On Weak McCoy Rings

E. Hashemi : Shahrood University of Technology

Abstract

In this note we introduce the notion of weak McCoy rings as a generalization of McCoy rings, and investigate their properties. Also we show that, if \( R \) is a semi-commutative ring, then \( R \) is weak McCoy if and only if \( R[x] \) is weak McCoy.

1. Introduction

Throughout this paper, all rings are associative with identity. For a commutative ring \( R \), McCoy [10] obtained the following result: If \( f(x)g(x) = 0 \) for some non-zero polynomials \( f(x), g(x) \in R[x] \), then \( f(x)c = 0 \) for some non-zero \( c \in R \). According to Nielsen [12], a ring \( R \) is called right McCoy whenever polynomials \( f(x), g(x) \in R[x] - \{0\} \) satisfy \( f(x)g(x) = 0 \), there exists a non-zero \( r \in R \) such that \( f(x)r = 0 \). Left McCoy rings are defined similarly. If a ring is both left and right McCoy, we say that the ring is a McCoy ring. It is well known that commutative rings are always McCoy rings [10], but it is not true for non-commutative rings (see [12]).

Recall that a ring \( R \) is called:

- reduced if \( a^2 = 0 \Rightarrow a = 0 \), for all \( a \in R \),
- reversible if \( ab = 0 \Rightarrow ba = 0 \), for all \( a, b \in R \),
- symmetric if \( abc = 0 \Rightarrow acb = 0 \), for all \( a, b, c \in R \),
- semi-commutative if \( ab = 0 \Rightarrow aRb = 0 \), for all \( a, b \in R \).

The following implications hold:

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eb_hashemi@yahoo.com
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Reduced \(\Rightarrow\) Symmetric \(\Rightarrow\) Reversible \(\Rightarrow\) Semi-commutative.

Reversible rings are McCoy rings by [12]. But the converse is not true; there exists a non-reversible McCoy ring (see [12]).

Motivated by the above, as a generalization of McCoy rings, in this paper we introduce the notion of weak McCoy rings and investigate their properties and extend several known results relating to McCoy rings to a general setting.

For a ring \(R\), we denote by \(\text{nil}(R)\) the set of all nilpotent elements of \(R\), by \(N_s(R)\) the prime radical of \(R\) and by \(M_n(R), U_n(R)\) and \(L_n(R)\) the \(n\times n\) matrix ring over \(R\), the \(n\times n\) upper and lower triangular matrix rings over \(R\) respectively.

2. On Weak McCoy rings

Definition 2.1. We say \(R\) is a weak McCoy ring if \(f(x)g(x)\in\text{nil}(R[x])\) implies \(f(x)c\in\text{nil}(R[x])\), for some non-zero \(c\in R\), where \(f(x)\) and \(g(x)\) are non-zero polynomials in \(R[x]\).

Remark 2.2. Since \(ab\) is nilpotent if and only if \(ba\) is nilpotent in a ring, hence the definition of weak McCoy rings is left-right symmetric.

Proposition 2.3. McCoy rings are weak McCoy.

Proof. Let \(R\) be a McCoy ring and \(f(x)g(x)\in\text{nil}(R[x])\) for non-zero polynomials \(f(x),g(x)\in R[x]\). Then there exists \(m,n\geq 1\), such that \((f(x)g(x))^m = (g(x)f(x))^m = 0\), and \((f(x)g(x))^{n-1},(g(x)f(x))^{m-1} \neq 0\). If \(f(x)g(x) = 0\) or \(g(x)f(x) = 0\), then the result follows from the definition of McCoy rings. Assume \(f(x)g(x) \neq 0 \neq g(x)f(x)\) and \(0 = (f(x)g(x))^n = f(x)(g(x)f(x)\ldots f(x)g(x)) = f(x)h(x)\).

If \(h(x) = g(x)f(x)\ldots f(x)g(x) \neq 0\), then \(f(x)c = 0\) for some non-zero \(c\in R\), since \(R\) is McCoy.

Let \(h(x) = g(x)(f(x)g(x)\ldots f(x)g(x)) = g(x)(f(x)g(x))^{n-1} = 0\). Since \((f(x)g(x))^{n-1} \neq 0\) and \(R\) is McCoy, there exists \(0 \neq d \in R\) such that \(g(x)d = 0\). Therefore \(f(x)c = 0\) or
\( g(x)d = 0 \) for some non-zero \( c,d \in R \). Hence \( f(x)c \in \text{nil}(R[x]) \) or \( dg(x) \in \text{nil}(R[x]) \) for some non-zero \( c,d \in R \). Therefore \( R \) is weak McCoy.

**Proposition 2.4.** Let \( R \) be a ring. Then \( U_n(R) \) and \( L_n(R) \) are weak McCoy for each \( n \geq 2 \).

**Proof.** Clearly \( U_n(R)[x] \cong U_n(R[x]) \) and for each \( A = \begin{bmatrix} 0 & f_{12} & \cdots & f_{1n} \\ 0 & 0 & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in U_n(R[x]) \),

\[
A^n = 0.
\]

Let \( 0 \neq A = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ 0 & f_{22} & \cdots & f_{2n} \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & f_{nn} \end{bmatrix} \in U_n(R[x]) \). Then

\[
A^n = 0.
\]

Hence \( U_n(R) \) is weak McCoy. By a similar argument one can show that \( L_n(R) \) is weak McCoy.

**Proposition 2.5.** Let \( R \) and \( S \) be rings and \( _RM_S \) a bimodule. Then \( \begin{bmatrix} R & M \\ 0 & S \end{bmatrix} \) is a weak McCoy ring.

**Proof.** Similarly, as used in Proposition 2.4 one can prove it.

The following example shows that \( U_n(R) \) and \( M_n(R) \) are neither left nor right McCoy for some \( n \geq 2 \).

**Example 2.6.** Let \( R \) be a ring. We show that \( U_4(R) \) and \( M_4(R) \) are neither right nor left McCoy. Let

\[
f(x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x \text{ and}
\]

\[
51
\]
If \( f(x)A = 0 \), for some \( A = [a_{ij}] \in M_4(R) \), then \( 0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \) and \( 0 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \) for \( x = 0 \). Let \( A = [a_{ij}] \) and \( 0 = \begin{bmatrix} -a_{21} & -a_{22} & -a_{23} & -a_{24} \\ 0 & 0 & 0 & 0 \\ -a_{41} & -a_{42} & -a_{43} & -a_{44} \\ 0 & 0 & 0 & 0 \end{bmatrix} \). Hence \( A = 0 \) and \( U_4(R) \) and \( M_4(R) \) are not right McCoy. If \( Bg(x) = 0 \) for some \( B \in M_4(R) \), then by a similar way as above, we can show \( B = 0 \). Therefore \( U_4(R) \) and \( M_4(R) \) are not left McCoy.

**Definition 2.7.** A ring \( R \) is called right Ore if given \( a, b \in R \) with \( b \) regular there exist \( a, b \in R \) with \( b \) regular such that \( ab = ba \). It is well-known that \( R \) is a right Ore ring if and only if the classical right quotient ring of \( R \) exists. We use \( C(R) \) to denote the set of all regular elements in \( R \).

**Theorem 2.8.** Let \( R \) be a right Ore ring with its classical right quotient ring \( Q \). If \( R \) is weak McCoy then \( Q \) is weak McCoy.

**Proof.** Let \( 0 \neq F(x) = \sum_{i=0}^{m} a_{ij} x^i \) and \( 0 \neq G(x) = \sum_{j=0}^{n} b_{ji} x^j \) with \( a_{ij}, b_{ji} \in R, \), \( u, v \in C(R) \) such that \( F(x)G(x) \in \text{nil}(Q[x]) \).

**Case 1.** \( F(x)G(x) = 0 \) or \( G(x)F(x) = 0 \). Assume that \( F(x)G(x) = 0 \). Since \( R \) is right Ore, there exists \( b_j \in R \) and \( u_j \in C(R) \) such that \( u_j b_j = b_j u_j \) for \( j = 1, \ldots, n \). Let \( f(x) = \sum_{i=0}^{m} a_{ij} x^i \) and \( g(x) = \sum_{j=0}^{n} b_{ji} x^j \). Then \( f(x)g(x) = 0 \). Since \( R \) is weak McCoy, there exists \( 0 \neq c \in R \) with \( f(x)c \in \text{nil}(R[x]) \subseteq \text{nil}(Q[x]) \). Hence \( F(x)uc = f(x)u^{-1}uc = f(x)c \in \text{nil}(Q[x]) \). If \( G(x)F(x) = 0 \), then by a similar argument we can show that \( G(x)v \) \( d \in \text{nil}(Q[x]) \) for some non-zero \( d \in R \).
Case 2. \( F(x)G(x) \neq 0 \) and \( G(x)F(x) \neq 0 \). Since \( F(x)G(x) \in \text{nil}(Q[x]) \), there exists \( n \geq 2 \) such that \( (F(x)G(x))^n = 0 \) and \( (F(x)G(x))^{n-1} \neq 0 \). Let \( (F(x)G(x))^{n-1} = F(x)H(x) \). If \( H(x) \neq 0 \), then by a similar argument as above there exists \( \alpha \in C(R) \), \( r \in R \) such that \( F(x)\alpha r \in \text{nil}(Q[x]) \). Now assume \( H(x) = G(x)F(x)G(x) \cdots F(x)G(x) = 0 \). Since \( (F(x)G(x))^{n-1} \neq 0 \) and \( R \) is weak McCoy, then by Case 1, there exists \( \beta \in C(R) \), \( s \in R \) such that \( G(x)\beta s = 0 \). Therefore \( Q \) is weak McCoy.

According to Bell [2], a ring \( R \) is called semi-commutative if \( ab = 0 \) implies \( aRb = 0 \). We say an ideal \( I \) is a semi-commutative ideal, if \( RI / I \) is a semi-commutative ring.

**Lemma 2.9.** Let \( R \) be a semi-commutative ring. If \( c_1c_2 \cdots c_k = 0 \) for some \( c_i \in R \), then \( c_1Rc_2Rc_3 \cdots Rc_k = 0 \).

**Proof.** By induction, let \( c_{k-1} = c_{k-1}c_k \). Then \( c_1c_2 \cdots c_{k-1} = 0 \) and by induction assumption, we have \( 0 = c_1Rc_2Rc_3 \cdots Rc_{k-1} = c_1Rc_2Rc_3 \cdots Rc_{k-1}c_k \). Hence, for all \( x \in c_1Rc_2Rc_3 \cdots Rc_{k-1}c_k \), we have \( xc_k = 0 \). It follows by hypothesis that \( xRc_k = 0 \). Thus \( c_1Rc_2Rc_3 \cdots Rc_k = 0 \), as desired.

**Lemma 2.10** (4, Lemma 2.5). Let \( R \) be a semi-commutative ring. Then \( \text{nil}(R) \) is a semi-commutative ideal of \( R \).

**Proof.** Let \( a, b \in \text{nil}(R) \). Then \( a^n = 0 = b^m \) for some \( m, n \geq 0 \). Each term of the expansion of \( (a+b)^{m+n+1} \) has the form \( x := (a^ib^j) \cdots (a^{i+s}b^{j+t}) \) where \( i, j \in N \cup \{0\} \). Since \( (i_1 + j_1) + (i_2 + j_2) + \cdots + (i_{m+n+1} + j_{m+n+1}) = \sum_{r=1}^{n} i_r + \sum_{s=1}^{m} j_s = m + n + 1 \), either \( \sum_{r=1}^{n} i_r \geq n \) or \( \sum_{s=1}^{m} j_s \geq m \). If \( \sum_{r=1}^{n} i_r \geq n \), then \( a^ia^j \cdots a^{i+s}b^{j+t} = 0 \). Thus \( (a^ib^j) \cdots (a^{i+s}b^{j+t}) = 0 \), by Lemma 2.9. If \( \sum_{r=1}^{n} i_r < n \), then \( \sum_{s=1}^{m} j_s \geq m \). Thus \( b^jb^k \cdots b^{j+t} = 0 \) and so \( (a^ib^j) \cdots (a^{i+s}b^{j+t}) = 0 \), by Lemma 2.9. Hence \( (a+b)^{m+n+1} = 0 \).

Now suppose that \( a^n = 0 \) and \( r \in R \). Then \( (ar)^n = 0 = (ra)^n \), by Lemma 2.9. Thus \( \text{nil}(R) \) is an ideal of \( R \).
Since $R/\text{nil}(R)$ is a reduced ring, hence it is a semi-commutative ring. Therefore $\text{nil}(R)$ is a semi-commutative ideal of $R$.

**Lemma 2.11.** Let $R$ be a semi-commutative ring. Then $\text{nil}(R[x]) = \text{nil}(R)[x]$.

**Proof.** Let $f(x) = a_0 + \ldots + a_n x^n \in \text{nil}(R[x])$. Then $f(x)^k = 0$, for some integer $k \geq 0$. Hence $a_n^k = 0$, and that $a_n \in \text{nil}(R)$. There exists $g(x), h(x) \in R[x]$ such that $f(x)^k = (a_0 + \ldots + a_n x^{n-1})^k + a_n g(x) + h(x)a_n$. Since $\text{nil}(R)[x]$ is an ideal of $R[x]$ and $a_n g(x), h(x)a_n, f(x)^k \in \text{nil}(R[x])$, we have $(a_0 + \ldots + a_n x^{n-1})^k \in \text{nil}(R)[x]$. Hence $a_{n-1}^k \in \text{nil}(R)$ and that $a_{n-1} \in \text{nil}(R)$. Continuing this process yields $a_0, \ldots, a_n \in \text{nil}(R)$. Therefore $\text{nil}(R[x]) \subseteq \text{nil}(R)[x]$.

Now, let $f(x) = a_0 + \ldots + a_n x^n \in \text{nil}(R)[x]$. Then $a_i^m = 0$, for some positive integer $m_i$. Let $k = m_0 + \ldots + m_n + 1$. Then $(f(x))^k = \sum (a_0^{i_0} (a_1 x)^{i_1} \ldots (a_n x^n)^{i_n}) \cdots (a_0^{i_0} (a_1 x)^{i_1} \ldots (a_n x^n)^{i_n})$, where $i_0, \ldots, i_n = 1$, for $r = 1, \ldots, k$ and $0 \leq i_r \leq 1$. Each coefficient of $f(x)^k$ is a sum of such elements $\gamma = (a_0^{i_1} \cdots (a_n)^{i_n}) \cdots (a_0^{i_1} \cdots (a_n)^{i_n})$, where $i_0 + \ldots + i_n = k$.

It can be easily checked that there exists $a_k \in \{a_0, \ldots, a_n\}$ such that $i_1 + \ldots + i_k \geq m_i$. Since $a_i^{m_i} = 0$ and $R$ is semi-commutative, $\gamma = 0$. Thus $(f(x))^k = 0$ and $\text{nil}(R)[x] \subseteq \text{nil}(R[x])$. Therefore $\text{nil}(R[x]) = \text{nil}(R)[x]$.

**Lemma 2.12.** Let $R$ be a semi-commutative ring. Then $\text{nil}(R[x][y]) = \text{nil}(R[x])[y]$.

**Proof.** By Lemma 2.11, $\text{nil}(R[x])$ is an ideal of $R[x]$. Since $R[x]/\text{nil}(R[x])$ is a reduced ring, hence $\text{nil}(R[x])$ is a semi-commutative ideal of $R[x]$, and that $\text{nil}(R[x])[y] \subseteq \text{nil}(R[x][y])$.

Now, let $F(y) = \sum_{i=0}^{m} f_i y^i \in \text{nil}(R[x][y])$, where $f_i = \sum_{s=0}^{p} a_s x^s \in R[x]$. Then $F(y)^n = 0$, for some positive integers $n$. As in the proof of [1], let $k = n \sum \deg f_i$, where the degree is as polynomial in $x$ and the degree of zero polynomial is taken to be 0. Then $(F(x^k))^n = 0$ and the set of coefficients of $F(x^k)$ is equal to the set of all coefficients of $f_i$, $0 \leq i \leq m$. Hence by Lemma 2.11, $a_j \in \text{nil}(R)$ for all $i, j$ and that $f_i \in \text{nil}(R[x])$, for each $i$. Thus $F(y) \in \text{nil}(R[x])[y]$. Therefore $\text{nil}(R[x][y]) = \text{nil}(R[x])[y]$.
If \( R \) is semi-commutative, then \( R[x] \) may not be semi-commutative, by [5, Example 2]). Here we will show that if \( R \) is semi-commutative, then \( R \) is weak McCoy if and only if \( R[x] \) is weak McCoy.

**Theorem 2.13.** If \( R \) is a semi-commutative ring, then \( R[x] \) is a weak McCoy ring if and only if \( R \) is weak McCoy.

**Proof.** Suppose that \( R \) is a weak McCoy ring. Let \( F(t) = \sum_{i=0}^{m} f_i t^i, \ G(t) = \sum_{j=0}^{n} g_j t^j \) be non-zero polynomials in \( R[x][t] \) such that \( F(t)G(t) \in \text{nil}(R[x][t]) \), where \( f_i = \sum_{\alpha} a_{i\alpha} x^{\alpha} \), \( g_j = \sum_{\beta} b_{j\beta} x^{\beta} \in R[x] \). As in the proof of [1], let \( k = \deg f_i + \deg g_j \), where the degree is as polynomial in \( x \) and the degree of zero polynomial is taken to be 0. Then \( F(x^k) = \sum_{i=0}^{m} f_i x^{ik}, \ G(x^k) = \sum_{j=0}^{n} g_j x^{jk} \in R[x] \), and the set of coefficients of the \( F(x^k) \) is (respectively \( G(x^k) \)) equal to the set of all coefficients of \( f_i, 0 \leq i \leq m \) (respectively \( g_j, 0 \leq j \leq n \)). Since \((F(t)G(t))^p = 0\), for some \( p \geq 1 \), and \( x \) commutes with elements of \( R \), \((F(x^k)G(x^k))^p = 0\). Since \( R \) is weak McCoy, there is \( 0 \neq r \in R \) such that \( F(x^k)r \in \text{nil}(R[x]) \) and \( a_{i\alpha} r \in \text{nil}(R), \ f_i r \in \text{nil}(R[x]) \) for \( 0 \leq i \leq m, \ 0 \leq s \leq p \), by Lemma 2.11. Hence \( F(t)r \in \text{nil}(R[x][t]) \), by Lemma 2.12. Therefore \( R[x] \) is weak McCoy.

Now suppose \( R[x] \) is a weak McCoy ring and \( f(t)g(t) \in \text{nil}(R[t]) \subseteq \text{nil}(R[x][t]) \). Since \( R[x] \) is weak McCoy, there exists \( 0 \neq h(x) \in R[x] \) such that \( f(t)h(x) \in \text{nil}(R[x][t]) \). Let \( h(x) = a_0 + \ldots + a_n x^n \in R[x] \) (\( a_0 \neq 0 \)). Then \( f(t)a_0 \in \text{nil}(R[t]) \), since \((f(t)h(x))^k = (f(t)a_0)^k + k_1 x + \ldots + k_{nk} x^{nk} \) with \( k_1, \ldots, k_{nk} \in R[t] \). Therefore \( R \) is weak McCoy.

**Theorem 2.14.** Let \( R \) be a ring and \( \Delta \) a multiplicatively closed subset of \( R \) consisting of central regular elements. Then \( R \) is weak McCoy if and only if \( \Delta^{-1}R \) is weak McCoy.
Proof. If $R$ is a weak McCoy ring, then by a similar way as used in Theorem 2.8, one can show that $\Delta^{-1}R$ is weak McCoy.

Conversely, let $\Delta^{-1}R$ be a weak McCoy ring. Let $f(x) = \sum_{j=0}^{m} a_j x^j$ and $g(x) = \sum_{j=0}^{n} b_j x^j$ be non-zero polynomials of $R[x]$ such that $f(x)g(x) \in \text{nil}(R[x])$. Since $\Delta^{-1}R$ is weak McCoy, $f(x)(c\alpha^{-1}) \in \text{nil}((\Delta^{-1}R)[x])$ for some non-zero $c\alpha^{-1} \in \Delta^{-1}R$. Thus $f(x)c \in \text{nil}(R[x])$ and $R$ is weak McCoy.

Corollary 2.15. Let $R$ be a ring. Then $R[x]$ is weak McCoy if and only if $R[x,x^{-1}]$ is weak McCoy.

Proof. Clearly $\Delta = \{1,x,x^2,\ldots\}$ is a multiplicatively closed subset of $R[x]$ consisting of central regular elements and $\Delta^{-1}R[x] = R[x,x^{-1}]$. Hence the proof follows from Theorem 2.14.

Theorem 2.16. The classes of weak McCoy rings are closed under direct limits.

Proof. Let $A = \{R_i, \alpha_{ij}\}$ be a direct system of weak McCoy rings $R_i$ for $i \in I$ and ring homomorphisms $\alpha_{ij} : R_i \rightarrow R_j$ for each $i \leq j$ with $\alpha_{ii}(1) = 1$, where $I$ is a directed partially ordered set. Let $R = \lim_{R_i}$ be the direct limit of $A$ with $\ell_i : R_i \rightarrow R$ and $\ell_j \alpha_{ij} = \ell_i$. We show that $R$ is weak McCoy ring. Let $a,b \in R$. Then $a = \ell_i(a_i), b = \ell_j(b_j)$ for some $i,j \in I$ and there is $k \in I$ such that $i \leq k, j \leq k$. Define $a + b = \ell_k(\alpha_{ik}(a_i) + \alpha_{jk}(b_j))$ and $ab = \ell_k(\alpha_{ik}(a_i)\alpha_{jk}(b_j))$, where $\alpha_{ik}(a_i), \alpha_{jk}(b_j) \in R_k$. Then $R$ forms a ring with $0 = \ell_i(0)$ and $1 = \ell_i(1)$. Let $f,g \in R[x]$ be non-zero polynomials such that $fg \in \text{nil}(R[x])$. There is $k \in I$ such that $f,g \in R_k[x]$. Hence $fg \in \text{nil}(R_k[x])$. Since $R_k$ is weak McCoy, there exists $0 \neq c_k \in R_k$ such that $f \alpha_k \in \text{nil}(R_k[x]).$ If $c = \ell_k(c_k),$ then $fc \in \text{nil}(R[x])$ with non-zero $c$. Therefore $R$ is weak McCoy.

Proposition 2.17. (1) Let $R$ be a ring. If there exists a non-zero ideal $I$ of $R$ such that $I[x] \subseteq \text{nil}(R[x])$, then $R$ is weak McCoy.
(2) Every non-semiprime ring is weak McCoy.

(3) Let $R$ be a ring with a non-zero nilpotent ideal. Then $Mat_n(R)$ ($n \geq 2$) is weak McCoy.

**Proof.** (1) Let $0 \neq f \in R[x]$. If $f \in I[x]$, then $fr \in \text{nil}(R[x])$ for all $r \in R$. If $f \notin I[x]$ then $fs \in I[x] \subseteq \text{nil}(R[x])$ for all non-zero $s \in I$. Thus $R$ is weak McCoy.

(2) Let $R$ be a ring with $N_*(R) \neq 0$. Since $0 \neq N_*(R)[x] = N_*(R[x]) \subseteq \text{nil}(R[x])$, $R$ is weak McCoy by (1).

(3) Since $Mat_n(R)$ is non-semiprime, hence by (1) $Mat_n(R)$ is weak McCoy.

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