Abstract
Let $R$ be a commutative ring with non-zero identity and $M$ be a unitary $R$-module.
Then the concept of quasi-secondary submodules of $M$ is introduced and some results concerning this class of submodules is obtained.

1. Introduction
Throughout this paper all rings are commutative with non-zero identity and all modules are unitary. In [4] L. Fuchs introduced and studied the concept of quasi-primary ideals (see also [5]). An ideal $I$ of a ring $R$ is called a quasi-primary ideal of $R$ if the radical of $I$ is a prime ideal of $R$. This concept then generalized to modules, i.e., the concept of quasi-primary submodules of a module introduced and developed in [3]. Here, we introduce the dual notation, that is, the quasi-secondary submodules of a module and obtain some results concerning this class of submodules. In section 2, we obtain some preliminary properties of quasi-secondary submodules. Section 3 is devoted to the quasi-secondary submodules of a multiplication module. Now we define some concepts which will be needed in sequel.

Let $M$ be an $R$-module and $N$ a submodule of it. The ideal $\{r \in R | rM \subseteq N\}$ will be denoted by $(N_R M)$; in particular $(0_R M)$ is called the annihilator of $M$. A non-zero submodule $N$ of $M$ is called a secondary (resp. second) submodule of $M$ if for each $r \in R$ the homothety $N \rightarrow N \overset{r}{\rightarrow}$ is surjective or nilpotent (resp. surjective or zero). In this case $\sqrt{(0_R N)}$ is a prime ideal, say $p$, and we call $N$ a $p$-secondary (resp. a $p$-second) submodule of $M$. We refer readers for more details concerning secondary (resp. second) submodulse to [9] (resp. [12]).

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An $R$-module $M$ is said to be a multiplication module if for each submodule $N$ of $M$ there exists an ideal $I$ of $R$ such that $N = IM$. It is easy to see that in this case $N = (N_M)M$. Also the ideal $\theta(M)$ is defined as $\theta(M) := \sum_{m \in M} (Rm)_N$. If $M$ is a multiplication module and $N$ is a submodule of it, then $M = \theta(M)M$ and $N = \theta(M)N$. (see [1]). An $R$-module $M$ is sum-irreducible if $M \neq 0$ and the sum of any two proper submodules of $M$ is always a proper submodule. Finally a proper submodule $N$ of an $R$-module $M$ is called a prime submodule if for each $r \in R$ the homothety $M_N \rightarrow M_N$ is either injective or zero. This implies that $\text{Ann}(M/N) = p$ is a prime ideal of $R$, and $N$ is said to be a $p$-prime submodule (c.f. [7], [8], [10] and [11]).

2. Quasi-Secondary Submodules

The starting point of this section is the definition of quasi-secondary submodules of a module.

Definition 2.1. Let $M$ be a non-zero $R$-module. Then the non-zero submodule $N$ of $M$ is said to be quasi-secondary if $\sqrt{(0_r \cap N)} = p$ where $p$ is a prime ideal of $R$. It is obvious that every secondary (or second) submodule of a module is a quasi-secondary submodule, but the converse is not true in general. For example, $2\mathbb{Z}$ is a 0-quasi-secondary submodule of the $\mathbb{Z}$-module $\mathbb{Z}$ but it is not 0-secondary (or 0-second) submodule. (Here $\mathbb{Z}$ denotes the set of all integers.)

Remark 2.2.

(i) Let $M$ be a non-zero $R$-module and $N$ a submodule of it such that $\sqrt{(0_r \cap N)} = m(m \in \text{Max} (R))$. Then $N$ is $m$-secondary ($m$-second).

(ii) Every quasi-secondary submodule of a module over a zero-dimentional ring (i.e., a ring in which every prime ideal is a maximal ideal) is secondary.

(iii) Every quasi-secondary submodule of a module over a D.V.R is secondary.

Definition 2.3. Let $M$ be an $R$-module and $N$ a submodule of $M$. An element $r$ of $R$ is called co-primal to $N$ if $rN = N$. Denote by $W(N)$ the set of all elements of $R$ that are not co-primal to $N$. The submodule $N$ is said to be a co-primal submodule of $M$ if $W(N)$ is an ideal of $R$. This ideal is always a prime ideal. In this case we say that $N$ is a $p$-co-primal submodule of $M$. The class of co-primal submodules of a module is a...
fairly large class. For example, all secondary (second) submodules are co-primal. Also it is easy to see that a sum-irreducible submodule of a module is co-primal. But, in general, a quasi-secondary submodule of a module may not be a co-primal submodule. (consider the Z-module Z_2). It is worth to mention that in [2] the term secondal is used for co-primal submodules. The next proposition characterizes those p-quasi-secondary submodules which are p-co-primal.

Proposition 2.4. Let N be a p-quasi-secondary submodule of an R-module M. Then N is a pco-primal submodule of M if and only if it is a p-secondary submodule of M.

Proof. \( \Rightarrow \) Let \( N \mapsto N \) be the R-endomorphism of N given by multiplication by r of R and \( rN \neq N \). Then by our assumption \( r \in p = \{ s \in R | sN \neq N \} \). On the other hand, \( p = \sqrt{\overline{0}_{R_n}} \) and so there exists a positive integer t such that \( r^t N = 0 \). The result follows. \( \Leftarrow \) Is obvious.

The proof of two next propositions is easy and so we state them without proof.

Proposition 2.5. Let M be a module over an integral domain and N be a 0- co-primal submodule of M. Then N is 0-secondary.

Proposition 2.6. Let M be an R-module and \( N_1, N_2, \ldots, N_t \) be submodules of M. Then
(i) Suppose that for \( i = 1, 2, N_i \) is \( p_i \)-quasi-secondary. Then \( N_1 + N_2 \) is quasi-secondary if and only if \( p_1 \subseteq p_2 \) or \( p_2 \subseteq p_1 \)
(ii) If \( N_1, \ldots, N_t \) are p-quasi-secondary, then \( N_1 + \cdots + N_t \) is a p-quasi-secondary submodule of M.
(iii) If \( N_1 + \cdots + N_2 \) is a p-quasi-secondary submodule of M. Then \( N_j \) is p-quasi-secondary for some \( j, 1 \leq j \leq t \).

3. Multiplication Modules

In this short section we give a property of quasi-secondary submodules of a multiplication module.

Lemma 3.1. Let M be a multiplication module and N be a p-quasi-secondary submodule of M. Then \( \theta(M) \not\subseteq p \).

Proof. Suppose that \( \theta(M) \subseteq p \) and \( 0 \neq n \in N \). Then \( Rn = \theta(M)Rn \subseteq pn \). Hence \( n = p_0n \) for some \( p_0 \in p \). By our assumption there exists a positive integer t such that \( p_0^t N = 0 \). Therefore \( n = p_0^t n = 0 \), a contradiction.
Theorem 3.2. Suppose that $M$ is a faithfulfull multiplication module and $N$ a $p$-quasi-secondary submodule of $M$. Then $pM$ is a prime submodule of $M$. In particular, if $p \in \text{max}(R)$, then $pM$ is a maximal submodule of $M$.

Proof. By Lemma 3.1, $\theta(M) \not\subseteq p$. Now suppose that $pM = M = RM$. Then by [1, Theorem 1.5] $R \cap \theta(M) = \theta(M) = p \cap \theta(M)$ and hence $\theta(M) \subseteq p$ which is a contradiction. Thus $pM \neq M$ and the result of the first part follows from [6, Lemma 2.4(2)]. The last part can be deduced from the first part and [6, Corollary 2.7).

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References