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On Quasi-cofaithful Ideals

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Abstract

We introduce quasi-cofaithful ideals which are a generalization of cofaithful ideals, and investigate their properties. We hold that a faithful ideal I is quasi-cofaithful if I contains a finitely generated faithful ideal I_1. We show that every faithful ideal of R is quasi-cofaithful if and only if every faithful ideal of $M_n(R)$ is quasi-cofaithful. Also we show that if R has the descending chain condition on right annihilators of right ideals, then each faithful ideal of R is quasi-cofaithful. For a u.p.-monoid M, it is shown that if R is a quasi-Baer ring, then each faithful ideal of R is quasi-cofaithful if and only if each faithful ideal of the monoid ring $R[M]$ is quasi-cofaithful.

1. Introduction

Throughout this paper R denotes an associative ring with identity and all modules are unitary. A module M_R is said to be *cofaithful* if there exists a finite subset $X \subseteq M$ such that $r_R(X) = \{r \in R \mid Xr = 0\} = 0$, or equivalently, if for some direct sum M^k of k copies of M_R there exists an exact sequence $0 \to R \to M^k$. Recall that an ideal (right ideal) I of a ring R is called *faithful* whenever $r_R(I) = 0$. For a nonempty subset X of R, $r_R(X)$, $< X >$ and $\ell_R(X)$ denote the right annihilator of X in R, the ideal generated by X and the left annihilator of X in R, respectively. Beachy and Blair [1] studied rings that satisfy the condition that every faithful right ideal I of R is cofaithful.

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Faith [5] introduced the class of right zip rings. Faith [5] called a ring R *right zip* provided that if the right annihilator of a subset X of R, $r_R(X)$, is zero, then $r_R(Y)=0$ for a finite subset $Y \subseteq X$; equivalently, for a left ideal L of R with $r_R(L)=0$, there exists a finitely generated left ideal $L_1 \subseteq L$ such that $r_R(L_1)=0$. *Left zip* rings are defined similarly. A ring R is *zip* if it is right and left zip.

If R is a right zip ring, then every faithful right ideal I of R is cofaithful. In fact, the class of right zip rings is a proper subclass of rings whose faithful right ideals are cofaithful. Note that a commutative ring R is zip if and only if every faithful (right) ideal of R is cofaithful. We denote the $n \times n$ full matrix ring over R by $M_n(R)$.

Beachy and Blair [1, Proposition 1.9] proved that if R is a commutative ring in which every faithful ideal is cofaithful, then every faithful ideal of $R[x]$ is cofaithful. Cedó [3, Example 1] proved that there exists right (or left) zip ring R such that $M_2(R)$ is not right (left) zip. Also, he proved that if R is a commutative zip ring, then $M_n(R)$ is a right zip ring. Zelmanowitz [13, Introduction] noted that any ring satisfying the descending chain condition on right annihilators is right zip, and he also showed that there exist commutative zip rings which do not satisfy the descending chain condition on (right) annihilators.

Faith [6, Corollary 8.5] proved that if R is a commutative zip ring and G is a finite abelian group, then the group ring $R[G]$ of G over R is zip.

A ring R is called *Armendariz* if whenever polynomials $f(x)=a_0+a_1x+\cdots+a_nx^n$, $g(x)=b_0+b_1x+\cdots+b_mx^m \in R[x]$ satisfy $f(x)g(x)=0$, then $a_ib_j=0$ for each i,j.

Hong, et al. [9] studied the extensions of noncommutative zip rings. If R is an Armendariz ring, then R is right zip if and only if $R[x]$ is right zip if and only if $R[x,x^{-1}]$ is right zip.

Motivated by results in Beachy and Blair [1], Cedó [3], Zelmanowitz [13] and Hong et al. [9], we investigate a generalization of cofaithful ideals which we call quasi-cofaithful ideals. We say a faithful ideal I is *quasi-cofaithful* if I contains a finitely generated faithful ideal I_1.

We show that every faithful ideal of R is quasi-cofaithful if and only if every faithful ideal of $M_n(R)$ is quasi-cofaithful. Also we show that if R has the descending chain condition on right annihilators of ideals, then each faithful ideal of R is quasi-cofaithful. For a u.p.-monoid M, it is shown that if R is a quasi-Baer ring, then each faithful ideal of R is quasi-cofaithful if and only if each faithful ideal of monoid ring $R[M]$ is quasi-cofaithful. As a corollary, we show that if R is a semiprime or quasi-Baer ring, then each faithful ideal of R is quasi-cofaithful if and only if each faithful ideal of $R[x]$ is quasi-cofaithful if and only if each faithful ideal of $R[x, x^{-1}]$ is quasi-cofaithful. Also we show that each faithful ideal of R is quasi-cofaithful if and only if for each $n \geq 2$, each faithful ideal of $R[x]/ < x^n >$ is quasi-cofaithful.

2. On quasi-cofaithful ideals

Definition 2.1. Let I be a faithful ideal of a ring R. We say that I is *quasi-cofaithful* whenever I contains a finitely generated faithful ideal I_1 of R.

Obviosly, cofaithful ideals are quasi-cofaithful. Also faithful ideals of right zip rings and faithful ideals of prime rings are quasi-cofaithful. If R is a commutative ring, then R is zip if and only if all faithful ideals of R are quasi-cofaithful.

Remark. Cedó [3, Example 1] proved that there exists right (or left) zip ring R such that $M_2(R)$ is not right (left) zip.

The following theorem shows that there exists non right zip ring R such that all faithful ideals of R are quasi-cofaithful.

Theorem 2.2. Let R be a ring and n a positive integer. Then each faithful ideal of R is quasi-cofaithful if and only if each faithful ideal of $M_n(R)$ is quasi-cofaithful.

Proof. Assume that each faithful ideal of R is quasi-cofaithful. Let J be a faithful ideal of $M_n(R)$. There exists a faithful ideal I of R such that $J = M_n(I)$. Since each faithful ideal of R is quasi-cofaithful, there exists a finite subset $B \subseteq I$ such that $r_R(< B >) = 0$. Then $r_{M_n(R)}(M_n(< B >)) = 0$. Since $< B >$ is a finitely generated ideal of R, hence $M_n(< B >)$ is a finitely generated ideal of $M_n(R)$. Therefore J is quasi-cofaithful.

Conversely, suppose that each faithful ideal of $M_n(R)$ is quasi-cofaithful. Let I be a faithful ideal of R. Then $M_n(I)$ is a faithful ideal of $M_n(R)$. Since each faithful ideal of $M_n(R)$ is quasi-cofaithful, there exist $A_1,\dots,A_m \in M_n(I)$ such that $r_{M_n(R)}(<A_1,\dots,A_m>)=0$. There exists an ideal I_1 of R such that $<A_1,\dots,A_m>=M_n(I_1)$. Clearly $I_1 \subseteq I$ and I_1 is a finitely generated ideal of R. If $a \in r_R(I_1)$, then $A_i M_n(R) I_{n\times n} a = 0$ for $i=1,\dots,m$, where $I_{n\times n}$ is the identity matrix of $M_n(R)$. Thus $I_{n\times n} a \in r_{M_n(R)}(<A_1,\dots,A_m>)=0$, and so $a=0$. Hence I is quasi-cofaithful.

Proposition 2.3. A ring R has the descending chain condition on right annihilators of ideals if and only if for each ideal I of R there exists a finitely generated ideal $I_1 \subseteq I$ such that $r_R(I_1)=r_R(I)$.

Proof. Assume that R has the descending chain condition on right annihilators of ideals. Let I be an ideal of R. Assume that $X=\{r_R(J) \mid J$ is a finitely generated ideal of R contained in $I\}$. Choose a finitely generated ideal I_1 of R contained in I such that $r_R(I_1)$ is a minimal element of X. We claim that $r_R(I_1)=r_R(I)$. Clearly, $r_R(I) \subseteq r_R(I_1)$. Now, let $a \in r_R(I_1)$ such that $a \notin r_R(I)$.Then $xa \neq o$, for some $x \in I$. Let $I_2 =< I_1, x >$ be the ideal of R generated by $I_1 \bigcup \{x\}$. Then I_2 is finitely generated and contained in I and $r_R(I_2) \subset r_R(I_1)$, which is a contradiction.

Conversely, let $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ be a descending chain of right annihilators of ideals of R. Let $I=\bigcup_{i=1}^{\infty} \ell_R(A_i)$. Clearly I is an ideal of R. Then there exists a subset $\{x_1,\dots,x_n\} \subseteq I$ such that $r_R(<x_1,\dots,x_n>)=r_R(I)$. There exists a positive integer k such that $\{x_1,\dots,x_n\} \subseteq \ell_R(A_i)$, for all $i \geq k$. But for $i \geq k$, $A_i = r_R(\ell_R(A_i)) \subseteq r_R(<x_1,\dots,x_n>) = r_R(I) \subseteq A_i$. Hence $A_i = r_R(I)$ for each $i \geq k$ and the chain terminates at A_k.

Theorem 2.4. Let R has the descending chain condition on right annihilators of ideals. Then each faithful ideal of R is quasi-cofaithful.

Proof. If follows from Proposition 2.3.

Proposition 2.5. Let R be a ring such that every faithful ideal is quasi-cofaithful.

1- R is not a direct product of infinitely many (nontrivial) rings.

2- If R is semiprime, then it contains no infinite direct sum of nonzero ideals.

Proof. (1). Assume that $R=\prod_{i\in I} R_i$, where I is an infinite set and R_i is a ring, for each $i\in I$. Let $A=\oplus_{i\in I} R_i$. Then A is a faithful ideal of R but not quasi-cofaithful.

(2). Assume that $A=\oplus_{i=1}^{\infty} A_i$ is an infinite direct sum of ideals of R. First we show that $A\bigcap r_R(A)=0$ and $A\oplus r_R(A)$ is a faithful ideal of R. Let $I=r_R(A\oplus r_R(A))$. Then $I\subseteq r_R(A)$, and $r_R(A)I=0$. Hence $I^2=0$ and so $I=0$, since R is semiprime. By a similar argument one can show that $A\bigcap r_R(A)=0$.

By assumption there exist $x_1,\dots,x_k\in A\oplus r_R(A)$, such that $r_R(x_1R+\cdots+x_kR)=0$. But there exists an integer n such that $x_i\in A_1\oplus\cdots\oplus A_n\oplus r_R(A)$ for all i, and so for any $0\neq y\in A_{n+1}$ we have $y\in r_R(x_1R+\cdots+x_kR)=0$, a contradiction.

We denote the $n\times n$ upper triangular matrix ring over a ring R by $T_n(R)$.

Remark. Hong et al. [9, Proposition 4] showed that there exists an $n\times n$ upper triangular matrix ring over a right zip ring which is not right zip for any $n\geq 2$.

The next theorem allows us to construct numerous examples of non right zip rings such that each faithful ideal is quasi-cofaithful. For each $1\leq i,j\leq n$, let E_{ij} be the matrix units.

Theorem 2.6. Let R be a ring and $n\geq 2$. Then each faithful ideal of R is quasi-cofaithful if and only if each faithful ideal of $T_n(R)$ is quasi-cofaithful.

Proof. Assume that each faithful ideal of $T_n(R)$ is quasi-cofaithful. Let I be a faithful ideal of R. Then $T_n(I)$ is a faithful ideal of $T_n(R)$ and so there exist $A_1,\dots,A_m\in T_n(I)$ such that $r_{T_n(R)}(<A_1,\dots,A_m>)=0$. Let $A_s=(a_{ij}^s)$ for $s=1,\dots,m$. Let $X=\{a_{ij}^s\mid s=1,\dots,m, 1\leq i,j\leq n\}$ and $J=<X>$. Since $r_{T_n(R)}(<A_1,\dots,A_m>)=0$, we have $r_R(J)=0$. Therefore I is quasi-cofaithful.

Conversely, assume that each faithful ideal of R is quasi-cofaithful. Let X be a faithful ideal of $T_n(R)$. Let $I_{1i}=\{a\in R\mid aE_{1i}\in X\}$ for $i=1,\dots,n$. Since $E_{11}XE_{ii}\subseteq X$ and $E_{11}XE_{1i}\subseteq X$, hence I_{1i} is an ideal of R and $I_{11}\subseteq I_{1i}$, for $i=1,\dots,n$. Thus $Y=I_{11}E_{11}+I_{11}E_{12}+\cdots+I_{11}E_{1n}\subseteq Y$ is an ideal of $T_n(R)$. Since X is a faithful ideal of $T_n(R)$, hence $r_R(I_{11})=0$. Since each faithful ideal of R is quasi-cofaithful, there exists a finitely generated faithful ideal J_{11} of R such that $J_{11}\subseteq I_{11}$. Then $J_{11}E_{11}+J_{11}E_{12}+\cdots+J_{11}E_{1n}\subseteq X$ is a finitely generated ideal of $T_n(R)$. One can easily check that $r_{T_n(R)}(J_{11}E_{11}+J_{11}E_{12}+\cdots+J_{11}E_{1n})=0$. Therefore X is quasi-cofaithful.

Now we consider a special kind of subring of $n\times n$ upper triangular matrix ring. For a ring R and $n\geq 2$, we consider the ring $R_n(R)=\left\{\begin{bmatrix} a & a_{12} & \cdots & a_{1n}\\ 0 & a & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & a\end{bmatrix} \mid a,a_{ij}\in R\right\}$.

Theorem 2.7. Let R be a ring and $n\geq 2$. Then each faithful ideal of R is quasi-cofaithful if and only if each faithful ideal of $R_n(R)$ is quasi-cofaithful.

Proof. Assume that each faithful ideal of R is quasi-cofaithful. Let J be a faithful ideal of $R_n(R)$. Let $I=\left\{a\in R \mid \begin{bmatrix} a & a_{12} & \cdots & a_{1n}\\ 0 & a & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0\end{bmatrix}\in J \ \textit{for some}\ a_{ij}\in R\right\}$. Clearly I is an ideal of R. If $b\in r_R(I)$, then $\begin{bmatrix} a & a_{12} & \cdots & a_{1n}\\ 0 & a & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & a\end{bmatrix}\begin{bmatrix} 0 & 0 & \cdots & b\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0\end{bmatrix}=0$ for each $\begin{bmatrix} a & a_{12} & \cdots & a_{1n}\\ 0 & a & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & a\end{bmatrix}\in J$. Hence $b=0$ and so $r_R(I)=0$. Since each faithful ideal of R is quasi-cofaithful, there exist $a_1,\ldots,a_m\in I$ such that $r_R(a_1R+\cdots+a_mR)=0$. Then $A_k=\begin{bmatrix} a_k & a_{k_{12}} & \cdots & a_{k_{1n}}\\ 0 & a_k & \cdots & a_{k_{2n}}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & a_k\end{bmatrix}\in J$ for some $a_{k_{ij}}\in R$, where $k=1,\ldots,m$. We claim that $r_{R_n(R)}(A_1R_n+\cdots+A_mR_n)=0$. Let $B=\begin{bmatrix} b & b_{12} & \cdots & b_{1n}\\ 0 & b & \cdots & b_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & b\end{bmatrix}\in r_{R_n(R)}(A_1R_n+\cdots+A_mR_n)$. Then $b\in r_R(a_1R+\cdots+a_mR)=0$. Thus $B=\begin{bmatrix} 0 & b_{12} & \cdots & b_{1n}\\ 0 & 0 & \cdots & b_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0\end{bmatrix}$. Since $A_iaE_{1j}B=0$ for each $a\in R$ and $j=2,\ldots,n$, we have $b_{jk}\in r_R(a_1R+\cdots+a_mR)=0$, for $j=2,\ldots,n$ and

$k=3,\dots,n$. Thus $B=\begin{bmatrix} 0 & b_{12} & \cdots & b_{1n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$. Since $B \in r_{R_n(R)}(A_1R_n+\cdots+A_mR_n)$, we have $b_{1j} \in r_R(a_1R+\cdots+a_mR)=0$, for $j=2,\dots,n$, and so $B=0$. Therefore J is quasi-cofaithful.

Conversely, assume that each faithful ideal of $R_n(R)$ is quasi-cofaithful. Let I be a faithful ideal of R. Assume that $J=\left\{\begin{bmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{bmatrix} \mid a, a_{ij} \in I\right\}$. Clearly J is an ideal of $R_n(R)$. We claim that J is faithful. Let $B=\begin{bmatrix} b & b_{12} & \cdots & b_{1n} \\ 0 & b & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b \end{bmatrix} \in r_{R_n(R)}(J)$.

Then $\begin{bmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{bmatrix}\begin{bmatrix} b & b_{12} & \cdots & b_{1n} \\ 0 & b & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b \end{bmatrix}=0$, for each $a \in I$. Hence $b, b_{ij} \in r_R(I)=0$ and so $r_{R_n(R)}(J)=0$. Since each faithful ideal of $R_n(R)$ is quasi-cofaithful, there exist $A_1=\begin{bmatrix} a_1 & a_{1_{12}} & \cdots & a_{1_{1n}} \\ 0 & a_1 & \cdots & a_{1_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_1 \end{bmatrix}, \cdots, A_k=\begin{bmatrix} a_k & a_{k_{12}} & \cdots & a_{k_{1n}} \\ 0 & a_k & \cdots & a_{k_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_k \end{bmatrix} \in J$ such that $r_{R_n(R)}(A_1R_n(R)+\cdots+A_kR_n(R))=0$. We claim that $r_R(a_1R+\cdots+a_kR)=0$. Let $b \in r_R(a_1R+\cdots+a_kR)$. Then $(A_1R_n(R)+\cdots+A_kR_n(R))\begin{bmatrix} 0 & 0 & \cdots & b \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}=0$, and so

$\begin{bmatrix} 0 & 0 & \cdots & b \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in r_{R_n(R)}(A_1R_n(R)+\cdots+A_kR_n(R))=0$. Therefore I is quasi-cofaithful.

Corollary 2.8. Let R be a ring and $n \geq 1$. Then each faithful ideal of R is quasi-cofaithful if and only if each faithful ideal of $\frac{R[x]}{<x^{n+1}>}$ is quasi-cofaithful.

Proof. Since $\frac{R[x]}{<x^{n+1}>} \cong \left\{ \begin{bmatrix} a & a_1 & \cdots & a_n \\ 0 & a & \cdots & a_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{bmatrix} \mid a, a_j \in R \right\}$, hence by a similar argument as used in the proof of Theorem 2.7, one can prove it.

According to Hirano [8], a ring R is called to be *quasi-Armendariz* if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n$, $g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x]$ satisfy $f(x)R[x]g(x) = 0$, then $a_iRb_j = 0$ for each i, j. If R is a commutative ring, then R is quasi-Armendariz if and only if it is Armendariz. It should be notied that *reduced rings* (i.e., rings with no nonzero nilpotent elements) are Armendariz and quasi-Armendariz. The following examples show that there exist quasi-Armendariz rings such that some of faithful ideals are not quasi-cofaithful.

Example 2.9. (1) Let F be a field and $R = \prod_{i=1}^{\infty} F_i$ where $F_i = F$ for all i. Then R is reduced and so it is quasi-Armendariz. Let $I = \oplus_{i=1}^{\infty} F_i$. Then I is a faithful ideal of R. But for each $a_1, \ldots, a_m \in I$, $r_R(a_1R + \cdots + a_mR) \neq 0$. Hence I is not quasi-cofaithful.

(2) Let R be the ring in (1). Then $M_n(R)$ and $T_n(R)$ are quasi-Armendariz, by [8, Theorem 3.12 and Corollary 3.15]. But $M_n(R)$ and $T_n(R)$ have faithful ideals which are not quasi-cofaithful, by Theorems 2.2 and 2.6.

According to [7], for a monoid M, a ring R is called M*-quasi-Armendariz*, if $\alpha = a_1g_1 + \cdots + a_ng_n$, $\beta = b_1h_1 + \cdots + b_mh_m \in R[M]$ satisfy $\alpha R[M]\beta = 0$, then $a_iRb_j = 0$ for each i, j. Clearly, for $M = N \cup \{0\}$, a ring R is M-quasi-Armendariz if and only if R is quasi-Armendariz.

Recall that a monoid M is called a *unique product* monoid (or *u.p.-monoid*) if for any two nonempty finite subsets $A, B \subseteq M$ there exists an element $g \in M$ uniquely presented in the form ab where $a \in A$ and $b \in B$. The class of u.p.-monoids is quite large and important. For example, this class includes the right or left ordered monoids, submonoids of a free group, and torsion-free nilpotent groups. Every u.p.-monoid M has no non-unity element of finite order. We use $R[M]$ to denote the monoid ring M over R.

An ordered set $(S,\leq)$ is called *artinian* if every strictly decreasing sequence of elements of S is finite, and $(S,\leq)$ is called *narrow* if every subset of pairwise orderincomparable elements of S is finite. It is easy to see that $(S,\leq)$ is artinian an narrow if and only if every nonempty subset of S contains at least one, but only a finite number, of minimal elements. For a partially ordered set X we write $\min X$ to denote the set of minimal elements of X.

Marks et. al. [10] introduced a new class of u.p.-monoids as follows: An ordered monoid $(M,\leq)$ is called an *artinian narrow unique product* monoid (or an *a.n.u.p.-monoid*, or simply a.n.u.p.) if for every two artinian and narrow subsets A and B of M there exists a u.p. element in the product AB. Also an ordered monoid $(M,\leq)$ is called a *minimal artinian narrow unique product* monoid (or a *m.a.n.u.p.-monoid*, or simply m.a.n.u.p.) if for every two artinian and narrow subsets A and B of M there exist $a\in \min A$ and $b\in \min B$ such that ab is a u.p. element of AB.

For an ordered monoid $(M,\leq)$, Marks, Mazurek and Ziembowski [10] proved the following implications:

$$
\begin{array}{c}
M \text{ is a commutative, torsion-free, cancellative monoid}\\
\Downarrow\\
(M,\leq) \text{ is quasitotally ordered}\\
\Downarrow\\
(M,\leq) \text{ is a m.a.n.u.p.-monoid}\\
\Downarrow\\
(M,\leq) \text{ is an a.n.u.p.-monoid}\\
\Downarrow\\
M \text{ is a u.p.-monoid}
\end{array}
$$

Also they showed that all of the implications in diagram above are irreversible.

Theorem 2.10. Let R be a ring and M a u.p.-monoid. If R is M-quasi-Armendariz, then each faithful ideal of R is quasi-cofaithful if and only if each faithful ideal of $R[M]$ is quasi-cofaithful.

Proof. Assume that each faithful ideal of R is quasi-cofaithful. Let J be a faithful ideal of $R[M]$. Let I be the set of all coefficients of elements of J in R. Then I is a faithful ideal of R, and by assumption there exist $a_1,\ldots,a_m\in I$ such that $r_R(a_1R+\cdots+a_mR)=0$. For each a_i there exists $\alpha_{a_i}\in J$ such that a_i is a one of the coefficients of α_{a_i}. We claim that $r_{R[M]}(\alpha_{a_1}R[M]+\cdots+\alpha_{a_m}R[M])=0$. Let $\beta=b_1h_1+\cdots+b_nh_n\in r_{R[M]}(\alpha_{a_1}R[M]+\cdots+\alpha_{a_m}R[M])$. Then $\alpha_{a_i}R[M]\beta=0$, for each $i=1,\ldots,m$. Hence $a_iRb_j=0$ for each i,j, since R is M-quasi-Armendariz. Thus $b_j\in r_R(a_1R+\cdots+a_mR)=0$, for each $j=1,\ldots,n$, and so $\beta=0$. Therefore J is quasi-cofaithful.

Conversely, assume that each faithful ideal of $R[M]$ is quasi-cofaithful. Let I be a faithful ideal of R. We show that $I[M]$ is a faithful ideal of $R[M]$. Let $\beta=b_1h_1+\cdots+b_nh_n\in r_{R[M]}(I[M])$. Then $Ib_i=0$ for each $i=1,\ldots,n$. Hence $b_i\in r_R(I)=0$ for each i and so $I[M]$ is faithful. There exist $\alpha_1,\ldots,\alpha_m\in I[M]$ such that $r_{R[M]}(\alpha_1R[M]+\cdots+\alpha_mR[M])=0$. Let $a_{11},a_{12},\ldots,a_{1n_1},a_{21},a_{22},\ldots,a_{m1},\ldots,a_{mn_m}$ be the all coefficients of $\alpha_1,\ldots,\alpha_m$ in R. We claim that $r_R(a_{11}R+\cdots+a_{mn_m}R)=0$. If $b\in r_R(a_{11}R+\cdots+a_{mn_m}R)$, then $(\alpha_1R[M]+\cdots+\alpha_mR[M])b=0$, and so $b=0$. Therefore I is quasi-cofaithful.

By [7, Proposition 1.7], if R is a semiprime ring and M a u.p.-monoid, then R is M-quasi-Armendariz. Hence we have the following result.

Corollary 2.11. Let R be a semiprime ring and M a u.p.-monoid. Then each faithful ideal of R is quasi-cofaithful if and only if each faithful ideal of $R[M]$ is quasi-cofaithful.

According to [4], a ring R is called *quasi-Baer* if the right annihilator of every right ideal of R is generated by an idempotent. Note that this definition is left-right symmetric. By [7, Corollary 2.4], if R is a quasi-Baer ring and M a u.p.-monoid, then R is M-quasi-Armendariz. Hence we have the following result.

Corollary 2.12. Let R be a quasi-Baer ring and M a u.p.-monoid. Then each faithful ideal of R is quasi-cofaithful if and only if each faithful ideal of $R[M]$ is quasi-cofaithful.

Note that by [2, Proposition 2.1] if R_α is quasi-Baer for each $\alpha\in\Lambda$, then $R=\prod_{\alpha\in\Lambda}R_\alpha$ is quasi-Baer. Hence Example 2.9(1) shows that there exists a quasi-Baer ring R such that has a faithful ideal which is not quasi-cofaithful.

Beachy and Blair [1, Proposition 1.9], proved that a commutative ring R is zip if and only if $R[x]$ is zip. In Theorem 2.10, if we consider $M=N\bigcup\{0\}$ or $M=Z$, then we have the following result.

Corollary 2.13. Let R be a quasi-Armendariz ring. Then the following are equivalent:

1- Each faithful ideal of R is quasi-cofaithful;

2- Each faithful ideal of $R[x]$ is quasi-cofaithful;

3- Each faithful ideal of $R[x, x^{-1}]$ is quasi-cofaithful.

Since quasi-Baer rings are quasi-Armendariz [8, Corollary 3.11], hence we have the following result.

Corollary 2.14. Let R be a quasi-Baer ring. Then the following are equivalent:

1- Each faithful ideal of R is quasi-cofaithful;

2- Each faithful ideal of $R[x]$ is quasi-cofaithful;

3- Each faithful ideal of $R[x, x^{-1}]$ is quasi-cofaithful.

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