H$ in which $C_m \simeq \prod_{p \in \pi(A)} B_p$ and $H \simeq \prod_{p \not\in \pi(A)} B_p$. In particular, if $\pi(A) = \pi(B)$ then this is equivalent to $B$ is a cyclic group.

Let $G$ be a finitely generated nilpotent group of class 2. Then $G/Z(G)$ is finitely generated abelian group and thus $G/Z(G) = \langle x_1 Z(G) \rangle \times \cdots \times \langle x_k Z(G) \rangle$ for some $x_1, \ldots, x_k \in G$.

Let $f \in \text{Hom}_{pw1}(G/Z(G), G')$. So $f(gZ(G)) \in [g, G]$ for all $g \in G$. In particular, for all $1 \leq i \leq k$ we have $f(x_i Z(G)) \in [x_i, G]$. Now we prove Theorem 1.

**Proof of Theorem 1.**

(i) By Proposition 1.2, we have $\text{Aut}_{pw1}(G) \simeq \text{Hom}_{pw1}(G/Z(G), G')$. It suffices to show that there exists a monomorphism from $\text{Hom}_{pw1}(G/Z(G), G')$ into $\prod_{i=1}^{k} \text{Hom}(\langle \overline{x_i} \rangle, [x_i, G])$. Let $f \in \text{Hom}_{pw1}(G/Z(G), G')$. Denote by $f_i$, the homomorphism induced by $f$ in $\langle \overline{x_i} \rangle$, for all $1 \leq i \leq k$. Since $G$ is a nilpotent group of class 2, we have $[a^m, b] = [a, b]^m = [a, b]^m$ for each $a, b \in G$ and $m \in \mathbb{Z}$. Consequently, $[x_i^m, G] \leq [x_i, G]$ for all $m \in \mathbb{Z}$ and $1 \leq i \leq k$. Therefore $f_i \in \text{Hom}(\langle \overline{x_i} \rangle, [x_i, G])$. Thus the map $\alpha$ sending any $f \in \text{Hom}_{pw1}(G/Z(G), G')$ to $\alpha(f) = (f_1, \ldots, f_k) \in \prod_{i=1}^{k} \text{Hom}(\langle \overline{x_i} \rangle, [x_i, G])$ is well defined. Now we prove that this map is a monomorphism. Since $(fg)_i = f_ig_i$ for each $f, g \in \text{Hom}_{pw1}(G/Z(G), G')$ and $1 \leq i \leq k$, $\alpha$ is homomorphism. Clearly, $\ker \alpha$ is trivial, this implies that $\alpha$ is monomorphism. Hence the proof of (i) is complete.

(ii) First we show that $[x_i, G]$ is finite if and only if $\langle \overline{x_i} \rangle$ is finite, and further
exp([x_i, G]) = exp(⟨x_i⟩) = |x_i|. For this, let [[x_i, G]] = n. Since G is a nilpotent group of class 2, we have [x_i^n, G] = [x_i, G]^n = 1 for all g ∈ G and so x_i^n ∈ Z(G). Hence ⟨x_i⟩ is finite and |x_i|/n. Conversely if |x_i| = m then x_i^m ∈ Z(G) and [x_i, G]^m = [x_i^m, G] = 1. Consequently [x_i, G] is finite and exp([x_i, G]) = n|m. Therefore in this case, m = n. Hence by Lemma 2.1, for all 1 ≤ i ≤ k we have Hom(⟨x_i⟩), [x_i, G]) = ⟨x_i⟩ if and only if [x_i, G] is cyclic.

Now from (i), we have a monomorphism from Aut_{pwi}(G) into \( \prod_{i=1}^{k} \text{Hom}(⟨x_i⟩), [x_i, G]) \) and therefore we conclude that there exists a monomorphism Aut_{pwi}(G) ≅ G/Z(G), this completes the proof of (ii).

If G is a d-group, then it is easy to see that the monomorphism defined in (i) is an isomorphism from Aut_{pwi}(G) into \( \prod_{i=1}^{k} \text{Hom}(⟨x_i⟩), [x_i, G]) \).

Finally to complete the proof, it is sufficient to show that if Aut_{pwi}(G) ≅ Inn(G), then [x_i, G] is cyclic for all 1 ≤ i ≤ k. Since Aut_{pwi}(G) ≅ Inn(G), by Proposition 1.2 we have G/Z(G) ≅ Hom_{pwi}(G/Z(G), G'). On the other hand, G is a d-group and hence

\[ \text{Hom}_{pwi}(G/Z(G), G') \cong \prod_{i=1}^{k} \text{Hom}(⟨x_i⟩), [x_i, G]) \]

It follows that

\[ G/Z(G) = ⟨x_1⟩ \times ... × ⟨x_k⟩ \cong \prod_{i=1}^{k} \text{Hom}(⟨x_i⟩), [x_i, G]). \]

Now we may assume that ⟨x_1⟩ × ... × ⟨x_n⟩ is the torsion part and ⟨x_{n+1}⟩ × ... × ⟨x_k⟩ is the torsion-free part of G/Z(G). Since for all 1 ≤ i ≤ n, exp([x_i, G]) = exp(⟨x_i⟩) = |x_i| and \( \prod_{i=1}^{n} \text{Hom}(⟨x_i⟩), [x_i, G]) \cong ⟨x_1⟩ \times ... × ⟨x_n⟩, \text{ Hom}(⟨x_i⟩), [x_i, G]) \cong ⟨x_i⟩ \) for all 1 ≤ i ≤ n and hence [x_i, G] is cyclic. Furthermore, we have

\[ \prod_{i=n+1}^{k} \text{Hom}(⟨x_i⟩), [x_i, G]) \cong ⟨x_{n+1}⟩ \times ... × ⟨x_k⟩ \cong \mathbb{Z}^{k-n}. \]

Now we have Hom(⟨x_i⟩), [x_i, G]) ≅ [x_i, G], since ⟨x_i⟩ ≅ \mathbb{Z} and hence \( \prod_{i=n+1}^{m} [x_i, G] \cong \mathbb{Z}^{k-n}. \). That is [x_i, G] ≅ \mathbb{Z} for all n + 1 ≤ i ≤ k. This implies that [x_i, G] is cyclic for all 1 ≤ i ≤ k, as required.

*Notice that if G is a finite group then, as a consequence of this result, we derive Theorem 3.5 and Corollary 3.6 of Yadav in [12].

The following corollary is an easy consequence of the above theorem.
Corollary 2.2 Let $G$ be a finitely generated nilpotent group of class 2 with cyclic commutator subgroup. Then there exists a monomorphism from $\text{Aut}_{\text{pwf}}(G)$ into $\text{Inn}(G)$ or equivalently $\text{Aut}_{\text{pwf}}(G)$ is isomorphic to a subgroup of $G/Z(G)$.

Remark 2.3 We keep here the notation used in Theorem 1.

(i) By the discussion of (ii) in Theorem 1, if $G'$ is finite cyclic, then $G/Z(G)$ is finite and $|\text{Aut}_{\text{pwf}}(G)| \leq |\text{Inn}(G)| = |G/Z(G)|$. On the other hand, $\text{Inn}(G) \leq \text{Aut}_{\text{pwf}}(G)$ conclude that $\text{Aut}_{\text{pwf}}(G) = \text{Inn}(G)$. Note that in this case, $G$ is not necessarily finite.

(ii) If $G'$ is infinite cyclic, it follows from the discussion of (ii) in Theorem 1, that $G/Z(G)$ is a free abelian group of finite rank, say $r(G/Z(G)) = k$. We certainly have $\text{Inn}(G) \leq \text{Aut}_{\text{pwf}}(G)$ and thus $r(\text{Inn}(G)) \leq r(\text{Aut}_{\text{pwf}}(G))$. Also $r(\text{Aut}_{\text{pwf}}(G)) \leq r(\text{Inn}(G))$, since $\text{Aut}_{\text{pwf}}(G)$ is isomorphic to a subgroup of $\text{Inn}(G)$. Therefore $\text{Aut}_{\text{pwf}}(G)$ and $\text{Inn}(G)$ have the same rank and hence $\text{Aut}_{\text{pwf}}(G) \simeq \text{Inn}(G)$.

Now it is easy to deduce Corollary 1 from Remark 2.3.

Remark 2.4 It is known that in a nilpotent groups of class 2, $\text{Inn}(G) \leq \text{Aut}_{\text{pwf}}(G) \leq C^*$. So $\text{Inn}(G) = \text{Aut}_{\text{pwf}}(G)$ when $\text{Inn}(G) = C^*$. In [1] we characterized all non torsion-free finitely generated groups in which $\text{Inn}(G) = C^*$. We proved that $\text{Inn}(G) = C^*$ if and only if $G$ is an abelian group or nilpotent of class 2 and $Z(G) \simeq C_m \times H \times \square^r$ in which $C_m \simeq \Pi_{p \in \pi(G/Z(G))} Z(G)_p$, $H \simeq \Pi_{p \in \pi(G/Z(G))} Z(G)_p$ and $r \geq 0$ is the torsion-free rank of $Z(G)$ and $G/Z(G)$ has finite exponent.

Hence if $G$ is nilpotent group of class 2, $Z(G) \simeq C_m \times H \times \square^r$ and $G/Z(G)$ has finite exponent then we have $\text{Inn}(G) = \text{Aut}_{\text{pwf}}(G)$. Notice that in this case, $G'$ is cyclic and the equality $\text{Inn}(G) = \text{Aut}_{\text{pwf}}(G)$ also follows from Corollary 1.

Recall that by Corollary 3.6 in [12], in a finite nilpotent group of class 2, if $G'$ is cyclic then $\text{Aut}_{\text{pwf}}(G) = \text{Inn}(G)$. But we cannot hope for a similar conclusion when $G$ is not finite.
For example, consider countably infinitely many copies $H_1, H_2, \ldots$ of a given nilpotent group $H$ of class 2 with cyclic commutator subgroup. Let $G$ (respectively, $\overline{G}$) be the direct product (the cartesian product) of the family $(H_i)_{i>0}$. Clearly, $G$ and $\overline{G}$ are nilpotent of class 2. For each integer $i > 0$, choose an element $a_i \in H_i$ which is not in the center of $H_i$. Then the inner automorphism of $\overline{G}$ defined by $\overline{a}((t_i)_{i>0}) = (a_i^{-1}t_ia_i)_{i>0}$ induces in $G$ a pointwise inner automorphism $a$ which is not inner (see [4]).

However, in a finitely generated nilpotent group of class 2 with cyclic commutator subgroup, we have $\text{Aut}_{\text{pwi}}(G) \cong \text{Inn}(G)$, by Corollary 1. So the structure of $\text{Aut}_{\text{pwi}}(G)$ is determined.

Furthermore it is fairly easy to deduce Corollary 2 from Remark 2.3.

We end this part of the paper with some examples of infinite groups $G$ satisfying the conditions of Corollary 1 and therefore $\text{Aut}_{\text{pwi}}(G) \cong \text{Inn}(G)$.

**Example 2.5** Let $G = \langle x_1, x_2, y_1, y_2; x_1^p = x_2^p = y_1^p = y_2^p = 1, [x_1, x_2] = y_1, [y_1, y_2] = [x_2, y_1] = 1; 1 \leq i, j \leq 2 \rangle$. Then $G$ satisfies the condition of Corollary 1. We have $G' = \langle y_1 \rangle \cong C_2$, $Z(G) = \langle y_1, y_2 \rangle \cong C_2 \times \mathbb{Z}$ and $G/Z(G) = \langle \overline{x_1}, \overline{x_2} \rangle \cong C_2 \times C_2$ and hence $\text{Aut}_{\text{pwi}}(G) = \text{Inn}(G)$.

**Example 2.6** Let $G = \langle x_1, x_2, x; [x_1, x_2] = x, [x_2, x] = 1; 1 \leq i \leq 2 \rangle$. Then $G$ satisfies the condition of Corollary 1. We have $G' = Z(G) = \langle x \rangle \cong \mathbb{Z}$ and $\frac{G}{Z(G)} = \langle \overline{x_1}, \overline{x_2} \rangle \cong \mathbb{Z} \times \mathbb{Z}$ . Hence $\text{Aut}_{\text{pwi}}(G) \cong \text{Inn}(G)$. It is easy to see that in this case every pointwise inner automorphism is inner and so $\text{Aut}_{\text{pwi}}(G) = \text{Inn}(G)$ (see [1, Example 3.4]).

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