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Quasi- Secondary Submodules

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Abstract

Let R be a commutative ring with non-zero identity and M be a unitary R-module. Then the concept of quasi-secondary submodules of M is introduced and some results concerning this class of submodules is obtained.

1. Introduction

Throughout this paper all rings are commutative with non-zero identity and all modules are unitary. In [4] L.Fuchs introduced and studied the concept of quasi-primary ideals (see also [5]). An ideal I of a ring R is called a *quasi-primary* ideal of R if the radical of I is a prime ideal of R. This concept then generalized to modules, i.e., the concept of quasi-primary submodules of a module introduced and developed in [3]. Here, we introduce the dual notation, that is, the quasi-secondary submodules of a module and obtain some results concerning this class of submodules. In section 2, we obtain some preliminary properties of quasi-secondary submodules. Section 3 is devoted to the quasi-secondary submodules of a multiplication module. Now we define some concepts which will be needed in sequel.

Let M be an R-module and N a submodule of it. The ideal $\{r \in R | rM \subseteq N\}$ will be denoted by $(N :_R M)$; in particular $(0 :_R M)$ is called the annihilator of M. A non-zero submodule N of M is called a *secondary* (*resp.second*) submodule of M if for each $r \in R$ the homothety $N \xrightarrow{r} N$ is surjective or nilpoten (resp. surjective or zero). In this case $\sqrt{(0 :_R N)}$ is a prime ideal, say p, and we call N a p-*secondary* (resp.a p- *second*) submodule of M. We refer readers for more details concerning secondary (resp.second) submodulse to [9] (resp. [12]).

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An R-module M is said to be a *multiplication* module if for each submodule N of M there exists an ideal I of R such that $N = IM$. It is easy to see that in this case $N = (N:_R M)M$. Also the ideal $\theta(M)$ is defined as $\theta(M) := \sum_{m \in M}(Rm:_R M)$. If M is a multiplication module and N is a submodule of it, then $M = \theta(M)M$ and $N = \theta\,(M)\,N$. (see [1]). An R-module M is *sum-irreducible* if $M \neq 0$ and the sum of any two proper submodules of M is always a proper submodule. Finally a proper submodule N of an R-module M is called a *prime submodule* if for each $r \in R$ the homothety $M/_N \xrightarrow{r} M/_N$ is either injective or zero. This implies that $Ann\left(M/_N\right) = p$ is a prime ideal of R , and N is said to be a *p-prime submodule* (c.f. [7], [8], [10] and [11]).

2. Quasi-Secondary Submodules

The starting point of this section is the definition of quasi-secondary submodules of a module.

Definitian 2.1. Let M be a non-zero R-module. Then the non-zero submodule N of M is said to be *quasi-secondary* if $\sqrt{(0:_R N)} = p$ where p is a prime ideal of R. It is obvious that every secondary (or second) submodule of a module is a quasi-secondary submodule, but the converse is not true in general. For example, 2Z is a 0-quasi-secondary submodule of the Z-module Z but it is not 0-secondary (or 0-second) submodule. (Here Z denotes the set of all integers.)

Remark 2.2.

(i) Let M be a non-zero R-module and N a submodule of it such that $\sqrt{(0:_R N)} = m(m \in Max\,(R))$. Then N is m-secondary (m-second).

(ii) Every quasi-secondary submodule of a module over a zero-dimentional ring (i.e. , a ring in which every prime ideal is a maximal ideal) is secondary.

(iii) Every quasi-secondary submodule of a module over a D.V.R is secondary.

Definition 2.3. Let M be an R-module and N a submodule of M. An element r of R is called *co-primal* to N if $rN = N$. Denote by $W(N)$ the set of all elements of R that are not co-primal to N. The submodule N is said to be a co-primal submodule of M if $W(N)$ is an ideal of R. This ideal is always a prime ideal. In this case we say that N is a *p-co-primal* submodule of M. The class of co-primal submodules of a module is a fairly large class. For example , all secondary (second) submodules are co-primal. Also it is easy to see that a sum-irreducible submodule of a module is co-primal. But, in general, a quasi-secondary submodule of a module may not be a co-primal submodule. (consider the Z-module Z.). It is worth to mention that in [2] the term secondal is used for co-primal submodules. The next proposition characterizes those p-quasi- secondary submodules which are p-co-primal.

Proposition 2 4. Let N be a p-quasi-secondary submodule of an R-module M. Then N is a pcoprimal submodule of M if and only if it is a p-secondary submodule of M .

Proof $\Rightarrow$) Let $N \xrightarrow{r} N$ be the R-endomorphism of N given by multiphication by r of R and $rN \neq N$. Then by our assumption $r \in p = \{ s \in R \mid s N \neq N\}$. On the other hand, $p = \sqrt{0 :_R N}$ and so there exists a positive integer t such that $r^t N = 0$. The result follows. $\Leftarrow$) Is obvious.

The proof of two next propositions is easy and so we state them without proof.

Proposition 2.5. Let M be a module over an integral domain and N be a 0- co-primal submodule of M .Then N is 0-secondary.

Proposition 2.6. Let M be an R-module and $N_1, N_2, \ldots, N_t$ be submodules of M. Then

(i) Suppose that for $i = 1, 2, N_i$ is p_i-quasi-secondary. Then $N_1 + N_2$ is quasi-secondary if and only if $p_1 \subseteq p_2$ or $p_2 \subseteq p_1$

(ii) If $N_1, \ldots, N_t$ are p-quasi-secondary, then $N_1 + \cdots + N_t$ is a p-quasi-secondary submodule of M.

(iii) If $N_1 + \cdots + N_2$ is a p-quasi-secondary submodule of M.Then N_j is p-quasi-secondary for some j , $1 \leq j \leq t$.

3. Multiplication Modules

In this short section we give a property of quasi-secondary submodules of a multiphication module .

Lemma 3.1. let M be a multiplication module and N be a p-quasi-secondary submodule of M. Then $\theta(M) \nsubseteq p$.

Proof. Suppose that $\theta(M) \subseteq p$ and $0 \neq n \in N$. Then $Rn = \theta(M)Rn \subseteq pn$. Hence $n = p_0 n$ for some $p_0 \in p$. By our assumption there exists a positive integer t such that $p_0^t N = 0$. Therefore $n = p_0^t n = 0$, a contradiction.

Theorem 3.2. Suppose that M is a faithfull multiplication module and N a p-quasi-secondary submodule of M. Then pM is a prime submodule of M.In particular, if $p \in \max(R)$, then pM is a maximal submodule of M.

Proof. By Lemma 3.1 , $\theta(M) \not\subseteq p$. Now suppose that $pM = M = RM$. Then by $[1, \text{Theorem1.5}]$ $R \cap \theta(M) = \theta(M) = p \cap \theta(M)$and hence $\theta(M) \subseteq p$which is a contradiction. Thus $pM \neq M$ and the result of the first part follows from [6, Lemma 2.4(2)]. The last part can be deduced from the first part and [6, Corollary 2.7]

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