On pointwise inner automorphisms of nilpotent groups of class 2

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Abstract
An automorphism \( \theta \) of a group \( G \) is pointwise inner if \( \theta(x) \) is conjugate to \( x \) for any \( x \in G \). The set of all pointwise inner automorphisms of group \( G \), denoted by \( \text{Aut}_{\text{pwi}}(G) \) form a subgroups of \( \text{Aut}(G) \) containing \( \text{Inn}(G) \). In this paper, we find a necessary and sufficient condition in certain finitely generated nilpotent groups of class 2 for which \( \text{Aut}_{\text{pwi}}(G) = \text{Inn}(G) \). We also prove that in a nilpotent group of class 2 with cyclic commutator subgroup \( \text{Aut}_{\text{pwi}}(G) = \text{Inn}(G) \) and the quotient \( \text{Aut}_{\text{pwi}}(G)/\text{Inn}(G) \) is torsion.

In particular if \( G' \) is a finite cyclic group then \( \text{Aut}_{\text{pwi}}(G) = \text{Inn}(G) \).

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Introduction

By definition, a pointwise inner automorphism of a group \( G \) is an automorphism \( \theta: G \to G \) such that \( t \) and \( \theta(t) \) are conjugate for any \( t \in G \). This notion appears in the famous book of Burnside [1, Note B, p 463]. Denote by \( \text{Aut}_{\text{pwi}}(G) \) the set of all pointwise inner automorphisms of \( G \).

Obviously, \( \text{Aut}_{\text{pwi}}(G) \) contains \( \text{Inn}(G) \), the group of all inner automorphisms of \( G \). These groups can coincide, for instance when \( G \) is \( S_n, A_n, SL_n(D) \) and \( GL_n(D) \) where \( D \) is an Euclidean domain (see [7], [10], [11]).

By a result of Grossman [5], it turns out that \( \text{Aut}_{\text{pwi}}(G) = \text{Inn}(G) \) when \( G \) is a free group. Endimioni in [4] proved that this property remains true in a free nilpotent group.

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Also, Yadav in [12] gave a sufficient condition for a finite $p$-group $G$ of nilpotent class 2 to be such that $\text{Aut}_{\text{pwi}}(G) = \text{Inn}(G)$. But the equality does not hold in general.

In fact, in 1911, Burnside posed the following question: Does there exist any finite group $G$ such that $G$ has a non-inner and pointwise inner automorphism? In 1913, Burnside himself gave an affirmative answer to this question [3]. Indeed, there are many examples of groups admitting a pointwise inner automorphism which is not inner (see, for instance [3], [4], [8], [9], [12] where these groups are besides nilpotent).

Segal also gave a subtle example. He constructed a finitely generated torsion-free nilpotent group $G$, in which $\text{Aut}_{\text{pwi}}(G)/\text{Inn}(G)$ contains an element of infinite order (see [9]).

In this paper, we study the pointwise inner automorphisms of a finitely generated nilpotent group of class 2 with cyclic commutator subgroup.

We introduce the following definition:

**Definition.** Let $G$ be a finitely generated nilpotent group of class 2. Then $G/Z(G)$ is finitely generated abelian group and thus $G/Z(G) = \langle x_1 Z(G) \rangle \times \ldots \times \langle x_k Z(G) \rangle$ for some $x_1, \ldots, x_k \in G$. The group $G$ is called d-group if the following distributive law holds in $G$,

$$[x_1^{\alpha_1} \ldots x_k^{\alpha_k}, G] = [x_1, G]^{\alpha_1} \ldots [x_k, G]^{\alpha_k}$$

where $\alpha_i \in \mathbb{Z}$ and $1 \leq i \leq k$.

Let $G$ be a 2-generator nilpotent group of class 2. It is straightforward to show that $G$ is a d-group.

To give an example of an infinite d-group, consider the group $G$ with the following presentation

$$G = \langle x_1, x_2, x_3, x_4, x : [x_i, x_j] = x^{m_{ij}}, [x_i, x] = 1; 1 \leq i \leq 4 \text{ and } i < j \rangle,$$

where $m_{ii+1} = 1$ for all $1 \leq i < 4$ and $m_{ij} = 0$ for all $i + 1 < j$. Then $G' = Z(G) = \langle x \rangle \cong \mathbb{Z}$ and $G/Z(G) = (x_1, x_2, x_3, x_4) \cong \mathbb{Z}^4$. A quick calculation shows that

$$[x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} x_4^{\alpha_4}, G] = [x_1, G]^{\alpha_1} [x_2, G]^{\alpha_2} [x_3, G]^{\alpha_3} [x_4, G]^{\alpha_4} = \langle x^\alpha \rangle,$$

Where $\alpha_i \in \mathbb{Z}$ for all $1 \leq i \leq 4$ and $\alpha = \gcd(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. Therefore $G$ is an infinite d-group.
Now we give a nilpotent group $G$ of class 2 which is not a d-group.

Let $G$ be a free nilpotent group of class 2 on 4 generators $a_1, a_2, a_3$ and $a_4$. If $c_{ij} = [a_i, a_j]$ for $1 \leq i < j \leq 4$, then the relations in $G$ are $[c_{ij}, a_k] = 1$ for $1 \leq i < j \leq 4$ and $1 \leq k \leq 4$, and their consequences. Macdonald in [6] proved that $c_{13} c_{24}$ is not a commutator. Therefore $G$ is not a d-group.

**Theorem 1.** Let $G$ be a finitely generated nilpotent group of class 2 and $G/Z(G) = \langle x_1 \rangle \times \cdots \times \langle x_k \rangle$.

(i) There exists a monomorphism $\text{Aut}_{\text{pwi}}(G) \hookrightarrow \prod_{i=1}^{k} \text{Hom}(\langle x_i \rangle, [x_i, G])$.

(ii) If $[x_i, G]$ is cyclic for all $1 \leq i \leq k$, then there exists a monomorphism $\text{Aut}_{\text{pwi}}(G) \hookrightarrow \text{Inn}(G)$.

In particular if $G$ is a d-group of class 2 then the monomorphisms in (i) and (ii) are isomorphism. Furthermore $\text{Aut}_{\text{pwi}}(G) \cong \text{Inn}(G)$ if and only if $[x_i, G]$ is cyclic for all $1 \leq i \leq k$.

Notice that if $G$ is a finite group then, as consequence of this result, we derive Theorem 3.5 and Corollary 3.6 of Yadav in [12].

In particular, we derive the following consequences of Theorem 1.

**Corollary 1.** Let $G$ be a finitely generated nilpotent group of class 2 in which $G'$ is cyclic, then $\text{Aut}_{\text{pwi}}(G) \cong \text{Inn}(G)$. In particular if $G'$ is finite, then $\text{Aut}_{\text{pwi}}(G) = \text{Inn}(G)$.

Recall that by Corollary 3.6 in [12], in a finite nilpotent group of class 2, if $G'$ is cyclic then $\text{Aut}_{\text{pwi}}(G) = \text{Inn}(G)$. But we cannot hope for a similar conclusion when $G$ is not finite. We will provide an example in the section 2. However, in a finitely generated nilpotent group of class 2, by Corollary 1 we have $\text{Aut}_{\text{pwi}}(G) \cong \text{Inn}(G)$. So the structure of $\text{Aut}_{\text{pwi}}(G)$ is determined.

**Corollary 2.** Let $G$ be a finitely generated nilpotent group of class 2. If the commutator subgroup of $G$ is cyclic, then $\text{Aut}_{\text{pwi}}(G)/\text{Inn}(G)$ is torsion.

Let $G$ be a group and $N$ be a non-trivial proper normal subgroup of $G$. The pair...
(G,N) is called a Camina pair if xN ≤ xG for all x ∈ G\N. A group G is called a Camina group if (G,G′) is a Camina pair.

Clearly, if G is a Camina group of class 2 then it is a d-group. So, as an immediate consequence of Theorem 1, one readily gets the following corollary.

**Corollary 3.** Let G be a finitely generated nilpotent group of class 2. If G is a Camina group then Aut_{pwi}(G) ≃ Inn(G) if and only if G′ is cyclic. Particularly, if G/Z(G) is finite, then Aut_{pwi}(G) = Inn(G) if and only if G′ is cyclic.

**Preliminary results**

Our notation is standard. Let G be a group, by C_m, G′ and Z(G), we denote the cyclic group of order m, the commutator subgroup and the center of G, respectively.

If x, y ∈ G, then x^y denotes the conjugate element y^{-1}xy ∈ G. For x ∈ G, x^G denotes the conjugacy class of x in G. The commutator of two elements x,y ∈ G is defined by [x,y] = x^{-1}y^{-1}xy and more generally, the left-normed commutator of n elements x_1, . . . , x_n is defined inductively by

\[ [x_1, . . . , x_{n-1}, x_n] = [x_1, . . . , x_{n-1}]^{-1}x_n^{-1}[x_1, . . . , x_{n-1}]x_n. \]

If H ≤ G, [x, H] denotes the set of all [x, h] for h ∈ H, this is a subgroup of G when G is of class 2. For any group H and abelian group K, Hom(H,K) denotes the group of all homomorphisms from H to K. Also C^* is the set of all central automorphisms of G fixing Z(G) elementwise.

Yadav in [12] shows that in a finite nilpotent group of class 2, there exists a monomorphism from Aut_{pwi}(G) into Hom(G/Z(G),G′). It turns out that this result remains true when G is an infinite nilpotent group of class 2.

For that, let G be a nilpotent group (finite or infinite) of class 2. Let α ∈ Aut_{pwi}(G).

Then the map g ↦ g^{-1}α(g) is a homomorphism from G into G′. This homomorphism sends Z(G) to 1. So it induces a homomorphism f_α: G/Z(G) → G′, sending gZ(G) to g^{-1}α(g), for any g ∈ G. Define

\[ \text{Hom}_{pwi}(G/Z(G), G′) = \{ f ∈ \text{Hom}\left( G/Z(G), G′ \right) : f(gZ(G)) ∈ [g, G] \text{ for all } g ∈ G \}. \]

To prove Aut_{pwi}(G) ≃ Hom_{pwi}(G/Z(G), G′), we use the following well-known result.
Lemma 1.1 Let N be a normal subgroup of a group G. Let \( \vartheta \) be an endomorphism of G such that \( \vartheta(N) \leq N \). Denote by \( \tilde{\vartheta} \) and \( \vartheta_0 \) the endomorphisms induced by \( \vartheta \) in \( G/N \) and \( N \), respectively. If \( \tilde{\vartheta} \) and \( \vartheta_0 \) are surjective (injective), then so is \( \vartheta \).

Proposition 1.2 Let G be a nilpotent group of class 2. Then the above map \( \varphi: \alpha \mapsto f_\alpha \) is an isomorphism from \( \text{Aut}_{\text{pw}}(G) \) into \( \text{Hom}_{\text{pw}}(G/Z(G), G') \).

Proof. Since for any \( \alpha \in \text{Aut}_{\text{pw}}(G) \), by the definition \( f_\alpha \in \text{Hom}_{\text{pw}}(G/Z(G), G') \), \( \varphi \) is well defined. Let \( \alpha_1, \alpha_2 \in \text{Aut}_{\text{pw}}(G) \) and \( g \in G \). We have \( \alpha_1(g^{-1} \alpha_2(g)) = g^{-1} \alpha_2(g) \) since \( g^{-1} \alpha_2(g) \in G' \leq Z(G) \). This implies that

\[
  f_{\alpha_1 \alpha_2}(g) = g^{-1} \alpha_1(\alpha_2(g)) = g^{-1} \alpha_1(gg^{-1} \alpha_2(g)) = g^{-1} \alpha_1(g) \cdot g^{-1} \alpha_2(g) = f_{\alpha_1}(g) \cdot f_{\alpha_2}(g).
\]

Hence \( \varphi \) is a homomorphism. Clearly, \( \varphi \) is injective. Now it suffices to show that \( \varphi \) is surjective.

Let \( f \) be any element of \( \text{Hom}_{\text{pw}}(G/Z(G), G') \). By Lemma 1.1 a quick calculation shows that \( \varphi(\alpha) = f \), where \( \alpha \) is an element of \( \text{Aut}_{\text{pw}}(G) \), sending \( g \in G \) to \( gf(gZ(G)) \). Then we have \( \text{Aut}_{\text{pw}}(G) \approx \text{Hom}_{\text{pw}}(G/Z(G), G') \).

* Note that if \( G \) is a nilpotent group of class 2 then \( \text{Aut}_{\text{pw}}(G) \approx \text{Hom}_{\text{pw}}(G/Z(G), G') \).

It is easy to see that in a Camina nilpotent group of class 2, \( \text{Hom}_{\text{pw}}(G/Z(G), G') = \text{Hom}(G/Z(G), G') \). Hence if \( G \) is a Camina group of class 2, then \( \text{Aut}_{\text{pw}}(G) \approx \text{Hom}(G/Z(G), G') \).

The following well-known facts will be used repeatedly.

Lemma 1.3 Let \( A, B \) and \( C \) be abelian groups.

(i) \( \text{Hom}(A \times B, C) \approx \text{Hom}(A, C) \times \text{Hom}(B, C) \).

(ii) \( \text{Hom}(A, B \times C) \approx \text{Hom}(A, B) \times \text{Hom}(A, C) \).

(iii) \( \text{Hom}(C_m, C_n) \approx C_d \) where \( d = \text{gcd}(m, n) \).

(iv) \( \text{Hom}(\mathbb{Z}, A) \approx A \).

(v) If \( A \) is torsion group and \( B \) is torsion-free group, then \( \text{Hom}(A, B) = 1 \).

(vi) If \( \text{gcd}(|A|, |B|) \neq 1 \), then \( \text{Hom}(A, B) \neq 1 \).
Main Result

Let $G$ be a finite abelian group. We denote by $G_p$, the $p$-primary component of $G$. Hence $G = \prod_{p \in \pi(G)} G_p$ where $\pi(G)$ denotes the set of all primes $p$ dividing $|G|$. To prove Theorem 1, we need the following Lemma.

Lemma 2.1 ([1, Corollary 1.4]) Let $A$ and $B$ be two finite abelian groups and $\exp(A) \mid \exp(B)$. Then $\text{Hom}(A, B) \cong A$ if and only if $B \cong C_m \times H$ in which $C_m = \prod_{p \in \pi(A)} B_p$ and $H \cong \prod_{p \notin \pi(A)} B_p$. In particular, if $\pi(A) = \pi(B)$ then this is equivalent to $B$ is a cyclic group.

Let $G$ be a finitely generated nilpotent group of class 2. Then $G/Z(G)$ is finitely generated abelian group and thus $G/Z(G) = \langle x_1 Z(G) \rangle \times \ldots \times \langle x_k Z(G) \rangle$ for some $x_1, \ldots, x_k \in G$.

Let $f \in \text{Hom}_{pw1}(G/Z(G), G')$. So $f(gZ(G)) \in [g, G]$ for all $g \in G$. In particular, for all $1 \leq i \leq k$ we have $f(x_i Z(G)) \in [x_i, G]$. Now we prove Theorem 1.

Proof of Theorem 1.

(i) By Proposition 1.2, we have $\text{Aut}_{pw1}(G) \cong \text{Hom}_{pw1}(G/Z(G), G')$. It suffices to show that there exists a monomorphism from $\text{Hom}_{pw1}(G/Z(G), G')$ into $\prod_{i=1}^k \text{Hom}(\langle x_i \rangle, [x_i, G])$. Let $f \in \text{Hom}_{pw1}(G/Z(G), G')$. Denote by $f_i$, the homomorphism induced by $f$ in $\langle x_i \rangle$, for all $1 \leq i \leq k$. Since $G$ is a nilpotent group of class 2, we have $[a^m, b] = [a^m, b^m] = [a, b^m]$ for each $a, b \in G$ and $m \in \mathbb{Z}$. Consequently, $[x_i^m, G] \leq [x_i, G]$ for all $m \in \mathbb{Z}$ and $1 \leq i \leq k$. Therefore $f_i \in \text{Hom}(\langle x_i \rangle, [x_i, G])$. Thus the map $\alpha$ sending any $f \in \text{Hom}_{pw1}(G/Z(G), G')$ to $\alpha(f) = (f_1, \ldots, f_k) \in \prod_{i=1}^k \text{Hom}(\langle x_i \rangle, [x_i, G])$ is well defined. Now we prove that this map is a monomorphism. Since $(fg)_i = f_i g_i$ for each $f, g \in \text{Hom}_{pw1}(G/Z(G), G')$ and $1 \leq i \leq k$, $\alpha$ is homomorphism. Clearly, $\ker\alpha$ is trivial, this implies that $\alpha$ is monomorphism. Hence the proof of (i) is complete.

(ii) First we show that $[x_i, G]$ is finite if and only if $\langle x_i \rangle$ is finite, and further
exp([x_i, G]) = exp([\langle x_i \rangle]) = |x_i|. For this, let |[x_i, G]| = n. Since G is a nilpotent 
group of class 2, we have [x_i^n, g] = [x_i, g]^n = 1 for all g ∈ G and so x_i^n ∈ Z(G). 
Hence \langle x_i \rangle is finite and |x_i|/n. Conversely if |x_i| = m then x_i^m ∈ Z(G) and 
[x_i, G]^m = [x_i^m, G] = 1. Consequently [x_i, G] is finite and exp([x_i, G]) = n|m. 
Therefore in this case, m = n. Hence by Lemma 2.1, for all 1 ≤ i ≤ k we have 
\text{Hom}(\langle x_i \rangle, [x_i, G]) ≈ \langle x_i \rangle if and only if [x_i, G] is cyclic. 

Now from (i), we have a monomorphism from Aut_{pw}(G) into \prod_{i=1}^k \text{Hom}(\langle x_i \rangle, [x_i, G]) 
and therefore we conclude that there exists a monomorphism Aut_{pw}(G) ↪ G/Z(G), this 
completes the proof of (ii).

If G is a d-group, then it is easy to see that the monomorphism defined in (i) is an 
isomorphism from Aut_{pw}(G) into \prod_{i=1}^k \text{Hom}(\langle x_i \rangle, [x_i, G]).

Finally to complete the proof, it is sufficient to show that if Aut_{pw}(G) ∼= Inn(G), then 
[x_i, G] is cyclic for all 1 ≤ i ≤ k. Since Aut_{pw}(G) ∼= Inn(G), by Proposition 1.2 we 
have G/Z(G) ∼= Hom_{pw}(G/Z(G), G'). On the other hand, G is a d-group and hence

\text{Hom}_{pw}(G/Z(G), G') ∼= \prod_{i=1}^k \text{Hom}(\langle x_i \rangle, [x_i, G]).

It follows that

G/Z(G) = \langle x_1 \rangle × ... × \langle x_k \rangle ≈ \prod_{i=1}^k \text{Hom}(\langle x_i \rangle, [x_i, G]).

Now we may assume that \langle x_1 \rangle × ... × \langle x_n \rangle is the torsion part and \langle x_{n+1} \rangle × ... × \langle x_k \rangle is 
the torsion-free part of G/Z(G). Since for all 1 ≤ i ≤ n, exp([x_i, G]) = exp(\langle x_i \rangle) = |x_i| 
and \prod_{i=1}^n \text{Hom}(\langle x_i \rangle, [x_i, G]) ≈ \langle x_1 \rangle × ... × \langle x_n \rangle, \text{Hom}(\langle x_i \rangle, [x_i, G]) ≈ \langle x_i \rangle for all 
1 ≤ i ≤ n and hence [x_i, G] is cyclic. Furthermore, we have

\prod_{i=n+1}^k \text{Hom}(\langle x_i \rangle, [x_i, G]) ≈ \langle x_{n+1} \rangle × ... × \langle x_k \rangle ≈ \mathbb{Z}^{k-n}.

Now we have Hom(\langle x_i \rangle, [x_i, G]) ≈ [x_i, G], since \langle x_i \rangle ≈ \mathbb{Z} and hence \prod_{i=n+1}^m [x_i, G] ≈ 
\mathbb{Z}^{k-n}. That is [x_i, G] ≈ \mathbb{Z} for all n + 1 ≤ i ≤ k. This implies that [x_i, G] is cyclic for all 
1 ≤ i ≤ k, as required.

*Notice that if G is a finite group then, as a consequence of this result, we derive 
Theorem 3.5 and Corollary 3.6 of Yadav in [12].

The following corollary is an easy consequence of the above theorem.
Corollary 2.2 Let $G$ be a finitely generated nilpotent group of class 2 with cyclic commutator subgroup. Then there exists a monomorphism from $\text{Aut}_{pwi}(G)$ into $\text{Inn}(G)$ or equivalently $\text{Aut}_{pwi}(G)$ is isomorphic to a subgroup of $G/Z(G)$.

Remark 2.3 We keep here the notation used in Theorem 1.

(i) By the discussion of (ii) in Theorem 1, if $G'$ is finite cyclic, then $G/Z(G)$ is finite and $|\text{Aut}_{pwi}(G)| \leq |\text{Inn}(G)| = |G/Z(G)|$. On the other hand, $\text{Inn}(G) \leq \text{Aut}_{pwi}(G)$ conclude that $\text{Aut}_{pwi}(G) \cong \text{Inn}(G)$. Note that in this case, $G$ is not necessarily finite.

(ii) If $G'$ is infinite cyclic, it follows from the discussion of (ii) in Theorem 1, that $G/Z(G)$ is a free abelian group of finite rank, say $r(G/Z(G)) = k$. We certainly have $\text{Inn}(G) \leq \text{Aut}_{pwi}(G)$ and thus $r(\text{Inn}(G)) \leq r(\text{Aut}_{pwi}(G))$. Also $r(\text{Aut}_{pwi}(G)) \leq r(\text{Inn}(G))$, since $\text{Aut}_{pwi}(G)$ is isomorphic to a subgroup of $\text{Inn}(G)$. Therefore $\text{Aut}_{pwi}(G)$ and $\text{Inn}(G)$ have the same rank and hence $\text{Aut}_{pwi}(G) \cong \text{Inn}(G)$.

Now it is easy to deduce Corollary 1 from Remark 2.3.

Remark 2.4 It is known that in a nilpotent groups of class 2, $\text{Inn}(G) \leq \text{Aut}_{pwi}(G) \leq C^*$. So $\text{Inn}(G) = \text{Aut}_{pwi}(G)$ when $\text{Inn}(G) = C^*$. In [1] we characterized all non torsion-free finitely generated groups in which $\text{Inn}(G) = C^*$. We proved that $\text{Inn}(G) = C^*$ if and only if $G$ is an abelian group or nilpotent of class 2 and $Z(G) \cong C_m \times H \times \mathbb{Z}^r$ in which $C_m \cong \prod_{p \in \pi(G/Z(G))} Z(G)_p$, $H \cong \prod_{p \in \pi(G/Z(G))} Z(G)_p$ and $r \geq 0$ is the torsion-free rank of $Z(G)$ and $G/Z(G)$ has finite exponent.

Hence if $G$ is nilpotent group of class 2, $Z(G) \cong C_m \times H \times \mathbb{Z}^r$ and $G/Z(G)$ has finite exponent then we have $\text{Inn}(G) = \text{Aut}_{pwi}(G)$. Notice that in this case, $G'$ is cyclic and the equality $\text{Inn}(G) = \text{Aut}_{pwi}(G)$ also follows from Corollary 1.

Recall that by Corollary 3.6 in [12], in a finite nilpotent group of class 2, if $G'$ is cyclic then $\text{Aut}_{pwi}(G) = \text{Inn}(G)$. But we cannot hope for a similar conclusion when $G$ is not finite.
For example, consider countably infinitely many copies $H_1, H_2, \ldots$ of a given nilpotent group $H$ of class 2 with cyclic commutator subgroup. Let $G$ (respectively, $\overline{G}$) be the direct product (the cartesian product) of the family $(H_i)_{i>0}$. Clearly, $G$ and $\overline{G}$ are nilpotent of class 2. For each integer $i > 0$, choose an element $a_i \in H_i$ which is not in the center of $H_i$. Then the inner automorphism of $\overline{G}$ defined by $\overline{a}((t_i)_{i>0}) = (a_i^{-1}t_ia_i)_{i>0}$ induces in $G$ a pointwise inner automorphism $\alpha$ which is not inner (see [4]).

However, in a finitely generated nilpotent group of class 2 with cyclic commutator subgroup, we have $\text{Aut}_{\text{pw}i}(G) \simeq \text{Inn}(G)$, by Corollary 1. So the structure of $\text{Aut}_{\text{pw}i}(G)$ is determined.

Furthermore it is fairly easy to deduce Corollary 2 from Remark 2.3.

We end this part of the paper with some examples of infinite groups $G$ satisfying the conditions of Corollary 1 and therefore $\text{Aut}_{\text{pw}i}(G) \simeq \text{Inn}(G)$.

**Example 2.5** Let $G = \langle x_1, y_1, y_2; x_1^p = x_2^p = y_1^p = 1, [x_1, x_2] = y_1, [y_1, y_2] = [x_1, y_1] = 1; 1 \leq i, j \leq 2 \rangle$. Then $G$ satisfies the condition of Corollary 1. We have $G' = \langle y_1 \rangle \simeq C_p$, $Z(G) = \langle y_1, y_2 \rangle \simeq C_p \times \mathbb{Z}$ and $G/Z(G) = \langle \overline{x_1}, \overline{x_2} \rangle \simeq C_p \times C_p$ and hence $\text{Aut}_{\text{pw}i}(G) = \text{Inn}(G)$.

**Example 2.6** Let $G = \langle x_1, x_2; [x_1, x_2] = x_1, [x_1, x] = 1; 1 \leq i \leq 2 \rangle$. Then $G$ satisfies the condition of Corollary 1. We have $G' = Z(G) = \langle x \rangle \simeq \mathbb{Z}$ and $\overline{G} = \langle \overline{x_1}, \overline{x_2} \rangle \simeq \mathbb{Z} \times \mathbb{Z}$. Hence $\text{Aut}_{\text{pw}i}(G) \simeq \text{Inn}(G)$. It is easy to see that in this case every pointwise inner automorphism is inner and so $\text{Aut}_{\text{pw}i}(G) = \text{Inn}(G)$ (see [1, Example 3.4]).

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