Abstract
Let $R$ be a commutative ring with non-zero identity and $M$ be a unitary $R$-module.
Then the concept of quasi-secondary submodules of $M$ is introduced and some results  concerning this class of submodules is obtained.

1. Introduction
Throughout this paper all rings are commutative with non-zero identity and all  modules are unitary. In [4] L.Fuchs introduced and studied the concept of quasi-primary  ideals (see also [5]). An ideal $I$ of a ring $R$ is called a quasi-primary ideal of $R$ if the  radical of $I$ is a prime ideal of $R$. This concept then generalized to modules, i.e., the  concept of quasi-primary submodules of a module introduced and developed in [3].
Here, we introduce the dual notation, that is, the quasi-secondary submodules of a  module and obtain some results concerning this class of submodules. In section 2, we  obtain some preliminary properties of quasi-secondary submodules. Section 3 is  devoted to the quasi-secondary submodules of a multiplication module. Now we define  some concepts which will be needed in sequel.
Let $M$ be an $R$-module and $N$ a submodule of it. The ideal $\{r \in R | rM \subseteq N\}$ will be  denoted by $(N_R M)$; in particular $(0_R M)$ is called the annihilator of $M$. A non-zero  submodule $N$ of $M$ is called a secondary (resp.second) submodule of $M$ if for each  $r \in R$ the homothety $N \rightarrow N$ is surjective or nilpoten (resp. surjective or zero). In this  case $\sqrt{(0_R N)}$ is a prime ideal, say $p$, and we call $N$ a $p$-secondary (resp.a $p$- second)  submodule of $M$. We refer readers for more details concerning secondary (resp.second)  submodulse to [9] (resp. [12]).

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An R-module $M$ is said to be a *multiplication* module if for each submodule $N$ of $M$, there exists an ideal $I$ of $R$ such that $N = IM$. It is easy to see that in this case $N = (N_r M) M$. Also the ideal $\theta(M)$ is defined as $\theta(M) := \sum_{m \in M} (Rm, \bar{M})$. If $M$ is a multiplication module and $N$ is a submodule of it, then $M = \theta(M)M$ and $N = \theta(M) N$. (see [1]). An R-module $M$ is *sum-irreducible* if $M \triangleleft 0$ and the sum of any two proper submodules of $M$ is always a proper submodule. Finally a proper submodule $N$ of an R-module $M$ is called a *prime submodule* if for each $r \in R$ the homothety $M/N \to M/N$ is either injective or zero. This implies that $Ann(M/N) = p$ is a prime ideal of $R$, and $N$ is said to be a *p-prime submodule* (c.f. [7], [8], [10] and [11]).

2. Quasi-Secondary Submodules

The starting point of this section is the definition of quasi-secondary submodules of a module.

Definition 2.1. Let $M$ be a non-zero R-module. Then the non-zero submodule $N$ of $M$ is said to be *quasi-secondary* if $\sqrt{(0 :_R N)} = p$ where $p$ is a prime ideal of $R$. It is obvious that every secondary (or second) submodule of a module is a quasi-secondary submodule, but the converse is not true in general. For example, $2Z$ is a 0-quasi-secondary submodule of the Z-module $Z$ but it is not 0-secondary (or 0-second) submodule. (Here $Z$ denotes the set of all integers.)

Remark 2.2.

(i) Let $M$ be a non-zero R-module and $N$ a submodule of $M$ such that $\sqrt{(0 :_R N)} = m(m \in Max(R))$. Then $N$ is $m$-secondary (m-second).

(ii) Every quasi-secondary submodule of a module over a zero-dimenstional ring (i.e., a ring in which every prime ideal is a maximal ideal) is secondary.

(iii) Every quasi-secondary submodule of a module over a D.V.R is secondary.

Definition 2.3. Let $M$ be an R-module and $N$ a submodule of $M$. An element $r$ of $R$ is called *co-primal* to $N$ if $rN = N$. Denote by $W(N)$ the set of all elements of $R$ that are not co-primal to $N$. The submodule $N$ is said to be a co-primal submodule of $M$ if $W(N)$ is an ideal of $R$. This ideal is always a prime ideal. In this case we say that $N$ is a *p-co-primal* submodule of $M$. The class of co-primal submodules of a module is a...
fairly large class. For example, all secondary (second) submodules are co-primal. Also it is easy to see that a sum-irreducible submodule of a module is co-primal. But, in general, a quasi-secondary submodule of a module may not be a co-primal submodule. (consider the Z-module Z.). It is worth to mention that in [2] the term secondal is used for co-primal submodules. The next proposition characterizes those p-quasi-secondary submodules which are p-co-primal.

Proposition 2.4. Let N be a p-quasi-secondary submodule of an R-module M. Then N is a pco-primal submodule of M if and only if it is a p-secondary submodule of M.

Proof $\Rightarrow$ Let $N \rightarrow N$ be the R-endomorphism of N given by multiplication by r of R and $rN \neq N$. Then by our assumption $r \in p = \{ s \in R \mid sN \neq N \}$. On the other hand, $p = \sqrt{0_R^N}$ and so there exists a positive integer t such that $r^t N = 0$. The result follows. $\Leftarrow$ Is obvious.

The proof of two next propositions is easy and so we state them without proof.

Proposition 2.5. Let M be a module over an integral domain and N be a 0-co-primal submodule of M. Then N is 0-secondary.

Proposition 2.6. Let M be an R-module and $N_1, N_2, ..., N_t$ be submodules of M. Then

(i) Suppose that for $i = 1, 2, N_i$ is $p_i$-quasi-secondary. Then $N_1 + N_2$ is quasi-secondary if and only if $p_1 \subseteq p_2$ or $p_2 \subseteq p_1$.

(ii) If $N_1, ..., N_t$ are p-quasi-secondary, then $N_1 + \cdots + N_t$ is a p-quasi-secondary submodule of M.

(iii) If $N_1 + \cdots + N_2$ is a p-quasi-secondary submodule of M. Then $N_j$ is p-quasi-secondary for some $j, 1 \leq j \leq t$.

3. Multiplication Modules

In this short section we give a property of quasi-secondary submodules of a multiplication module.

Lemma 3.1. let M be a multiplication module and N be a p-quasi-secondary submodule of M. Then $\theta(M) \subseteq p$.

Proof. Suppose that $\theta(M) \subseteq p$ and $0 \neq n \in N$. Then $Rn = \theta(M)Rn \subseteq pn$. Hence $n = p_0n$ for some $p_0 \in p$. By our assumption there exists a positive integer t such that $p_0^t N = 0$. Therefore $n = p_0^tn = 0$, a contradiction.
Theorem 3.2. Suppose that \( M \) is a faithfull multiplication module and \( N \) a \( p \)-quasi-secondary submodule of \( M \). Then \( pM \) is a prime submodule of \( M \). In particular, if \( p \in \text{max}(R) \), then \( pM \) is a maximal submodule of \( M \).

Proof. By Lemma 3.1, \( \theta(M) \not\subset p \). Now suppose that \( pM = M = RM \). Then by [1,Theorem1.5] \( R \cap \theta(M) = \theta(M) = p \cap \theta(M) \) and hence \( \theta(M) \subset p \), which is a contradiction. Thus \( pM \neq M \) and the result of the first part follows from [6, Lemma 2.4(2)]. The last part can be deduced from the first part and [6, Corollary 2.7].

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References