On Weak McCoy Rings

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Abstract

In this note we introduce the notion of weak McCoy rings as a generalization of McCoy rings, and investigate their properties. Also we show that, if $R$ is a semi-commutative ring, then $R$ is weak McCoy if and only if $R[x]$ is weak McCoy.

1. Introduction

Throughout this paper, all rings are associative with identity. For a commutative ring $R$, McCoy [10] obtained the following result: If $f(x)g(x) = 0$ for some non-zero polynomials $f(x), g(x) \in R[x]$, then $f(x)c = 0$ for some non-zero $c \in R$. According to Nielsen [12], a ring $R$ is called right McCoy whenever polynomials $f(x), g(x) \in R[x] - \{0\}$ satisfy $f(x)g(x) = 0$, there exists a non-zero $r \in R$ such that $f(x)r = 0$. Left McCoy rings are defined similarly. If a ring is both left and right McCoy, we say that the ring is a McCoy ring. It is well known that commutative rings are always McCoy rings [10], but it is not true for non-commutative rings (see [12]).

Recall that a ring $R$ is called:

- reduced if $a^2 = 0 \Rightarrow a = 0$, for all $a \in R$,
- reversible if $ab = 0 \Rightarrow ba = 0$, for all $a, b \in R$,
- symmetric if $abc = 0 \Rightarrow acb = 0$, for all $a, b, c \in R$,
- semi-commutative if $ab = 0 \Rightarrow aRb = 0$, for all $a, b \in R$.

The following implications hold:

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reduced $\Rightarrow$ symmetric $\Rightarrow$ reversible $\Rightarrow$ semi-commutative.

Reversible rings are McCoy rings by [12]. But the converse is not true; there exists a non-reversible McCoy ring (see [12]).

Motivated by the above, as a generalization of McCoy rings, in this paper we introduce the notion of weak McCoy rings and investigate their properties and extend several known results relating to McCoy rings to a general setting.

For a ring $R$, we denote by $\text{nil}(R)$ the set of all nilpotent elements of $R$, by $N_*(R)$ the prime radical of $R$ and by $M_n(R), U_n(R)$ and $L_n(R)$ the $n \times n$ matrix ring over $R$, the $n \times n$ upper and lower triangular matrix rings over $R$ respectively.

2. On Weak McCoy rings

Definition 2.1. We say $R$ is a weak McCoy ring if \( f(x)g(x) \in \text{nil}(R[x]) \) implies \( f(x)c \in \text{nil}(R[x]) \), for some non-zero \( c \in R \), where \( f(x) \) and \( g(x) \) are non-zero polynomials in \( R[x] \).

Remark 2.2. Since \( ab \) is nilpotent if and only if \( ba \) is nilpotent in a ring, hence the definition of weak McCoy rings is left-right symmetric.

Proposition 2.3. McCoy rings are weak McCoy.

Proof. Let $R$ be a McCoy ring and \( f(x)g(x) \in \text{nil}(R[x]) \) for non-zero polynomials \( f(x), g(x) \in R[x] \). Then there exists \( m,n \geq 1 \), such that \( (f(x)g(x))^n = (g(x)f(x))^m = 0 \), and \( (f(x)g(x))^{n-1}, (g(x)f(x))^{m-1} \neq 0 \). If \( f(x)g(x) = 0 \) or \( g(x)f(x) = 0 \), then the result follows from the definition of McCoy rings. Assume \( f(x)g(x) \neq 0 \neq g(x)f(x) \) and \( 0 = (f(x)g(x))^n = f(x)(g(x)f(x)\ldots f(x)g(x)) = f(x)h(x) \).

If \( h(x) = g(x)f(x)\ldots f(x)g(x) = 0 \), then \( f(x)c = 0 \) for some non-zero \( c \in R \), since $R$ is McCoy.

Let \( h(x) = g(x)(f(x)g(x)\ldots f(x)g(x)) = g(x)(f(x)g(x))^{n-1} = 0 \). Since \( (f(x)g(x))^{n-1} \neq 0 \) and $R$ is McCoy, there exists \( 0 \neq d \in R \) such that \( g(x)d = 0 \). Therefore \( f(x)c = 0 \) or
Let \( R \) be a ring. Then \( U_n(R) \) and \( L_n(R) \) are weak McCoy for each \( n \geq 2 \).

**Proof.** Clearly \( U_n(R)[x] \cong U_n(R[x]) \) and for each 

\[
A = \begin{bmatrix}
0 & f_{12} & \cdots & f_{1n} \\
0 & 0 & \cdots & f_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0
\end{bmatrix} \in U_n(R[x]),
\]

\[
A^n = 0.
\]

Let \( 0 \neq A = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\
0 & f_{22} & \cdots & f_{2n} \\
0 & 0 & \ddots & \vdots \\
0 & 0 & \cdots & f_{nn}\end{bmatrix} \in U_n(R[x]). \) Then

\[
\begin{bmatrix}
0 & 1 & \cdots & 1 \\
0 & 0 & \cdots & 1 \\
0 & 0 & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{bmatrix} = \begin{bmatrix}
g_{11} & \cdots & g_{1n} \\
0 & 0 & \cdots & g_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
0 & 1 & \cdots & 1 \\
0 & 0 & \cdots & 1 \\
0 & 0 & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{bmatrix}^n = 0.
\]

Hence \( U_n(R) \) is weak McCoy. By a similar argument one can show that \( L_n(R) \) is weak McCoy.

**Proposition 2.5.** Let \( R \) and \( S \) be rings and \( _R M_S \) a bimodule. Then \( \begin{bmatrix} R & M \\ 0 & S \end{bmatrix} \) is a weak McCoy ring.

**Proof.** Similarly, as used in Proposition 2.4 one can prove it.

The following example shows that \( U_n(R) \) and \( M_n(R) \) are neither left nor right McCoy for some \( n \geq 2 \).

**Example 2.6.** Let \( R \) be a ring. We show that \( U_4(R) \) and \( M_4(R) \) are neither right nor left McCoy. Let

\[
f(x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x
\]

and
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\[ g(x) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad x \in U_4(R)[x] \subseteq M_4(R)[x]. \] Then \( f(x)g(x) = 0. \)

If \( f(x)A = 0 \), for some \( A = [a_{ij}] \in M_4(R) \), then \( 0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix} \)

and \( 0 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} A = \begin{bmatrix} -a_{21} & -a_{22} & -a_{23} & -a_{24} \\ -a_{41} & a_{42} & -a_{43} & -a_{44} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \). Hence \( A = 0 \) and \( U_4(R) \) and \( M_4(R) \) are not right McCoy. If \( Bg(x) = 0 \) for some \( B \in M_4(R) \), then by a similar way as above, we can show \( B = 0 \). Therefore \( U_4(R) \) and \( M_4(R) \) are not left McCoy.

**Definition 2.7.** A ring \( R \) is called right Ore if given \( a, b \in R \) with \( b \) regular there exist \( a_i, b_i \in R \) with \( b_i \) regular such that \( ab_i = ba_i \). It is well-known that \( R \) is a right Ore ring if and only if the classical right quotient ring of \( R \) exists. We use \( C(R) \) to denote the set of all regular elements in \( R \).

**Theorem 2.8.** Let \( R \) be a right Ore ring with its classical right quotient ring \( Q \). If \( R \) is weak McCoy then \( Q \) is weak McCoy.

**Proof.** Let \( 0 \neq F(x) = \sum_{i=0}^{m} a_i u^{-i} x^i \) and \( 0 \neq G(x) = \sum_{j=0}^{n} b_j v^{-j} x^j \) with \( a_i, b_j \in R, u, v \in C(R) \) such that \( F(x)G(x) \in \text{nil}(Q[x]) \).

**Case 1.** \( F(x)G(x) = 0 \) or \( G(x)F(x) = 0 \). Assume that \( F(x)G(x) = 0 \). Since \( R \) is right Ore, there exists \( b_j' \in R \) and \( u_i \in C(R) \) such that \( u^{-1} b_j = b_j' u_i^{-1} \) for \( j = 1, \ldots, n \). Let \( f(x) = \sum_{i=0}^{m} a_i x^i \) and \( g(x) = \sum_{j=0}^{n} b_j x^j \). Then \( f(x)g(x) = 0 \). Since \( R \) is weak McCoy, there exists \( 0 \neq c \in R \) with \( f(x)c \in \text{nil}(R[x]) \subseteq \text{nil}(Q[x]) \). Hence \( F(x)uc = f(x)u^{-1}uc = f(x)c \in \text{nil}(Q[x]) \).

If \( G(x)F(x) = 0 \), then by a similar argument we can show that \( G(x)d \in \text{nil}(Q[x]) \) for some non-zero \( d \in R \).

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Case 2. \( F(x)G(x) \neq 0 \) and \( G(x)F(x) \neq 0 \). Since \( F(x)G(x) \in \text{nil}(Q[x]) \), there exists \( n \geq 2 \) such that \( (F(x)G(x))^n = 0 \) and \( (F(x)G(x))^{n-1} \neq 0 \). Let \( (F(x)G(x))^n = F(x)H(x) \). If \( H(x) \neq 0 \), then by a similar argument as above there exists \( \alpha \in C(R) \), \( r \in R \) such that \( F(x)\alpha r \in \text{nil}(Q[x]) \). Now assume \( H(x) = G(x)F(x)G(x) \cdots F(x)G(x) = 0 \). Since \( (F(x)G(x))^{n-1} \neq 0 \) and \( R \) is weak McCoy, then by Case 1, there exists \( \beta \in C(R) \), \( s \in R \) such that \( G(x)\beta s = 0 \). Therefore \( Q \) is weak McCoy.

According to Bell [2], a ring \( R \) is called semi-commutative if \( ab = 0 \) implies \( aRb = 0 \). We say an ideal \( I \) is a semi-commutative ideal, if \( IR/\) is a semi-commutative ring.

Lemma 2.9. Let \( R \) be a semi-commutative ring. If \( c_1c_2 \cdots c_k = 0 \) for some \( c_i \in R \), then \( c_1Rc_2Rc_3 \cdots Rc_k = 0 \).

Proof. By induction, let \( c_{k-1} = c_{k-1}c_k \). Then \( c_1c_2 \cdots c_{k-1} = 0 \) and by induction assumption, we have \( 0 = c_1Rc_2Rc_3 \cdots Rc_{k-1} = c_1Rc_2Rc_3 \cdots Rc_{k-1}c_k \). Hence, for all \( x \in c_1Rc_2Rc_3 \cdots Rc_{k-1} \), we have \( xc_k = 0 \). It follows by hypothesis that \( xRc_k = 0 \). Thus \( c_1Rc_2Rc_3 \cdots Rc_k = 0 \), as desired.

Lemma 2.10 (4, Lemma 2.5). Let \( R \) be a semi-commutative ring. Then \( \text{nil}(R) \) is a semi-commutative ideal of \( R \).

Proof. Let \( a, b \in \text{nil}(R) \). Then \( a^n = 0 = b^m \) for some \( m, n \geq 0 \). Each term of the expansion of \( (a+b)^{mn+1} \) has the form \( x = (a^ib^j) \cdots (a^ib^j) \) where \( i, j \in N \cup \{0\} \). Since \( (i_1 + i_2 + i_3) + \cdots + (i_{m+n+1} + j_{m+n+1}) = \sum_{r=1}^{n} i_r + \sum_{s=1}^{m} j_s = m + n + 1 \), either \( \sum_{r=1}^{n} i_r \geq n \) or \( \sum_{s=1}^{m} j_s \geq m \). If \( \sum_{r=1}^{n} i_r \geq n \), then \( a^i a^j \cdots a^{i_{m+n+1}} = 0 \). Thus \( (a^ib^j) \cdots (a^ib^j) = 0 \), by Lemma 2.9. If \( \sum_{r=1}^{n} i_r < n \), then \( \sum_{r=1}^{n} i_r \geq m \). Thus \( b^j b^j \cdots b^{j_{m+n+1}} = 0 \) and so \( (a^ib^j) \cdots (a^ib^j) = 0 \), by Lemma 2.9. Hence \( (a+b)^{mn+1} = 0 \).

Now suppose that \( a^n = 0 \) and \( r \in R \). Then \( (ar)^n = 0 = (ra)^n \), by Lemma 2.9. Thus \( \text{nil}(R) \) is an ideal of \( R \).
Since \( R/\text{nil}(R) \) is a reduced ring, hence it is a semi-commutative ring. Therefore \( \text{nil}(R) \) is a semi-commutative ideal of \( R \).

**Lemma 2.11.** Let \( R \) be a semi-commutative ring. Then \( \text{nil}(R[x]) = \text{nil}(R)[x] \).

**Proof.** Let \( f(x) = a_0 + \ldots + a_n x^n \in \text{nil}(R[x]) \). Then \( f(x)^k = 0 \), for some integer \( k \geq 0 \). Hence \( a_n^k = 0 \), and that \( a_n \in \text{nil}(R) \). There exists \( g(x), h(x) \in R[x] \) such that \( f(x)^k = (a_0 + \ldots + a_{n-1} x^{n-1})^k + a_n g(x) + h(x)a_n \). Since \( \text{nil}(R)[x] \) is an ideal of \( R[x] \) and \( a_n g(x), h(x)a_n, f(x)^k \in \text{nil}(R)[x] \), we have \( (a_0 + \ldots + a_{n-1} x^{n-1})^k \in \text{nil}(R)[x] \). Hence \( a_{n-1}^k \in \text{nil}(R) \) and that \( a_{n-1} \in \text{nil}(R) \). Continuing this process yields \( a_0, \ldots, a_n \in \text{nil}(R) \). Therefore \( \text{nil}(R[x]) \subseteq \text{nil}(R)[x] \).

Now, let \( f(x) = a_0 + \ldots + a_n x^n \in \text{nil}(R[x]) \). Then \( a_i m_i = 0 \), for some positive integer \( m_i \). Let \( k = m_0 + \ldots + m_n + 1 \). Then \( (f(x))^k = \sum (a_0^{i_0} a_1^{i_1} \ldots a_n^{i_n}) (x^r)^{i_0} (a_1 x^{i_1}) \ldots (a_n x^n)^{i_n} \), where \( i_0, \ldots, i_n = 1 \), for \( r = 1, \ldots, k \) and \( 0 \leq i_i \leq 1 \). Each coefficient of \( f(x)^k \) is a sum of such elements \( \gamma = (a_0^{i_0} \ldots a_n^{i_n}) \ldots (a_0^{i_0} \ldots a_n^{i_n}) \), where \( i_0 + \ldots + i_n = 1 \).

It can be easily checked that there exists \( a_k \in \{a_0, \ldots, a_n\} \) such that \( i_0 + \ldots + i_k \geq m_i \). Since \( a_i m_i = 0 \) and \( R \) is semi-commutative, \( \gamma = 0 \). Thus \( (f(x))^k = 0 \) and \( \text{nil}(R)[x] \subseteq \text{nil}(R)[x] \). Therefore \( \text{nil}(R[x]) = \text{nil}(R)[x] \).

**Lemma 2.12.** Let \( R \) be a semi-commutative ring. Then \( \text{nil}(R[x][y]) = \text{nil}(R[x])[y] \).

**Proof.** By Lemma 2.11, \( \text{nil}(R[x]) \) is an ideal of \( R[x] \). Since \( R[x]/\text{nil}(R[x]) \) is a reduced ring, hence \( \text{nil}(R[x]) \) is a semi-commutative ideal of \( R[x] \), and that \( \text{nil}(R[x])[y] \subseteq \text{nil}(R[x][y]) \).

Now, let \( F(y) = \sum_{i=0}^m f_i y^i \in \text{nil}(R[x][y]) \), where \( f_i = \sum_{i=0}^n a_i x^i \in R[x] \). Then \( F(y)^n = 0 \), for some positive integers \( n \). As in the proof of [1], let \( k = n \sum \text{deg } f_i \), where the degree is as polynomial in \( x \) and the degree of zero polynomial is taken to be \( 0 \). Then \( (F(x^k))^n = 0 \) and the set of coefficients of \( F(x^k) \) is equal to the set of all coefficients of \( f_i \), \( 0 \leq i \leq m \). Hence by Lemma 2.11, \( a_j \in \text{nil}(R) \) for all \( i, j \) and that \( f_i \in \text{nil}(R[x]) \), for each \( i \). Thus \( F(y) \in \text{nil}(R[x][y]) \). Therefore \( \text{nil}(R[x][y]) = \text{nil}(R[x])[y] \).
If \( R \) is semi-commutative, then \( R[x] \) may not be semi-commutative, by [5, Example 2]). Here we will show that if \( R \) is semi-commutative, then \( R \) is weak McCoy if and only if \( R[x] \) is weak McCoy.

**Theorem 2.13.** If \( R \) is a semi-commutative ring, then \( R[x] \) is a weak McCoy ring if and only if \( R \) is weak McCoy.

**Proof.** Suppose that \( R \) is a weak McCoy ring. Let \( F(t) = \sum_{i=0}^{m} f_i t^i \), \( G(t) = \sum_{j=0}^{n} g_j t^j \) be non-zero polynomials in \( R[x][t] \) such that \( F(t)G(t) \in \text{nil}(R[x][t]) \), where \( f_i = \sum_{a} a_i x^i \), \( g_j = \sum_{b} b_j x^j \in R[x] \). As in the proof of [1], let \( k = \sum \deg f_i + \sum \deg g_j \), where the degree is as polynomial in \( x \) and the degree of zero polynomial is taken to be 0. Then \( F(x^k) = \sum_{i=0}^{m} f_i x^{ik} \), \( G(x^k) = \sum_{j=0}^{n} g_j x^{jk} \in R[x] \), and the set of coefficients of the \( F(x^k) \) is (respectively \( G(x^k) \)) equal to the set of all coefficients of \( f_i \), \( 0 \leq i \leq m \) (respectively \( g_j \), \( 0 \leq j \leq n \)). Since \( (F(t)G(t))^p = 0 \), for some \( p \geq 1 \), and \( x \) commutes with elements of \( R \), \( (F(x^k)G(x^k))^p = 0 \). Since \( R \) is weak McCoy, there is \( 0 \neq r \in R \) such that \( F(x^k)r \in \text{nil}(R[x]) \) and \( a_i r \in \text{nil}(R) \), \( f_i r \in \text{nil}(R[x]) \) for \( 0 \leq i \leq m \), \( 0 \leq s \leq p \), by Lemma 2.11. Hence \( F(t)r \in \text{nil}(R[x][t]) \), by Lemma 2.12. Therefore \( R[x] \) is weak McCoy.

Now suppose \( R[x] \) is a weak McCoy ring and \( f(t)g(t) \in \text{nil}(R[t]) \). Since \( R[x] \) is weak McCoy, there exists \( 0 \neq h(x) \in R[x] \) such that \( f(t)h(x) \in \text{nil}(R[x][t]) \). Let \( h(x) = a_0 + \ldots + a_n x^n \in R[x] \) \( (a_0 \neq 0) \). Then \( f(t) a_0 \in \text{nil}(R[t]) \), since \( (f(t)h(x))^k = (f(t)a_0)^k + h_1 x + \ldots + h_{mk} x^{mk} \) with \( k_1, \ldots, k_{mk} \in R[t] \). Therefore \( R \) is weak McCoy.

**Theorem 2.14.** Let \( R \) be a ring and \( \Delta \) a multiplicatively closed subset of \( R \) consisting of central regular elements. Then \( R \) is weak McCoy if and only if \( \Delta^{-1}R \) is weak McCoy.
Proof. If \( R \) is a weak McCoy ring, then by a similar way as used in Theorem 2.8, one can show that \( \Delta^{-1}R \) is weak McCoy.

Conversely, let \( \Delta^{-1}R \) be a weak McCoy ring. Let \( f(x) = \sum_{j=0}^{m} a_j x^j \) and \( g(x) = \sum_{j=0}^{n} b_j x^j \) be non-zero polynomials of \( R[x] \) such that \( f(x)g(x) \in \text{nil}(R[x]) \). Since \( \Delta^{-1}R \) is weak McCoy, \( f(x)(c\alpha^{-1}) \in \text{nil}((\Delta^{-1}R)[x]) \) for some non-zero \( c\alpha^{-1} \in \Delta^{-1}R \). Thus \( f(x)c \in \text{nil}(R[x]) \) and \( R \) is weak McCoy.

Corollary 2.15. Let \( R \) be a ring. Then \( R[x] \) is weak McCoy if and only if \( R[x,x^{-1}] \) is weak McCoy.

Proof. Clearly \( \Delta = \{1, x, x^2, \ldots\} \) is a multiplicatively closed subset of \( R[x] \) consisting of central regular elements and \( \Delta^{-1}R[x] = R[x,x^{-1}] \). Hence the proof follows from Theorem 2.14.

Theorem 2.16. The classes of weak McCoy rings are closed under direct limits.

Proof. Let \( A = \{R_i, \alpha_{ij}\} \) be a direct system of weak McCoy rings \( R_i \) for \( i \in I \) and ring homomorphisms \( \alpha_{ij} : R_i \to R_j \) for each \( i \leq j \) with \( \alpha_{ii}(1) = 1 \), where \( I \) is a directed partially ordered set. Let \( R = \varinjlim R_i \) be the direct limit of \( A \) with \( \ell_i : R_i \to R \) and \( \ell_j \circ \alpha_{ij} = \ell_i \). We show that \( R \) is weak McCoy ring. Let \( a, b \in R \). Then \( a = \ell_i(a_i), b = \ell_j(b_j) \) for some \( i, j \in I \) and there is \( k \in I \) such that \( i \leq k, j \leq k \). Define \( a + b = \ell_k(\alpha_{ik}(a_i) + \alpha_{jk}(b_j)) \) and \( ab = \ell_k(\alpha_{ik}(a_i)\alpha_{jk}(b_j)) \), where \( \alpha_{ik}(a_i), \alpha_{jk}(b_j) \in R_k \). Then \( R \) forms a ring with \( 0 = \ell_i(0) \) and \( 1 = \ell_i(1) \). Let \( f, g \in R[x] \) be non-zero polynomials such that \( fg \in \text{nil}(R[x]) \). There is \( k \in I \) such that \( f, g \in R_k[x] \). Hence \( fg \in \text{nil}(R_k[x]) \). Since \( R_k \) is weak McCoy, there exists \( 0 \neq c_k \in R_k \) such that \( fc_k \in \text{nil}(R_k[x]) \). If \( c = \ell_k(c_k) \), then \( fc \in \text{nil}(R[x]) \) with non-zero \( c \). Therefore \( R \) is weak McCoy.

Proposition 2.17. (1) Let \( R \) be a ring. If there exists a non-zero ideal \( I \) of \( R \) such that \( I[x] \subseteq \text{nil}(R[x]) \), then \( R \) is weak McCoy.
(2) Every non-semiprime ring is weak McCoy.

(3) Let \( R \) be a ring with a non-zero nilpotent ideal. Then \( \text{Mat}_n(R) \) \((n \geq 2)\) is weak McCoy.

**Proof.** (1) Let \( 0 \neq f \in R[x] \). If \( f \in I[x] \), then \( fr \in \text{nil}(R[x]) \) for all \( r \in R \). If \( f \notin I[x] \) then \( fs \in I[x] \subseteq \text{nil}(R[x]) \) for all non-zero \( s \in I \). Thus \( R \) is weak McCoy.

(2) Let \( R \) be a ring with \( N_+(R) \neq 0 \). Since \( 0 \neq N_+(R)[x] = N_+(R[x]) \subseteq \text{nil}(R[x]) \), \( R \) is weak McCoy by (1).

(3) Since \( \text{Mat}_n(R) \) is non-semiprime, hence by (1) \( \text{Mat}_n(R) \) is weak McCoy.

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**References**


