On Weak McCoy Rings

E. Hashemi : Shahrood University of Technology

Abstract

In this note we introduce the notion of weak McCoy rings as a generalization of McCoy rings, and investigate their properties. Also we show that, if \( R \) is a semi-commutative ring, then \( R \) is weak McCoy if and only if \( R[x] \) is weak McCoy.

1. Introduction

Throughout this paper, all rings are associative with identity. For a commutative ring \( R \), McCoy [10] obtained the following result: If \( f(x)g(x) = 0 \) for some non-zero polynomials \( f(x) \), \( g(x) \in R[x] \), then \( f(x)c = 0 \) for some non-zero \( c \in R \). According to Nielsen [12], a ring \( R \) is called right McCoy whenever polynomials \( f(x), g(x) \in R[x] - \{0\} \) satisfy \( f(x)g(x) = 0 \), there exists a non-zero \( r \in R \) such that \( f(x)r = 0 \). Left McCoy rings are defined similarly. If a ring is both left and right McCoy, we say that the ring is a McCoy ring. It is well known that commutative rings are always McCoy rings [10], but it is not true for non-commutative rings (see [12]).

Recall that a ring \( R \) is called:

- reduced if \( a^2 = 0 \Rightarrow a = 0 \), for all \( a \in R \),
- reversible if \( ab = 0 \Rightarrow ba = 0 \), for all \( a, b \in R \),
- symmetric if \( abc = 0 \Rightarrow acb = 0 \), for all \( a, b, c \in R \),
- semi-commutative if \( ab = 0 \Rightarrow aRb = 0 \), for all \( a, b \in R \).

The following implications hold:

2000 Mathematics Subject Classification: 16N40, 16U80

Keywords: Semi-commutative ring, McCoy ring, Polynomial ring, Reversible ring, Matrix ring, Classical quotient ring

Received 5 Nov. 2008 Revised 19 Oct. 2009

eb_hashemi@yahoo.com
On Weak McCoy Rings

E. Hashemi

reduced \(\Rightarrow\) symmetric \(\Rightarrow\) reversible \(\Rightarrow\) semi-commutative.

Reversible rings are McCoy rings by [12]. But the converse is not true; there exists a non-reversible McCoy ring (see [12]).

Motivated by the above, as a generalization of McCoy rings, in this paper we introduce the notion of weak McCoy rings and investigate their properties and extend several known results relating to McCoy rings to a general setting.

For a ring \(R\), we denote by \(\text{nil}(R)\) the set of all nilpotent elements of \(R\), by \(N_s(R)\) the prime radical of \(R\) and by \(M_n(R), U_n(R)\) and \(L_n(R)\) the \(n\times n\) matrix ring over \(R\), the \(n\times n\) upper and lower triangular matrix rings over \(R\) respectively.

2. On Weak McCoy rings

**Definition 2.1.** We say \(R\) is a weak McCoy ring if \(f(x)g(x)\in \text{nil}(R[x])\) implies \(f(x)c\in \text{nil}(R[x])\), for some non-zero \(c\in R\), where \(f(x)\) and \(g(x)\) are non-zero polynomials in \(R[x]\).

**Remark 2.2.** Since \(ab\) is nilpotent if and only if \(ba\) is nilpotent in a ring, hence the definition of weak McCoy rings is left-right symmetric.

**Proposition 2.3.** McCoy rings are weak McCoy.

**Proof.** Let \(R\) be a McCoy ring and \(f(x)g(x)\in \text{nil}(R[x])\) for non-zero polynomials \(f(x), g(x)\in R[x]\). Then there exists \(m,n\geq 1\), such that \((f(x)g(x))^m = (g(x)f(x))^n = 0\), and \((f(x)g(x))^{n-1}, (g(x)f(x))^{m-1} \neq 0\). If \(f(x)g(x) = 0\) or \(g(x)f(x) = 0\), then the result follows from the definition of McCoy rings. Assume \(f(x)g(x) \neq 0 \neq g(x)f(x)\) and \(0 = (f(x)g(x))^m = f(x)(g(x)f(x)...f(x)g(x)) = f(x)h(x)\).

If \(h(x) = g(x)f(x)...f(x)g(x) \neq 0\), then \(f(x)c = 0\) for some non-zero \(c\in R\), since \(R\) is McCoy.

Let \(h(x) = g(x)(f(x)g(x)...f(x)g(x)) = g(x)(f(x)g(x))^{n-1} = 0\). Since \((f(x)g(x))^{n-1} \neq 0\) and \(R\) is McCoy, there exists \(0 \neq d \in R\) such that \(g(x)d = 0\). Therefore \(f(x)c = 0\) or

---

50
$g(x)d = 0$ for some non-zero $c, d \in R$. Hence $f(x)c \in \text{nil}(R[x])$ or $dg(x) \in \text{nil}(R[x])$ for some non-zero $c, d \in R$. Therefore $R$ is weak McCoy.

**Proposition 2.4.** Let $R$ be a ring. Then $U_n(R)$ and $L_n(R)$ are weak McCoy for each $n \geq 2$.

**Proof.** Clearly $U_n(R)[x] \cong U_n(R[x])$ and for each $A = \begin{bmatrix} 0 & f_{12} & \cdots & f_{1n} \\ 0 & 0 & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in U_n(R[x])$, $A^n = 0$. Let $0 \neq A = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ 0 & f_{22} & \cdots & f_{2n} \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & f_{nn} \end{bmatrix} \in U_n(R[x])$. Then $A^n \neq 0$. Hence $U_n(R)$ is weak McCoy. By a similar argument one can show that $L_n(R)$ is weak McCoy.

**Proposition 2.5.** Let $R$ and $S$ be rings and $R M_S$ a bimodule. Then $\begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$ is a weak McCoy ring.

**Proof.** Similarly, as used in Proposition 2.4 one can prove it.

The following example shows that $U_n(R)$ and $M_n(R)$ are neither left nor right McCoy for some $n \geq 2$.

**Example 2.6.** Let $R$ be a ring. We show that $U_4(R)$ and $M_4(R)$ are neither right nor left McCoy. Let $f(x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x$ and
On Weak McCoy Rings

E. Hashemi

If \( f(x)A = 0 \), for some \( A = [a_j] \in M_4(R) \), then \( 0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} A = \begin{bmatrix} a_{i1} & a_{i2} & a_{i3} & a_{i4} \\ 0 & 0 & 0 & 0 \\ a_{j1} & a_{j2} & a_{j3} & a_{j4} \\ 0 & 0 & 0 & 0 \end{bmatrix} \). Hence \( A = 0 \) and \( U_4(R) \) and \( M_4(R) \) are not right McCoy. If \( Bg(x) = 0 \) for some \( B \in M_4(R) \), then by a similar way as above, we can show \( B = 0 \). Therefore \( U_4(R) \) and \( M_4(R) \) are not left McCoy.

**Definition 2.7.** A ring \( R \) is called **right Ore** if given \( Rba \in R \), with \( b \) regular there exist \( Rba \in R \), with \( b \) regular such that \( ba = ab \). It is well-known that \( R \) is a right Ore ring if and only if the classical right quotient ring of \( R \) exists. We use \( R_c \) to denote the set of all regular elements in \( R \).

**Theorem 2.8.** Let \( R \) be a right Ore ring with its classical right quotient ring \( Q \). If \( R \) is weak McCoy then \( Q \) is weak McCoy.

**Proof.** Let \( 0 \neq F(x) = \sum_{i=0}^{m} a_i u^{-i} x^i \) and \( 0 \neq G(x) = \sum_{j=0}^{n} b_j v^{-j} x^j \) with \( a_i, b_j \in R, u, v \in C(R) \) such that \( F(x)G(x) \in \text{nil}(Q[x]) \).

**Case 1.** \( F(x)G(x) = 0 \) or \( G(x)F(x) = 0 \). Assume that \( F(x)G(x) = 0 \). Since \( R \) is right Ore, there exists \( b_j \in R \) and \( u_i \in C(R) \) such that \( u^{-1}b_j = b_j u_i^{-1} \) for \( j = 1, \ldots, n \). Let \( f(x) = \sum_{i=0}^{m} a_i x^i \) and \( g(x) = \sum_{j=0}^{n} b_j x^j \). Then \( f(x)g(x) = 0 \). Since \( R \) is weak McCoy, there exists \( 0 \neq c \in R \) with \( f(x)c \in \text{nil}(R[x]) \subseteq \text{nil}(Q[x]) \). Hence \( F(x)uc = f(x)u^{-1}uc = f(x)c \in \text{nil}(Q[x]) \). If \( G(x)F(x) = 0 \), then by a similar argument we can show that \( G(x)v \in \text{nil}(Q[x]) \) for some non-zero \( d \in R \).
Case 2. $F(x)G(x) \neq 0$ and $G(x)F(x) \neq 0$. Since $F(x)G(x) \in \text{nil}(Q[x])$, there exists $n \geq 2$ such that $(F(x)G(x))^n = 0$ and $(F(x)G(x))^{n-1} \neq 0$. Let $(F(x)G(x))^n = F(x)H(x)$. If $H(x) \neq 0$, then by a similar argument as above there exists $\alpha \in C(R)$, $r \in R$ such that $F(x)\alpha r \in \text{nil}(Q[x])$. Now assume $H(x) = G(x)F(x)G(x)\ldots F(x)G(x) = 0$. Since $(F(x)G(x))^{n-1} \neq 0$ and $R$ is weak McCoy, then by Case 1, there exists $\beta \in C(R)$, $s \in R$ such that $G(x)\beta s = 0$. Therefore $Q$ is weak McCoy.

According to Bell [2], a ring $R$ is called semi-commutative if $ab = 0$ implies $aRb = 0$. We say an ideal $I$ is a semi-commutative ideal, if $R/I$ is a semi-commutative ring.

**Lemma 2.9.** Let $R$ be a semi-commutative ring. If $c_1c_2\cdots c_k = 0$ for some $c_i \in R$, then $c_1Rc_2Rc_3\cdots Rc_k = 0$.

**Proof.** By induction, let $c_{k-1} = c_1c_2\cdots c_{k-1}$. Then $c_1c_2\cdots c_{k-1} = 0$ and by induction assumption, we have $0 = c_1Rc_2Rc_3\cdots Rc_{k-1} = c_1Rc_2Rc_3\cdots Rc_{k-1}c_k$. Hence, for all $x \in c_1Rc_2Rc_3\cdots Rc_{k-1}$, we have $xc_k = 0$. It follows by hypothesis that $xRc_k = 0$. Thus $c_1Rc_2Rc_3\cdots Rc_k = 0$, as desired.

**Lemma 2.10** (4, Lemma 2.5). Let $R$ be a semi-commutative ring. Then $\text{nil}(R)$ is a semi-commutative ideal of $R$.

**Proof.** Let $a, b \in \text{nil}(R)$. Then $a^n = 0 = b^m$ for some $m, n \geq 0$. Each term of the expansion of $(a+b)^{m+n+1}$ has the form $x = (a^ib^j)\cdots (a_i^{b_{j+1}})$ where $i_r, j_s \in N \cup \{0\}$. Since $(i_1 + j_1) + (i_2 + j_2) + \ldots + (i_{m+n+1} + j_{m+n+1}) = \sum_{r=1}^{m} i_r + \sum_{s=1}^{n} j_s = m + n + 1$, either $\sum_{i=1}^{m} i_r \geq n$ or $\sum_{j=1}^{n} j_s \geq m$. If $\sum_{i=1}^{m} i_r \geq n$, then $a^i a^j \cdots a_{m+n+1} = 0$. Thus $(a^ib^j)\cdots (a_{m+n+1}b_{j+1}) = 0$, by Lemma 2.9. If $\sum_{i=1}^{m} i_r < n$, then $\sum_{j=1}^{n} j_s \geq m$. Thus $b^j b^j \cdots b_{j+1} = 0$ and so $(a^ib^j)\cdots (a_{m+n+1}b_{j+1}) = 0$, by Lemma 2.9. Hence $(a+b)^{m+n+1} = 0$.

Now suppose that $a^n = 0$ and $R \in R$. Then $(ar)^n = 0 = (ra)^n$, by Lemma 2.9. Thus $\text{nil}(R)$ is an ideal of $R$. 

53
Since $R/\text{nil}(R)$ is a reduced ring, hence it is a semi-commutative ring. Therefore $\text{nil}(R)$ is a semi-commutative ideal of $R$.

**Lemma 2.11.** Let $R$ be a semi-commutative ring. Then $\text{nil}(R[x]) = \text{nil}(R)[x]$.

**Proof.** Let $f(x) = a_0 + \ldots + a_n x^n \in \text{nil}(R[x])$. Then $f(x)^k = 0$, for some integer $k \geq 0$. Hence $a_n^k = 0$, and that $a_n \in \text{nil}(R)$. There exists $g(x), h(x) \in R[x]$ such that $f(x)^k = (a_0 + \ldots + a_{n-1} x^{n-1})^k + a_n g(x) + h(x) a_n$. Since $\text{nil}(R)[x]$ is an ideal of $R[x]$ and $a_n g(x), h(x) a_n, f(x)^k \in \text{nil}(R)[x]$, we have $(a_0 + \ldots + a_{n-1} x^{n-1})^k \in \text{nil}(R)[x]$. Hence $a_{n-1}^k \in \text{nil}(R)$ and that $a_{n-1} \in \text{nil}(R)$. Continuing this process yields $a_0, \ldots, a_n \in \text{nil}(R)$. Therefore $\text{nil}(R[x]) \subseteq \text{nil}(R)[x]$.

Now, let $f(x) = a_0 + \ldots + a_n x^n \in \text{nil}(R)[x]$. Then $a_i^m = 0$, for some positive integer $m_i$. Let $k = m_0 + \ldots + m_n + 1$. Then $(f(x))^k = \sum (a_0 i_0 (a_i x i_0) \ldots (a_n x i_n)) \cdots (a_0 i_0 (a_i x i_0) \ldots (a_n x i_n))$, where $i_0 + \ldots + i_n = 1$, for $r = 1, \ldots, k$ and $0 \leq i_n \leq 1$. Each coefficient of $f(x)^k$ is a sum of such elements $\gamma = (a_0 h_0 (a_n h_n) \ldots (a_0 h_n) \ldots (a_i h_i) \ldots (a_n h_n))$, where $i_0 + \ldots + i_n = 1$.

It can be easily checked that there exists $a_i \in \{a_0, \ldots, a_n\}$ such that $i_0 + \ldots + i_k \geq m_i$. Since $a_i^m = 0$ and $R$ is semi-commutative, $\gamma = 0$. Thus $(f(x))^k = 0$ and $\text{nil}(R)[x] \subseteq \text{nil}(R[x])$. Therefore $\text{nil}(R[x]) = \text{nil}(R)[x]$.

**Lemma 2.12.** Let $R$ be a semi-commutative ring. Then $\text{nil}(R[x][y]) = \text{nil}(R[x])[y]$.

**Proof.** By Lemma 2.11, $\text{nil}(R[x])$ is an ideal of $R[x]$. Since $R[x]/\text{nil}(R[x])$ is a reduced ring, hence $\text{nil}(R[x])$ is a semi-commutative ideal of $R[x]$, and that $\text{nil}(R[x][y]) \subseteq \text{nil}(R[x][y])$.

Now, let $F(y) = \sum_{i=0}^{m} f_i y^i \in \text{nil}(R[x][y])$, where $f_i = \sum_{s=0}^{p} a_s x^s \in R[x]$. Then $F(y)^n = 0$, for some positive integers $n$. As in the proof of [1], let $k = n \sum \deg f_i$, where the degree is as polynomial in $x$ and the degree of zero polynomial is taken to be 0. Then $(F(x))^n = 0$ and the set of coefficients of $F(x)$ is equal to the set of all coefficients of $f_i$, $0 \leq i \leq m$. Hence by Lemma 2.11, $a_j \in \text{nil}(R)$ for all $i, j$ and that $f_i \in \text{nil}(R[x])$, for each $i$. Thus $F(y) \in \text{nil}(R[x][y])$. Therefore $\text{nil}(R[x][y]) = \text{nil}(R[x])[y]$. 

54
If $R$ is semi-commutative, then $R[x]$ may not be semi-commutative, by [5, Example 2]). Here we will show that if $R$ is semi-commutative, then $R$ is weak McCoy if and only if $R[x]$ is weak McCoy.

Theorem 2.13. If $R$ is a semi-commutative ring, then $R[x]$ is a weak McCoy ring if and only if $R$ is weak McCoy.

**Proof.** Suppose that $R$ is a weak McCoy ring. Let $F(t) = \sum_{i=0}^{m} f_i t^i$, $G(t) = \sum_{j=0}^{n} g_j t^j$ be non-zero polynomials in $R[x][t]$ such that $F(t)G(t) \in \text{nil}(R[x][t])$, where $f_i = \sum_{x^i} a_{i,x^i}$, $g_j = \sum_{x^j} b_{j,x^j} \in R[x]$. As in the proof of [1], let $k = \sum \deg f_i + \sum \deg g_j$, where the degree is as polynomial in $x$ and the degree of zero polynomial is taken to be 0. Then $F(x^k) = \sum_{i=0}^{m} f_i x^{ik}$, $G(x^k) = \sum_{j=0}^{n} g_j x^{jk} \in R[x]$, and the set of coefficients of the $F(x^k)$ is (respectively $G(x^k)$) equal to the set of all coefficients of $f_i$, $0 \leq i \leq m$ (respectively $g_j$, $0 \leq j \leq n$). Since $(F(t)G(t))^p = 0$, for some $p \geq 1$, and $x$ commutes with elements of $R$, $(F(x^k)G(x^k))^p = 0$. Since $R$ is weak McCoy, there is $0 \neq r \in R$ such that $F(x^k)r \in \text{nil}(R[x])$ and $a_i r \in \text{nil}(R)$, $f_i r \in \text{nil}(R[x])$ for $0 \leq i \leq m$, $0 \leq s \leq p$, by Lemma 2.11. Hence $F(t)r \in \text{nil}(R[x][t])$, by Lemma 2.12. Therefore $R[x]$ is weak McCoy.

Now suppose $R[x]$ is a weak McCoy ring and $f(t)g(t) \in \text{nil}(R[t]) \subseteq \text{nil}(R[x][t])$. Since $R[x]$ is weak McCoy, there exists $0 \neq h(x) \in R[x]$ such that $f(t)h(x) \in \text{nil}(R[x][t])$. Let $h(x) = a_0 + \ldots + a_n x^n \in R[x]$ ($a_0 \neq 0$). Then $f(t)a_0 \in \text{nil}(R[t])$, since $(f(t)h(x))^k = (f(t)a_0)^k + k_1 x + \ldots + k_{nk} x^{nk}$ with $k_1, \ldots, k_{nk} \in R[t]$. Therefore $R$ is weak McCoy.

Theorem 2.14. Let $R$ be a ring and $\Delta$ a multiplicatively closed subset of $R$ consisting of central regular elements. Then $R$ is weak McCoy if and only if $\Delta^{-1}R$ is weak McCoy.
Proof. If $R$ is a weak McCoy ring, then by a similar way as used in Theorem 2.8, one can show that $\Delta^{-1}R$ is weak McCoy.

Conversely, let $\Delta^{-1}R$ be a weak McCoy ring. Let $f(x) = \sum_{j=0}^{m} a_j x^j$ and $g(x) = \sum_{j=0}^{n} b_j x^j$ be non-zero polynomials of $R[x]$ such that $f(x)g(x) \in \text{nil}(R[x])$. Since $\Delta^{-1}R$ is weak McCoy, $f(x)(c\alpha^{-1}) \in \text{nil}((\Delta^{-1}R)[x])$ for some non-zero $c\alpha^{-1} \in \Delta^{-1}R$. Thus $f(x)c \in \text{nil}(R[x])$ and $R$ is weak McCoy.

Corollary 2.15. Let $R$ be a ring. Then $R[x]$ is weak McCoy if and only if $R[x,x^{-1}]$ is weak McCoy.

Proof. Clearly $\Delta = \{1, x, x^2,\ldots\}$ is a multiplicatively closed subset of $R[x]$ consisting of central regular elements and $\Delta^{-1}R[x] = R[x,x^{-1}]$. Hence the proof follows from Theorem 2.14.

Theorem 2.16. The classes of weak McCoy rings are closed under direct limits.

Proof. Let $A = \{R_i, \alpha_{ij}\}$ be a direct system of weak McCoy rings $R_i$ for $i \in I$ and ring homomorphisms $\alpha_{ij} : R_i \to R_j$ for each $i \leq j$ with $\alpha_{ii}(1) = 1$, where $I$ is a directed partially ordered set. Let $R = \lim R_i$ be the direct limit of $A$ with $\ell_j : R_i \to R$ and $\ell_j \circ \alpha_{ij} = \ell_i$. We show that $R$ is weak McCoy ring. Let $a, b \in R$. Then $a = \ell_i(a_i)$, $b = \ell_j(b_j)$ for some $i, j \in I$ and there is $k \in I$ such that $i \leq k, j \leq k$. Define $a + b = \ell_k(\alpha_{ik}(a_i) + \alpha_{jk}(b_j))$ and $ab = \ell_k(\alpha_{ik}(a_i)\alpha_{jk}(b_j))$, where $\alpha_{ik}(a_i), \alpha_{jk}(b_j) \in R_k$. Then $R$ forms a ring with $0 = \ell_i(0)$ and $1 = \ell_i(1)$. Let $f, g \in R[x]$ be non-zero polynomials such that $fg \in \text{nil}(R[x])$. There is $k \in I$ such that $f, g \in R_k[x]$. Hence $fg \in \text{nil}(R_k[x])$. If $R_k$ is weak McCoy, there exists $0 \neq c_k \in R_k$ such that $fc_k \in \text{nil}(R_k[x])$. If $c = \ell_k(c_k)$, then $fc \in \text{nil}(R[x])$ with non-zero $c$. Therefore $R$ is weak McCoy.

Proposition 2.17. (1) Let $R$ be a ring. If there exists a non-zero ideal $I$ of $R$ such that $I[x] \subseteq \text{nil}(R[x])$, then $R$ is weak McCoy.

56
(2) Every non-semiprime ring is weak McCoy.

(3) Let \( R \) be a ring with a non-zero nilpotent ideal. Then \( \text{Mat}_n(R) \) \((n \geq 2)\) is weak McCoy.

**Proof.** (1) Let \( 0 \neq f \in R[x] \). If \( f \in \text{I}[x] \), then \( fr \in \text{nil}(R[x]) \) for all \( r \in R \). If \( f \not\in \text{I}[x] \) then \( fs \in \text{I}[x] \subseteq \text{nil}(R[x]) \) for all non-zero \( s \in \text{I} \). Thus \( R \) is weak McCoy.

(2) Let \( R \) be a ring with \( \text{N}_1(R) \neq 0 \). Since \( 0 \neq \text{N}_1(R)[x] = \text{N}_1(R[x]) \subseteq \text{nil}(R[x]) \), \( R \) is weak McCoy by (1).

(3) Since \( \text{Mat}_n(R) \) is non-semiprime, hence by (1) \( \text{Mat}_n(R) \) is weak McCoy.

**Acknowledgement**

The author thanks the anonymous referees for their valuable suggestions, which simplified the proof of Lemma 2.10. This research is supported by the Shahrood University of Technology.

**References**


