A comparison between the Homotopy Perturbation Method and Adomian's Decomposition Method for Solving Nonlinear Volterra Integral Equations

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Abstract

In this paper, we conduct a comparative study between the homotopy perturbation method (HPM) and Adomian's decomposition method (ADM) for analytic treatment of nonlinear Volterra integral equations, and we show that the HPM with a specific convex homotopy is equivalent to the ADM for these type of equations.

1. Introduction

Nonlinear phenomena, that appear in many applications in scientific fields, such as fluid dynamic, solid state physics, plasma physics, mathematical biology and chemical kinetics, can be modeled by PDEs and by integral equations. The concepts of integral equations have motivated a huge size of research work in recent years. Several analytical and numerical methods were used such as, the direct computation method, the series solution method, the successive approximation method, the ADM and the HPM to integral equations. However, the analytical solutions methods are not easy to use and require tedious works and knowledge, the numerical methods such as the HPM and the ADM has been proved to be effective and reliable for handling differential, ordinary and partial, and integral equations, linear or nonlinear [1-5]. The ADM and the HPM are powerful methods that consider the approximate solution of nonlinear problems as an infinite series converging to the exact solution [6-9].

Several authors have previously compared these methods and attained acceptable results. For

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example, Abbasbandy in [10] compared the HPM and the ADM for the quadratic Riccati differential equation. In another work, he introduced a comparison between the ADM and iterated He's homotopy perturbation method [11]. Özis et al [12] compared the ADM and the HPM for certain nonlinear problems. And Li introduced a comparison between the ADM and the HPM, which showed that these methods are equivalent for solving nonlinear equations [13].

In this paper, firstly we explain the ADM and the HPM to solve nonlinear Volterra integral equations in sections 2 and 3, respectively. Then we show that the HPM with a specific convex homotopy for solving nonlinear Volterra integral equations is equivalent to the ADM.

2. The ADM for Nonlinear Volterra Integral Equations

Consider the following nonlinear Volterra integral equations

$$u(x)=f(x)+\int_0^x k(x,t)T(u(t))\,dt, \tag{1}$$

where the functions k, f and T are given, and u the solution to be determined. We assume that (1) has the unique solution. According to the ADM [14, 15], the solution $u(x)$ is represented by the decomposition series

$$u(x)=\sum_{n=0}^{\infty} u_n(x), \tag{2}$$

and the nonlinear part of Eq. (1) is represented by the decomposition series

$$T\big(u(t)\big)=\sum_{n=0}^{\infty} A_n(t), \tag{3}$$

where the $A_n's$ are Adomian's polynomials [16, 17] that are defined by the following formula

$$A_{n(t)}=\frac{1}{n!}\left[\frac{d^n}{d\lambda^n}T\left(\sum_{i=0}^{\infty}\lambda^i u_i(t)\right)\right]_{\lambda=0}, \qquad n = 0, 1, 2, \dots \tag{4}$$

Substituting (2) and (3) into (1), we obtain

$$\sum_{n=0}^{\infty} u_n(x)=f(x)+\int_0^x k(x,t)\Big(\sum_{n=0}^{\infty} A_n(t)\Big)dt. \tag{5}$$

Each term of the series in (5) is given by the recurrence relation

$$u_0(x)=f(x), \tag{6}$$

$$u_{n+1}(x)=\int_0^x k(x,t)\big(A_n(t)\big)dt. \qquad n = 0, 1, \cdots \tag{7}$$

In practice, not all terms of the series in (2) need be determined and hence, the solution will be approximated by the truncated series

$$\varphi_k(x)=\sum_{n=0}^{k-1} u_n(x) \quad \text{with} \quad \lim_{k\to\infty}\varphi_k(x)=u(x). \tag{8}$$

3. The HPM for Nonlinear Volterra Integral Equations

In this section, we apply the HPM to solve nonlinear Volterra integral equations. To do this, we consider Eq. (1) as

$$L(u(x)) = u(x) - f(x) - \int_0^x k(x,t)T(u(t))\,dt, \tag{9}$$

we can define homotopy $H(u(x),p)$ with properties

$$H(u(x),0) = F(u(x)), \qquad H(u(x),1) = L(u(x)), \tag{10}$$

where

$$F(u(x)) = u(x) - f(x). \tag{11}$$

Classically, we choose a convex homotopy by

$$H(u(x),p) = (1-p)F(u(x)) + pL(u(x)), \tag{12}$$

where p is embedding parameter. The embedding parameter p monotonically increases from zero to unit as trivial problem $H(v(x),0) = F(v(x))$ is continuously deformed to orginal problem $H(v(x),1) = L(v(x))$ where v(x) is a solution of (1). The HPM uses the homotopy parameter p as an expanding parameter to obtain [18]

$$u(x) = u_0(x) + pu_1(x) + p^2u_2(x) + \cdots \tag{13}$$

when $p \to 1$, (13) becomes the approximate solution of Eq. (1), i.e.,

$$v(x) = \lim_{p\to 1} u(x) = u_0(x) + u_1(x)\, u_2(x) + \cdots. \tag{14}$$

4. Equivalence between the HPM and the ADM for Nonlinear Volterra Integral Equations

In this section, we investigate the equivalence of the convergent HPM and the convergent ADM for the solution of nonlinear Volterra integral equations. We show that the HPM is equivalent to the ADM with a specific convex homotopy and vice versa. This fact is shown in the following theorem.

Theorem. The homotopy perturbation method is equivalent to Adomian's decompose-tion method, for nonlinear Volterra integral equations, with the homotopy $H(u(x),p)$ given by

$$H(u(x),p) = (1-p)F(u(x)) + pL(u(x)) = 0, \tag{15}$$

where

$$F(u(x)) = u(x) - f(x), \tag{16}$$

$$L(u(x)) = u(x) - f(x) - \int_0^x k(x,t)T(u(t))dt. \tag{17}$$

Proof. At first, we show that the HPM with the specific convex homotopy (15) for the nonlinear Volterra integral equation (1) is the ADM. According to homotopy equation (15), the embedding parameter p monotonically increases from zero to one as the trivial problem $F(u(x)) = 0$ is continuously deformed to the original problem $L(u(x)) = 0$.

Putting (16) and (17) into (15) gives

$$H(u(x),p) = u(x) - f(x) - p\int_0^x k(x,t)T(u(t))\,dt = 0, \tag{18}$$

or

$$u(x) = f(x) + p\int_0^x k(x,t)T(u(t))dt. \tag{19}$$

By substituting (13) into (19), we obtain

$$\sum_{n=0}^{\infty} p^n u_n(x) = f(x) + p\int_0^x k(x,t)T\left(\sum_{n=0}^{\infty} p^n u_n(t)\right)dt. \tag{20}$$

On the other hand

$$T\left(\sum_{n=0}^{\infty} p^n u_n(t)\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{d^n}{dp^n}\left[T\left(\sum_{j=0}^{\infty} p^j u_j(t)\right)\right]_{p=0} p^n = \sum_{n=0}^{\infty} p^n A_n(t). \tag{21}$$

inserting (21) into (20) and equating the coefficients of p for the same power, we find that

$$p^0 : u_0(x) = f(x), \tag{22}$$

$$p^{n+1} : u_{n+1}(x) = \int_0^x k(x,t)\left(\sum_{n=0}^{\infty} A_n(t)\right)dt. \qquad n = 0, 1, 2, \ldots \tag{23}$$

Thus the solution of the nonlinear Volterra integral equation (1) is given by

$$v(x) = \lim_{p\to 1}\sum_{n=0}^{\infty} p^n u_n(x) = u_0(x) + u_1(x) + u_2(x) + \cdots. \tag{24}$$

Note that the recurrence relations (22) and (23) obtained by the HPM are equal to recurrence relations obtained by the ADM as can be seen in (6) and (7). Then, the HPM with the convex homotopy (15) for the nonlinear Volterra integral equation (1) is the ADM.

Conversely, we show that the ADM for the nonlinear Volterra integral equation (1) is the HPM with the convex homotopy (15). To do this, let $u(p) = \sum_{n=0}^{\infty} p^n u_n(t)$, then

$$\lim_{p\to 1} u(p) = u(1) = \sum_{n=0}^{\infty} u_n(x). \tag{25}$$

From (2) and (25), one gets

$$u(x) = \lim_{p\to 1} u(p). \tag{26}$$

The convergence of series (3) implies

$$T(u(t)) = \sum_{n=0}^{\infty} A_n(t) = \lim_{p\to 1}\sum_{n=0}^{\infty} p^n A_n(t). \tag{27}$$

Applying the Taylor's expansion of a function about $t = 0$, from (4) we have

$$\sum_{n=0}^{\infty} p^n A_n(t) = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{d^n}{dp^n}\left[T\left(\sum_{i=0}^{\infty} p^i u_i(t)\right)\right]_{p=0} p^n = T\left(\sum_{i=0}^{\infty} p^i u_i(t)\right) = T(u(p)). \tag{28}$$

So, from (27) and (28) we deduce

$$T\big(u(t)\big) = \lim_{p\to 1} T(u(p)). \tag{29}$$

We shall construct a homotopy $H(u(x),p)$ such that $H(u(x),0) = F(u(x))$ and $H(u(x),1) = L(u(x))$. In the view of (6) and (7) and the above discussion, we have

$$u(p) = \sum_{n=0}^{\infty} p^n u_n(x)\, u_0(x) + p\sum_{n=0}^{\infty} p^n u_{n+1}(x) \tag{30}$$

$$= f(x) + p\sum_{n=0}^{\infty} p^n \int_0^x k(x,t)\big(A_n(t)\big)dt \tag{31}$$

$$= f(x) + p\int_0^x k(x,t)\Big(\sum_{n=0}^{\infty} p^n A_n(t)\Big)dt \tag{32}$$

$$= f(x) + p\int_0^x k(x,t)T(u(p))dt. \tag{33}$$

An equivalent expression of (33) is

$$u(p) - f(x) - p\int_0^x k(x,t)T(u(p))dt = 0. \tag{34}$$

Now, considering (34), we define the homotopy $H(u(p),p)$ as

$$H(u(p),p) = u(p) - f(p) - p\int_0^p k(p,t)T\big(u(p)\big)dt = 0. \tag{35}$$

Eq. (35) can be written in following form

$$H(u(x),p) = (1-p)F\big(u(x)\big) + pL\big(u(x)\big) = 0, \tag{36}$$

where $F(u(x))$ and $L(u(x))$ are as (16) and (17). Considering (30), we see that the power series $\sum_{n=0}^{\infty} p^n u_n(x)$ is the solution of the Eq. (36) and as p approaches 1, it becomes the approximate solution of Eq. (1). This shows that Adomian's decomposition method is the same as the homotopy perturbation method with the homotopy $H(u(x),p)$ given by (36). By this way, the proof of theorem is completed.

5. Conclusion

The ADM and the HPM are powerful methods which consider the approximate solution of a nonlinear Volterra integral equations as an infinite series converging to the exact solution. By theoretical analysis of the two methods, we have proven that the HPM is equivalent to the ADM with a specific convex homotopy for nonlinear Volterra integral equations. Note that, with the similar convex homotopy can proven that the HPM is equivalent to the ADM for Fredholm integral equations.

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