On Weak McCoy Rings

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Abstract

In this note we introduce the notion of weak McCoy rings as a generalization of McCoy rings, and investigate their properties. Also we show that, if $R$ is a semi-commutative ring, then $R$ is weak McCoy if and only if $R[x]$ is weak McCoy.

1. Introduction

Throughout this paper, all rings are associative with identity. For a commutative ring $R$, McCoy [10] obtained the following result: If $f(x)g(x) = 0$ for some non-zero polynomials $f(x), g(x) \in R[x]$, then $f(x)c = 0$ for some non-zero $c \in R$. According to Nielsen [12], a ring $R$ is called right McCoy whenever polynomials $f(x), g(x) \in R[x] - \{0\}$ satisfy $f(x)g(x) = 0$, there exists a non-zero $r \in R$ such that $f(x)r = 0$. Left McCoy rings are defined similarly. If a ring is both left and right McCoy, we say that the ring is a McCoy ring. It is well known that commutative rings are always McCoy rings [10], but it is not true for non-commutative rings (see [12]).

Recall that a ring $R$ is called:
- reduced if $a^2 = 0 \Rightarrow a = 0$, for all $a \in R$,
- reversible if $ab = 0 \Rightarrow ba = 0$, for all $a, b \in R$,
- symmetric if $abc = 0 \Rightarrow acb = 0$, for all $a, b, c \in R$,
- semi-commutative if $ab = 0 \Rightarrow aRb = 0$, for all $a, b \in R$.

The following implications hold:

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reduced $\Rightarrow$ symmetric $\Rightarrow$ reversible $\Rightarrow$ semi-commutative.

Reversible rings are McCoy rings by [12]. But the converse is not true; there exists a non-reversible McCoy ring (see [12]).

Motivated by the above, as a generalization of McCoy rings, in this paper we introduce the notion of weak McCoy rings and investigate their properties and extend several known results relating to McCoy rings to a general setting.

For a ring $R$, we denote by $\text{nil}(R)$ the set of all nilpotent elements of $R$, by $N_s(R)$ the prime radical of $R$ and by $M_n(R)$, $U_n(R)$ and $L_n(R)$ the $n \times n$ matrix ring over $R$, the $n \times n$ upper and lower triangular matrix rings over $R$ respectively.

2. On Weak McCoy rings

Definition 2.1. We say $R$ is a weak McCoy ring if $f(x)g(x) \in \text{nil}(R[x])$ implies $f(x)c \in \text{nil}(R[x])$, for some non-zero $c \in R$, where $f(x)$ and $g(x)$ are non-zero polynomials in $R[x]$.

Remark 2.2. Since $ab$ is nilpotent if and only if $ba$ is nilpotent in a ring, hence the definition of weak McCoy rings is left-right symmetric.

Proposition 2.3. McCoy rings are weak McCoy.

Proof. Let $R$ be a McCoy ring and $f(x)g(x) \in \text{nil}(R[x])$ for non-zero polynomials $f(x), g(x) \in R[x]$. Then there exists $m, n \geq 1$, such that $(f(x)g(x))^n = (g(x)f(x))^m = 0$, and $(f(x)g(x))^{n-1}, (g(x)f(x))^{m-1} \neq 0$. If $f(x)g(x) = 0$ or $g(x)f(x) = 0$, then the result follows from the definition of McCoy rings. Assume $f(x)g(x) \neq 0 \neq g(x)f(x)$ and $0 = (f(x)g(x))^n = f(x)(g(x)f(x)\ldots f(x)g(x)) = f(x)h(x)$.

If $h(x) = g(x)f(x)\ldots f(x)g(x) \neq 0$, then $f(x)c = 0$ for some non-zero $c \in R$, since $R$ is McCoy.

Let $h(x) = g(x)(f(x)g(x)\ldots f(x)g(x)) = g(x)(f(x)g(x))^{n-1} = 0$. Since $(f(x)g(x))^{n-1} \neq 0$ and $R$ is McCoy, there exists $0 \neq d \in R$ such that $g(x)d = 0$. Therefore $f(x)c = 0$ or
$g(x)d = 0$ for some non-zero $c,d \in R$. Hence $f(x)c \in \text{nil}(R[x])$ or $dg(x) \in \text{nil}(R[x])$ for some non-zero $c,d \in R$. Therefore $R$ is weak McCoy.

**Proposition 2.4.** Let $R$ be a ring. Then $U_n(R)$ and $L_n(R)$ are weak McCoy for each $n \geq 2$.

**Proof.** Clearly $U_n(R)[x] \cong U_n(R(x))$ and for each $A = \begin{bmatrix} 0 & f_{12} & \cdots & f_{1n} \\ 0 & 0 & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in U_n(R[x])$, $A^n = 0$. Let $0 \neq A = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ 0 & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_{nn} \end{bmatrix} \in U_n(R[x])$. Then

$$A^n = 0.$$ Let $0 \neq A = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} = 0$. Hence $U_n(R)$ is weak McCoy. By a similar argument one can show that $L_n(R)$ is weak McCoy.

**Proposition 2.5.** Let $R$ and $S$ be rings and $_RM_S$ a bimodule. Then $egin{bmatrix} R & M \\ 0 & S \end{bmatrix}$ is a weak McCoy ring.

**Proof.** Similarly, as used in Proposition 2.4 one can prove it.

The following example shows that $U_n(R)$ and $M_n(R)$ are neither left nor right McCoy for some $n \geq 2$.

**Example 2.6.** Let $R$ be a ring. We show that $U_4(R)$ and $M_4(R)$ are neither right nor left McCoy. Let $f(x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}x$ and
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g(x) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x \in U_4(R)[x] \subseteq M_4(R)[x]. \text{ Then } f(x)g(x) = 0.
\]

If \( f(x)A = 0 \), for some \( A = [a_j] \in M_4(R) \), then \( 0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} a_{i_1} & a_{i_2} & a_{i_3} & a_{i_4} \\ 0 & 0 & 0 & 0 \\ a_{j_1} & a_{j_2} & a_{j_3} & a_{j_4} \\ 0 & 0 & 0 & 0 \end{bmatrix} \). Hence \( A = 0 \) and \( U_4(R) \) and \( M_4(R) \) are not right McCoy. If \( Bg(x) = 0 \) for some \( B \in M_4(R) \), then by a similar way as above, we can show \( B = 0 \). Therefore \( U_4(R) \) and \( M_4(R) \) are not left McCoy.

**Definition 2.7.** A ring \( R \) is called right Ore if given \( a, b \in R \) with \( b \) regular there exist \( a_i, b_i \in R \) with \( b_i \) regular such that \( ab_i = ba_i \). It is well-known that \( R \) is a right Ore ring if and only if the classical right quotient ring of \( R \) exists. We use \( C(R) \) to denote the set of all regular elements in \( R \).

**Theorem 2.8.** Let \( R \) be a right Ore ring with its classical right quotient ring \( Q \). If \( R \) is weak McCoy then \( Q \) is weak McCoy.

**Proof.** Let \( 0 \neq F(x) = \sum_{i=0}^{m} a_i u^{-i} x^i \) and \( 0 \neq G(x) = \sum_{j=0}^{n} b_j v^{-j} x^j \) with \( a_i, b_j \in R, u, v \in C(R) \) such that \( F(x)G(x) \in nil(Q[x]) \).

**Case 1.** \( F(x)G(x) = 0 \) or \( G(x)F(x) = 0 \). Assume that \( F(x)G(x) = 0 \). Since \( R \) is right Ore, there exists \( b_j \in R \) and \( u_i \in C(R) \) such that \( u_i^{-1} b_j = b_j u_i^{-1} \) for \( j = 1, \ldots, n \). Let \( f(x) = \sum_{i=0}^{m} a_i x^i \) and \( g(x) = \sum_{j=0}^{n} b_j x^j \). Then \( f(x)g(x) = 0 \). Since \( R \) is weak McCoy, there exists \( 0 \neq c \in R \) with \( f(x)c \in nil(R[x]) \subseteq nil(Q[x]) \). Hence \( F(x)uc = f(x)u^{-1} uc = f(x)c \in nil(Q[x]) \).

If \( G(x)F(x) = 0 \), then by a similar argument we can show that \( G(x)v d \in nil(Q[x]) \) for some non-zero \( d \in R \).
Case 2. $F(x)G(x) \neq 0$ and $G(x)F(x) \neq 0$. Since $F(x)G(x) \in \text{nil}(Q[x])$, there exists $n \geq 2$ such that $(F(x)G(x))^n = 0$ and $(F(x)G(x))^{n-1} \neq 0$. Let $(F(x)G(x))^n = F(x)H(x)$. If $H(x) \neq 0$, then by a similar argument as above there exists $\alpha \in C(R)$, $r \in R$ such that $F(x)\alpha r \in \text{nil}(Q[x])$. Now assume $H(x) = G(x)F(x)G(x) \ldots F(x)G(x) = 0$. Since $(F(x)G(x))^{n-1} \neq 0$ and $R$ is weak McCoy, then by Case 1, there exists $\beta \in C(R)$, $s \in R$ such that $G(x)\beta s = 0$. Therefore $Q$ is weak McCoy.

According to Bell [2], a ring $R$ is called semi-commutative if $ab = 0$ implies $aRb = 0$. We say an ideal $I$ is a semi-commutative ideal, if $RI/I$ is a semi-commutative ring.

**Lemma 2.9.** Let $R$ be a semi-commutative ring. If $c_1c_2 \cdots c_k = 0$ for some $c_i \in R$, then $c_1Rc_2 \cdots Rc_k = 0$.

**Proof.** By induction, let $c_{k-1} = c_{k-1}c_k$. Then $c_1c_2 \cdots c_{k-1} = 0$ and by induction assumption, we have $0 = c_1Rc_2 \cdots Rc_{k-1} = c_1Rc_2 \cdots Rc_{k-1}c_k$. Hence, for all $x \in c_1Rc_2 \cdots Rc_{k-1}$, we have $xc_k = 0$. It follows by hypothesis that $xRc_k = 0$. Thus $c_1Rc_2 \cdots Rc_k = 0$, as desired.

**Lemma 2.10** (4, Lemma 2.5). Let $R$ be a semi-commutative ring. Then $\text{nil}(R)$ is a semi-commutative ideal of $R$.

**Proof.** Let $a, b \in \text{nil}(R)$. Then $a^n = 0 = b^m$ for some $m, n \geq 0$. Each term of the expansion of $(a+b)^{m+n+1}$ has the form $x := (a^{i_1}b^{j_1}) \cdots (a^{i_{m+n+1}}b^{j_{m+n+1}})$ where $i_r, j_s \in N \cup \{0\}$. Since $\sum_{r=1}^m i_r + \sum_{s=1}^n j_s = m + n + 1$, either $\sum_{r=1}^m i_r \geq n$ or $\sum_{s=1}^n j_s \geq m$. If $\sum_{r=1}^m i_r \geq n$, then $a^{i_1}a^{i_2} \cdots a^{i_{m+n+1}} = 0$. Thus $(a^{h_1}b^{j_1}) \cdots (a^{h_{m+n+1}}b^{j_{m+n+1}}) = 0$, by Lemma 2.9. If $\sum_{r=1}^m i_r < n$, then $\sum_{s=1}^n j_s \geq m$. Thus $b^{j_1}b^{j_2} \cdots b^{j_{m+n+1}} = 0$ and so $(a^{h_1}b^{j_1}) \cdots (a^{h_{m+n+1}}b^{j_{m+n+1}}) = 0$, by Lemma 2.9. Hence $(a+b)^{m+n+1} = 0$.

Now suppose that $a^n = 0$ and $r \in R$. Then $(ar)^n = 0 = (ra)^n$, by Lemma 2.9. Thus $\text{nil}(R)$ is an ideal of $R$.  

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Since \( R/\text{nil}(R) \) is a reduced ring, hence it is a semi-commutative ring. Therefore \( \text{nil}(R) \) is a semi-commutative ideal of \( R \).

**Lemma 2.11.** Let \( R \) be a semi-commutative ring. Then \( \text{nil}(R[x]) = \text{nil}(R)[x] \).

**Proof.** Let \( f(x) = a_0 + \ldots + a_n x^n \in \text{nil}(R[x]) \). Then \( f(x)^k = 0 \), for some integer \( k \geq 0 \). Hence \( a_n^k = 0 \), and that \( a_n \in \text{nil}(R) \). There exists \( g(x), h(x) \in R[x] \) such that \( f(x)^k = (a_0 + \ldots + a_{n-1} x^{n-1})^k + a_n g(x) + h(x)a_n \). Since \( \text{nil}(R)[x] \) is an ideal of \( R[x] \) and \( a_n g(x), h(x)a_n, f(x)^k \in \text{nil}(R)[x] \), we have \( (a_0 + \ldots + a_{n-1} x^{n-1})^k \in \text{nil}(R)[x] \). Hence \( a_{n-1}^k \in \text{nil}(R) \) and that \( a_{n-1} \in \text{nil}(R) \). Continuing this process yields \( a_0, \ldots, a_n \in \text{nil}(R) \).

Therefore \( \text{nil}(R[x]) \subseteq \text{nil}(R)[x] \).

Now, let \( f(x) = a_0 + \ldots + a_n x^n \in \text{nil}(R)[x] \). Then \( a_i^m = 0 \), for some positive integer \( m_i \). Let \( k = m_0 + \ldots + m_n + 1 \). Then \( f(x)^k = \sum (a_0^{i_0} (a_1 x)^{i_1} \ldots (a_n x^n)^{i_n}) = \ldots (a_0^{i_0} (a_1 x)^{i_1} \ldots (a_n x^n)^{i_n}) \), where \( i_0 + \ldots + i_n = 1 \), for \( r = 1, \ldots, k \) and \( 0 \leq i_r \leq 1 \). Each coefficient of \( f(x)^k \) is a sum of such elements \( \gamma = (a_0)^{i_0} \ldots (a_n)^{i_n} \), where \( i_0 + \ldots + i_n = 1 \).

It can be easily checked that there exists \( a_i \in \{a_0, \ldots, a_n\} \) such that \( i_0 + \ldots + i_n \geq m_i \). Since \( a_i^m = 0 \) and \( R \) is semi-commutative, \( \gamma = 0 \). Thus \( f(x)^k = 0 \) and \( \text{nil}(R)[x] \subseteq \text{nil}(R)[x] \). Therefore \( \text{nil}(R[x]) = \text{nil}(R)[x] \).

**Lemma 2.12.** Let \( R \) be a semi-commutative ring. Then \( \text{nil}(R[x])[y] = \text{nil}(R[x])[y] \).

**Proof.** By Lemma 2.11, \( \text{nil}(R[x]) \) is an ideal of \( R[x] \). Since \( R[x]/\text{nil}(R[x]) \) is a reduced ring, hence \( \text{nil}(R[x]) \) is a semi-commutative ideal of \( R[x] \), and that \( \text{nil}(R[x])[y] \subseteq \text{nil}(R[x])[y] \).

Now, let \( F(y) = \sum_{i=0}^{m} f_i y^i \in \text{nil}(R[x])[y] \), where \( f_i = \sum_{s=0}^{p} a_s x^s \in R[x] \). Then \( F(y)^n = 0 \), for some positive integers \( n \). As in the proof of [1], let \( k = n \sum \deg f_i \), where the degree is as polynomial in \( x \) and the degree of zero polynomial is taken to be 0. Then \( (F(x^k))^n = 0 \) and the set of coefficients of \( F(x^k) \) is equal to the set of all coefficients of \( f_i \), \( 0 \leq i \leq m \). Hence by Lemma 2.11, \( a_j \in \text{nil}(R) \) for all \( i, j \) and that \( f_i \in \text{nil}(R[x]) \), for each \( i \). Thus \( F(y) \in \text{nil}(R[x])[y] \). Therefore \( \text{nil}(R[x])[y] = \text{nil}(R[x])[y] \).
If $R$ is semi-commutative, then $R[x]$ may not be semi-commutative, by [5, Example 2]). Here we will show that if $R$ is semi-commutative, then $R$ is weak McCoy if and only if $R[x]$ is weak McCoy.

**Theorem 2.13.** If $R$ is a semi-commutative ring, then $R[x]$ is a weak McCoy ring if and only if $R$ is weak McCoy.

**Proof.** Suppose that $R$ is a weak McCoy ring. Let $F(t) = \sum_{i=0}^{m} f_i t^i$, $G(t) = \sum_{j=0}^{n} g_j t^j$ be non-zero polynomials in $R[x][t]$ such that $F(t)G(t) \in \text{nil}(R[x][t])$, where $f_i = \sum_{a_i} a_i x^i$, $g_j = \sum_{b_j} b_j x^j \in R[x]$. As in the proof of [1], let $k = \sum \deg f_i + \sum \deg g_j$, where the degree is as polynomial in $x$ and the degree of zero polynomial is taken to be 0. Then $F(x^k) = \sum_{i=0}^{m} f_i x^{ik}, G(x^k) = \sum_{j=0}^{n} g_j x^{jk} \in R[x]$, and the set of coefficients of the $F(x^k)$ is (respectively $G(x^k)$) equal to the set of all coefficients of $f_i$, $0 \leq i \leq m$ (respectively $g_j$, $0 \leq j \leq n$). Since $(F(t)G(t))^p = 0$, for some $p \geq 1$, and $x$ commutes with elements of $R$, $(F(x^k)G(x^k))^p = 0$. Since $R$ is weak McCoy, there is $0 \neq r \in R$ such that $F(x^k)r \in \text{nil}(R[x])$ and $a_i r \in \text{nil}(R)$, $f_i r \in \text{nil}(R[x])$ for $0 \leq i \leq m$, $0 \leq s \leq p$, by Lemma 2.11. Hence $F(t)r \in \text{nil}(R[x][t])$, by Lemma 2.12. Therefore $R[x]$ is weak McCoy.

Now suppose $R[x]$ is a weak McCoy ring and $f(t)g(t) \in \text{nil}(R[t]) \subseteq \text{nil}(R[x][t])$. Since $R[x]$ is weak McCoy, there exists $0 \neq h(x) \in R[x]$ such that $f(t)h(x) \in \text{nil}(R[x][t])$. Let $h(x) = a_0 + \ldots + a_n x^n \in R[x]$ ($a_0 \neq 0$). Then $f(t)a_0 \in \text{nil}(R[t])$, since $(f(t)h(x))^k = (f(t)a_0)^k + \ldots + k_{nk} x^{nk}$ with $k_1, \ldots, k_{nk} \in R[t]$. Therefore $R$ is weak McCoy.

**Theorem 2.14.** Let $R$ be a ring and $\Delta$ a multiplicatively closed subset of $R$ consisting of central regular elements. Then $R$ is weak McCoy if and only if $\Delta^{-1}R$ is weak McCoy.
**Proof.** If $R$ is a weak McCoy ring, then by a similar way as used in Theorem 2.8, one can show that $\Delta^{-1}R$ is weak McCoy.

Conversely, let $\Delta^{-1}R$ be a weak McCoy ring. Let $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j$ be non-zero polynomials of $R[x]$ such that $f(x)g(x) \in \text{nil}(R[x])$. Since $\Delta^{-1}R$ is weak McCoy, $f(x)c \alpha^{-1} \in \text{nil}(\Delta^{-1}R[x])$ for some non-zero $c \alpha^{-1} \in \Delta^{-1}R$. Thus $f(x)c \in \text{nil}(R[x])$ and $R$ is weak McCoy.

**Corollary 2.15.** Let $R$ be a ring. Then $R[x]$ is weak McCoy if and only if $R[x,x^{-1}]$ is weak McCoy.

**Proof.** Clearly $\Delta = \{1, x, x^2, \ldots\}$ is a multiplicatively closed subset of $R[x]$ consisting of central regular elements and $\Delta^{-1}R[x] = R[x,x^{-1}]$. Hence the proof follows from Theorem 2.14.

**Theorem 2.16.** The classes of weak McCoy rings are closed under direct limits.

**Proof.** Let $A = \{R_i, \alpha_i\}$ be a direct system of weak McCoy rings $R_i$ for $i \in I$ and ring homomorphisms $\alpha_i : R_i \to R_j$ for each $i \leq j$ with $\alpha_i(1) = 1$, where $I$ is a directed partially ordered set. Let $R = \lim R_i$ be the direct limit of $A$ with $\ell_i : R_i \to R$ and $\ell_j \alpha_i = \ell_i$. We show that $R$ is weak McCoy ring. Let $a, b \in R$. Then $a = \ell_i(a_i), b = \ell_j(b_j)$ for some $i, j \in I$ and there is $k \in I$ such that $i \leq k, j \leq k$.

Define $a + b = \ell_k(\alpha_k(a_i) + \alpha_k(b_j))$ and $ab = \ell_k(\alpha_k(a_i)\alpha_k(b_j))$, where $\alpha_k(a_i), \alpha_k(b_j) \in R_k$. Then $R$ forms a ring with $0 = \ell_i(0)$ and $1 = \ell_i(1)$. Let $f, g \in R[x]$ be non-zero polynomials such that $fg \in \text{nil}(R[x])$. There is $k \in I$ such that $f, g \in R_k[x]$. Hence $fg \in \text{nil}(R_k[x])$. Since $R_k$ is weak McCoy, there exists $0 \neq c_k \in R_k$ such that $fc_k \in \text{nil}(R_k[x])$. If $c = \ell_k(c_k)$, then $fc \in \text{nil}(R[x])$ with non-zero $c$. Therefore $R$ is weak McCoy.

**Proposition 2.17.** (1) Let $R$ be a ring. If there exists a non-zero ideal $I$ of $R$ such that $I[x] \subseteq \text{nil}(R[x])$, then $R$ is weak McCoy.
(2) Every non-semiprime ring is weak McCoy.

(3) Let $R$ be a ring with a non-zero nilpotent ideal. Then $Mat_n(R)$ $(n \geq 2)$ is weak McCoy.

**Proof.** (1) Let $0 \neq f \in R[x]$. If $f \in I[x]$, then $fr \in \text{nil}(R[x])$ for all $r \in R$. If $f \notin I[x]$ then $fs \in I[x] \subseteq \text{nil}(R[x])$ for all non-zero $s \in I$. Thus $R$ is weak McCoy.

(2) Let $R$ be a ring with $N_*(R) \neq 0$. Since $0 \neq N_*(R)[x] = N_*(R[x]) \subseteq \text{nil}(R[x])$, $R$ is weak McCoy by (1).

(3) Since $Mat_n(R)$ is non-semiprime, hence by (1) $Mat_n(R)$ is weak McCoy.

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**References**


