Quasi-Secondary Submodules

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Abstract

Let $R$ be a commutative ring with non-zero identity and $M$ be a unitary $R$-module.

Then the concept of quasi-secondary submodules of $M$ is introduced and some results concerning this class of submodules is obtained.

1. Introduction

Throughout this paper all rings are commutative with non-zero identity and all modules are unitary. In [4] L. Fuchs introduced and studied the concept of quasi-primary ideals (see also [5]). An ideal $I$ of a ring $R$ is called a quasi-primary ideal of $R$ if the radical of $I$ is a prime ideal of $R$. This concept then generalized to modules, i.e., the concept of quasi-primary submodules of a module introduced and developed in [3]. Here, we introduce the dual notation, that is, the quasi-secondary submodules of a module and obtain some results concerning this class of submodules. In section 2, we obtain some preliminary properties of quasi-secondary submodules. Section 3 is devoted to the quasi-secondary submodules of a multiplication module. Now we define some concepts which will be needed in sequel.

Let $M$ be an $R$-module and $N$ a submodule of it. The ideal $\{ r \in R \mid rM \subseteq N \}$ will be denoted by $(N_R)_R M$; in particular $(0_R)_R M$ is called the annihilator of $M$. A non-zero submodule $N$ of $M$ is called a secondary (resp. second) submodule of $M$ if for each $r \in R$ the homothety $N \rightarrow N^r$ is surjective or nilpotent (resp. surjective or zero). In this case $\sqrt{(0_R)_R N}$ is a prime ideal, say $p$, and we call $N$ a $p$-secondary (resp. a $p$-second) submodule of $M$. We refer readers for more details concerning secondary (resp. second) submodule to [9] (resp. [12]).

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An R-module $M$ is said to be a *multiplication* module if for each submodule $N$ of $M$ there exists an ideal $I$ of $R$ such that $N = IM$. It is easy to see that in this case $N = (N \cap N^r)M$. Also the ideal $\theta(M)$ is defined as $\theta(M) = \sum_{m \in M}(Rm \cap N^r)$. If $M$ is a multiplication module and $N$ is a submodule of it, then $M = \theta(M)M$ and $N = \theta(M)N$. (see [1]). An R-module $M$ is *sum-irreducible* if $M \not\subset 0$ and the sum of any two proper submodules of $M$ is always a proper submodule. Finally a proper submodule $N$ of an $R$-module $M$ is called a *prime submodule* if for each $r \in R$ the homothety $M/N \rightarrow M/N$ is either injective or zero. This implies that $\text{Ann}(M/N) = p$ is a prime ideal of $R$, and $N$ is said to be a *$p$-prime submodule* (c.f. [7], [8], [10] and [11]).

## 2. Quasi-Secondary Submodules

The starting point of this section is the definition of quasi-secondary submodules of a module.

**Definition 2.1.** Let $M$ be a non-zero R-module. Then the non-zero submodule $N$ of $M$ is said to be *quasi-secondary* if $\sqrt{(0 :_R^N)} = p$ where $p$ is a prime ideal of $R$. It is obvious that every secondary (or second) submodule of a module is a quasi-secondary submodule, but the converse is not true in general. For example, $2Z$ is a 0-quasi-secondary submodule of the $Z$-module $Z$ but it is not 0-secondary (or 0-second) submodule. (Here $Z$ denotes the set of all integers.)

**Remark 2.2.**

(i) Let $M$ be a non-zero R-module and $N$ a submodule of it such that $\sqrt{(0 :_R^N)} = m(m \in \text{Max}(R))$. Then $N$ is $m$-secondary (m-second).

(ii) Every quasi-secondary submodule of a module over a zero-dimentional ring (i.e., a ring in which every prime ideal is a maximal ideal) is secondary.

(iii) Every quasi-secondary submodule of a module over a D.V.R is secondary.

**Definition 2.3.** Let $M$ be an R-module and $N$ a submodule of $M$. An element $r$ of $R$ is called *co-primal* to $N$ if $rN = N$. Denote by $W(N)$ the set of all elements of $R$ that are not co-primal to $N$. The submodule $N$ is said to be a co-primal submodule of $M$ if $W(N)$ is an ideal of $R$. This ideal is always a prime ideal. In this case we say that $N$ is a *$p$-co-primal* submodule of $M$. The class of co-primal submodules of a module is a
fairly large class. For example, all secondary (second) submodules are co-primal. Also it is easy to see that a sum-irreducible submodule of a module is co-primal. But, in general, a quasi-secondary submodule of a module may not be a co-primal submodule. (consider the Z-module $\mathbb{Z}$.) It is worth to mention that in [2] the term secondal is used for co-primal submodules. The next proposition characterizes those $p$-quasi-secondary submodules which are $p$-co-primal.

Proposition 2.4. Let $N$ be a $p$-quasi-secondary submodule of an $R$-module $M$. Then $N$ is a $p$-co-primal submodule of $M$ if and only if it is a $p$-secondary submodule of $M$.

Proof $\Rightarrow$ Let $N \rightarrow R$ be the $R$-endomorphism of $N$ given by multiplication by $r$ of $R$ and $rN \neq N$. Then by our assumption $r \in p = \{ s \in R | sN \neq N \}$. On the other hand, $p = \sqrt{0_R}N$ and so there exists a positive integer $t$ such that $r^t N = 0$. The result follows. $\Leftarrow$ Is obvious.

The proof of two next propositions is easy and so we state them without proof.

Proposition 2.5. Let $M$ be a module over an integral domain and $N$ be a $0$-co-primal submodule of $M$. Then $N$ is $0$-secondary.

Proposition 2.6. Let $M$ be an $R$-module and $N_1, N_2, ..., N_t$ be submodules of $M$. Then

(i) Suppose that for $i = 1, 2, N_i$ is $p_i$-quasi-secondary. Then $N_1 + N_2$ is quasi-secondary if and only if $p_1 \subseteq p_2$ or $p_2 \subseteq p_1$.

(ii) If $N_1, ..., N_t$ are $p$-quasi-secondary, then $N_1 + ... + N_t$ is a $p$-quasi-secondary submodule of $M$.

(iii) If $N_1 + ... + N_2$ is a $p$-quasi-secondary submodule of $M$. Then $N_j$ is $p$-quasi-secondary for some $j, 1 \leq j \leq t$.

3. Multiplication Modules

In this short section we give a property of quasi-secondary submodules of a multiplication module.

Lemma 3.1. let $M$ be a multiplication module and $N$ be a $p$-quasi-secondary submodule of $M$. Then $\theta(M) \nsubseteq p$.

Proof. Suppose that $\theta(M) \subseteq p$ and $0 \neq n \in N$. Then $Rn = \theta(M)Rn \subseteq pn$. Hence $n = p_0 n$ for some $p_0 \in p$. By our assumption there exists a positive integer $t$ such that $p_0^t N = 0$. Therefore $n = p_0^t n = 0$, a contradiction.
**Theorem 3.2.** Suppose that $M$ is a faithful multiplication module and $N$ a $p$-quasi-secondary submodule of $M$. Then $pM$ is a prime submodule of $M$. In particular, if $p \in \text{max}(R)$, then $pM$ is a maximal submodule of $M$.

**Proof.** By Lemma 3.1, $\theta(M) \nsubseteq p$. Now suppose that $pM = M = RM$. Then by [1, Theorem 1.5] $R \cap \theta(M) = \theta(M) = p \cap \theta(M)$ and hence $\theta(M) \subseteq p$ which is a contradiction. Thus $pM \neq M$ and the result of the first part follows from [6, Lemma 2.4(2)]. The last part can be deduced from the first part and [6, Corollary 2.7].

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**References**