On Weak McCoy Rings

E. Hashemi : Shahrood University of Technology

Abstract

In this note we introduce the notion of weak McCoy rings as a generalization of McCoy rings, and investigate their properties. Also we show that, if $R$ is a semi-commutative ring, then $R$ is weak McCoy if and only if $R[x]$ is weak McCoy.

1. Introduction

Throughout this paper, all rings are associative with identity. For a commutative ring $R$, McCoy [10] obtained the following result: If $f(x)g(x) = 0$ for some non-zero polynomials $f(x), g(x) \in R[x]$, then $f(x)c = 0$ for some non-zero $c \in R$. According to Nielsen [12], a ring $R$ is called right McCoy whenever polynomials $f(x), g(x) \in R[x] - \{0\}$ satisfy $f(x)g(x) = 0$, there exists a non-zero $r \in R$ such that $f(x)r = 0$. Left McCoy rings are defined similarly. If a ring is both left and right McCoy, we say that the ring is a McCoy ring. It is well known that commutative rings are always McCoy rings [10], but it is not true for non-commutative rings (see [12]).

Recall that a ring $R$ is called:

- reduced if $a^2 = 0 \Rightarrow a = 0$, for all $a \in R$,
- reversible if $ab = 0 \Rightarrow ba = 0$, for all $a, b \in R$,
- symmetric if $abc = 0 \Rightarrow acb = 0$, for all $a, b, c \in R$,
- semi-commutative if $ab = 0 \Rightarrow aRb = 0$, for all $a, b \in R$.

The following implications hold:

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eb_hashemi@yahoo.com
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reduced ⇒ symmetric ⇒ reversible ⇒ semi-commutative.

Reversible rings are McCoy rings by [12]. But the converse is not true; there exists a non-reversible McCoy ring (see [12]).

Motivated by the above, as a generalization of McCoy rings, in this paper we introduce the notion of weak McCoy rings and investigate their properties and extend several known results relating to McCoy rings to a general setting.

For a ring \( R \), we denote by \( \text{nil}(R) \) the set of all nilpotent elements of \( R \), by \( N_s(R) \) the prime radical of \( R \) and by \( M_n(R) \), \( U_n(R) \) and \( L_n(R) \) the \( n \times n \) matrix ring over \( R \), the \( n \times n \) upper and lower triangular matrix rings over \( R \) respectively.

2. On Weak McCoy rings

**Definition 2.1.** We say \( R \) is a weak McCoy ring if \( f(x)g(x) \in \text{nil}(R[x]) \) implies \( f(x)c \in \text{nil}(R[x]) \), for some non-zero \( c \in R \), where \( f(x) \) and \( g(x) \) are non-zero polynomials in \( R[x] \).

**Remark 2.2.** Since \( ab \) is nilpotent if and only if \( ba \) is nilpotent in a ring, hence the definition of weak McCoy rings is left-right symmetric.

**Proposition 2.3.** McCoy rings are weak McCoy.

**Proof.** Let \( R \) be a McCoy ring and \( f(x)g(x) \in \text{nil}(R[x]) \) for non-zero polynomials \( f(x), g(x) \in R[x] \). Then there exists \( m, n \geq 1 \), such that \( (f(x)g(x))^n = (g(x)f(x))^m = 0 \), and \( (f(x)g(x))^{n-1}, (g(x)f(x))^{m-1} \neq 0 \). If \( f(x)g(x) = 0 \) or \( g(x)f(x) = 0 \), then the result follows from the definition of McCoy rings. Assume \( f(x)g(x) \neq 0 \neq g(x)f(x) \) and \( 0 = (f(x)g(x))^n = f(x)(g(x)f(x) \ldots f(x)g(x)) = f(x)h(x) \).

If \( h(x) = g(x)f(x) \ldots f(x)g(x) \neq 0 \), then \( f(x)c = 0 \) for some non-zero \( c \in R \), since \( R \) is McCoy.

Let \( h(x) = g(x)(f(x)g(x) \ldots f(x)g(x)) = g(x)(f(x)g(x))^{n-1} = 0 \). Since \( (f(x)g(x))^{n-1} \neq 0 \) and \( R \) is McCoy, there exists \( 0 \neq d \in R \) such that \( g(x)d = 0 \). Therefore \( f(x)c = 0 \) or...
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\[ g(x)d = 0 \text{ for some non-zero } c, d \in R. \text{ Hence } f(x)c \in \text{nil}(R[x]) \text{ or } dg(x) \in \text{nil}(R[x]) \]

for some non-zero \( c, d \in R \). Therefore \( R \) is weak McCoy.

**Proposition 2.4.** Let \( R \) be a ring. Then \( U_n(R) \) and \( L_n(R) \) are weak McCoy for each \( n \geq 2 \).

**Proof.** Clearly \( U_n(R)[x] \cong U_n(R[x]) \) and for each \( A = \begin{bmatrix} 0 & f_{12} & \cdots & f_{1n} \\ 0 & 0 & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in U_n(R[x]), \)

\[ A^n = 0. \]

Let \( 0 \neq A = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ 0 & f_{22} & \cdots & f_{2n} \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & f_{nn} \end{bmatrix} \in U_n(R[x]). \]

Then \[ A^n = 0. \]

\[ U_n(R) \text{ is weak McCoy. By a similar argument one can show that } L_n(R) \text{ is weak McCoy.} \]

**Proposition 2.5.** Let \( R \) and \( S \) be rings and \( R \cdot M \cdot S \) a bimodule. Then \( \begin{bmatrix} R & M \\ 0 & S \end{bmatrix} \) is a weak McCoy ring.

**Proof.** Similarly, as used in Proposition 2.4 one can prove it.

The following example shows that \( U_n(R) \) and \( M_n(R) \) are neither left nor right McCoy for some \( n \geq 2 \).

**Example 2.6.** Let \( R \) be a ring. We show that \( U_4(R) \) and \( M_4(R) \) are neither right nor left McCoy.

Let \( f(x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \) and
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\[ g(x) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x \in U_4(R)[x] \subseteq M_4(R)[x]. \] Then \( f(x)g(x) = 0. \)

If \( f(x)A = 0 \), for some \( A = [a_{ij}] \in M_4(R) \), then \( 0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \). Hence \( A = 0 \) and \( U_4(R) \)

and \( M_4(R) \) are not right McCoy. If \( Bg(x) = 0 \) for some \( B \in M_4(R) \), then by a similar way as above, we can show \( B = 0 \). Therefore \( U_4(R) \) and \( M_4(R) \) are not left McCoy.

Definition 2.7. A ring \( R \) is called right Ore if given \( Rba \in R \) with \( b \) regular there exist \( R_{ba} \in R \) such that \( ba_{ab} = 1 \). It is well-known that \( R \) is a right Ore ring if and only if the classical right quotient ring of \( R \) exists. We use \( C(R) \) to denote the set of all regular elements in \( R \).

Theorem 2.8. Let \( R \) be a right Ore ring with its classical right quotient ring \( Q \). If \( R \) is weak McCoy then \( Q \) is weak McCoy.

Proof. Let \( 0 \neq F(x) = \sum_{i=0}^{m} a_i x^i \) and \( 0 \neq G(x) = \sum_{j=0}^{n} b_j x^j \) with \( a_i, b_j \in R, u, v \in C(R) \) such that \( F(x)G(x) \in \text{nil}(Q[x]) \).

Case 1. \( F(x)G(x) = 0 \) or \( G(x)F(x) = 0 \). Assume that \( F(x)G(x) = 0 \). Since \( R \) is right Ore, there exists \( b_j \in R \) and \( u_i \in C(R) \) such that \( u^{-1}b_j = b_j u^{-1} \) for \( j = 1, \ldots, n \). Let \( f(x) = \sum_{i=0}^{m} a_i x^i \) and \( g(x) = \sum_{j=0}^{n} b_j x^j \). Then \( f(x)g(x) = 0 \). Since \( R \) is weak McCoy, there exists \( 0 \neq c \in R \) with \( f(x)c \in \text{nil}(R[x]) \subseteq \text{nil}(Q[x]) \). Hence \( F(x)uc = f(x)u^{-1}uc = f(x)c \in \text{nil}(Q[x]) \).

If \( G(x)F(x) = 0 \), then by a similar argument we can show that \( G(x)v \in \text{nil}(Q[x]) \) for some non-zero \( d \in R \).

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Case 2. $F(x)G(x) \neq 0$ and $G(x)F(x) \neq 0$. Since $F(x)G(x) \in \text{nil}(Q[x])$, there exists $n \geq 2$ such that $(F(x)G(x))^n = 0$ and $(F(x)G(x))^{n-1} \neq 0$. Let $(F(x)G(x))^n = F(x)H(x)$. If $H(x) \neq 0$, then by a similar argument as above there exists $\alpha \in C(R)$, $r \in R$ such that $F(x)\alpha r \in \text{nil}(Q[x])$. Now assume $H(x) = G(x)F(x)G(x)\ldots F(x)G(x) = 0$. Since $(F(x)G(x))^{n-1} \neq 0$ and $R$ is weak McCoy, then by Case 1, there exists $\beta \in C(R)$, $s \in R$ such that $G(x)\beta s = 0$. Therefore $Q$ is weak McCoy.

According to Bell [2], a ring $R$ is called semi-commutative if $ab = 0$ implies $aRb = 0$. We say an ideal $I$ is a semi-commutative ideal, if $RI/\text{nil}(R)$ is a semi-commutative ring.

**Lemma 2.9.** Let $R$ be a semi-commutative ring. If $c_1c_2\cdots c_k = 0$ for some $c_i \in R$, then $c_1Rc_2\cdots Rc_k = 0$.

**Proof.** By induction, let $c_k = c_k^{k-1}c_1$. Then $c_1c_2\cdots c_k = 0$ and by induction assumption, we have $0 = c_1Rc_2\cdots Rc_{k-1} = c_1Rc_2\cdots Rc_{k-1}c_k$. Hence, for all $x \in c_1Rc_2\cdots Rc_{k-1}$, we have $xc_k = 0$. It follows by hypothesis that $xRc_k = 0$. Thus $c_1Rc_2\cdots Rc_k = 0$, as desired.

**Lemma 2.10** (4, Lemma 2.5). Let $R$ be a semi-commutative ring. Then $\text{nil}(R)$ is a semi-commutative ideal of $R$.

**Proof.** Let $a, b \in \text{nil}(R)$. Then $a^n = 0 = b^n$ for some $m, n \geq 0$. Each term of the expansion of $(a+b)^{m+n}$ has the form $x = (a^ib^j)\cdots (a^{i_{m+n+1}}b^{j_{m+n+1}})$ where $i, j \in \mathbb{N} \cup \{0\}$. Since $(i_1 + i_2 + \cdots + i_{m+n+1}) = \sum_{i=1}^{m} i_j = m + n + 1$, either $\sum_{i=1}^{m} i_j \geq n$ or $\sum_{j=1}^{n} j_i \geq m$. If $\sum_{i=1}^{m} i_j \geq n$, then $a^i_1a^i_2\cdots a^i_{m+n+1} = 0$. Thus $(a^h b^j)\cdots (a^{i_{m+n+1}}b^{j_{m+n+1}}) = 0$, by Lemma 2.9. If $\sum_{i=1}^{m} i_j < n$, then $\sum_{j=1}^{n} j_i \geq m$. Thus $b^j b^j \cdots b^{j_{m+n+1}} = 0$ and so $(a^h b^j)\cdots (a^{i_{m+n+1}}b^{j_{m+n+1}}) = 0$, by Lemma 2.9. Hence $(a+b)^{m+n+1} = 0$.

Now suppose that $a^n = 0$ and $r \in R$. Then $(ar)^n = 0 = (ra)^n$, by Lemma 2.9. Thus $\text{nil}(R)$ is an ideal of $R$. 

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Since $R/\text{nil}(R)$ is a reduced ring, hence it is a semi-commutative ring. Therefore $\text{nil}(R)$ is a semi-commutative ideal of $R$.

**Lemma 2.11.** Let $R$ be a semi-commutative ring. Then $\text{nil}(R[x]) = \text{nil}(R)[x]$.

**Proof.** Let $f(x) = a_0 + \ldots + a_n x^n \in \text{nil}(R[x])$. Then $f(x)^k = 0$, for some integer $k \geq 0$. Hence $a_n^k = 0$, and that $a_n \in \text{nil}(R)$. There exists $g(x), h(x) \in R[x]$ such that $f(x)^k = (a_0 + \ldots + a_{n-1} x^{n-1})^k + a_n g(x) + h(x) a_n$. Since $\text{nil}(R)[x]$ is an ideal of $R[x]$ and $a_n g(x), h(x) a_n, f(x)^k \in \text{nil}(R)[x]$, we have $(a_0 + \ldots + a_{n-1} x^{n-1})^k \in \text{nil}(R)[x]$. Hence $a_{n-1}^k \in \text{nil}(R)$ and that $a_{n-1} \in \text{nil}(R)$. Continuing this process yields $a_0, \ldots, a_n \in \text{nil}(R)$. Therefore $\text{nil}(R[x]) \subseteq \text{nil}(R)[x]$.

Now, let $f(x) = a_0 + \ldots + a_n x^n \in \text{nil}(R)[x]$. Then $a_i^m = 0$, for some positive integer $m_i$. Let $k = m_0 + \ldots + m_n + 1$. Then $(f(x))^k = \sum (a_0^{i_0} (a_i x)^{i_1} \ldots (a_n x^n)^{i_n}) \cdot \ldots \cdot (a_0^{i_0} (a_n x^n)^{i_n})$, where $i_0 + \ldots + i_r = 1$, for $r = 1, \ldots, k$ and $0 \leq i_r \leq 1$. Each coefficient of $f(x)^k$ is a sum of such elements $\gamma = (a_0^{i_0} \ldots (a_i x)^{i_1} \ldots (a_n x^n)^{i_n})$, where $i_0 + \ldots + i_r = 1$.

It can be easily checked that there exists $a_i \in \{a_0, \ldots, a_n\}$ such that $i_0 + \ldots + i_k \geq m_i$. Since $a_i^m = 0$ and $R$ is semi-commutative, $\gamma = 0$. Thus $(f(x))^k = 0$ and $\text{nil}(R)[x] \subseteq \text{nil}(R[x])$. Therefore $\text{nil}(R[x]) = \text{nil}(R)[x]$.

**Lemma 2.12.** Let $R$ be a semi-commutative ring. Then $\text{nil}(R[x][y]) = \text{nil}(R[x])[y]$.

**Proof.** By Lemma 2.11, $\text{nil}(R[x])$ is an ideal of $R[x]$. Since $R[x]/\text{nil}(R[x])$ is a reduced ring, hence $\text{nil}(R[x])$ is a semi-commutative ideal of $R[x]$, and that $\text{nil}(R[x])[y] \subseteq \text{nil}(R[x][y])$.

Now, let $F(y) = \sum_{i=0}^m f_i y^i \in \text{nil}(R[x][y])$, where $f_i = \sum_{s=0}^p a_x x^s \in R[x]$. Then $F(y)^n = 0$, for some positive integers $n$. As in the proof of [1], let $k = n \sum \deg f_i$, where the degree is as polynomial in $x$ and the degree of zero polynomial is taken to be 0. Then $(F(x^k))^n = 0$ and the set of coefficients of $F(x^k)$ is equal to the set of all coefficients of $f_i$, $0 \leq i \leq m$. Hence by Lemma 2.11, $a_{ij} \in \text{nil}(R)$ for all $i, j$ and that $f_i \in \text{nil}(R[x])$, for each $i$. Thus $F(y) \in \text{nil}(R[x][y])$. Therefore $\text{nil}(R[x][y]) = \text{nil}(R[x])[y]$.
If $R$ is semi-commutative, then $R[x]$ may not be semi-commutative, by [5, Example 2]. Here we will show that if $R$ is semi-commutative, then $R$ is weak McCoy if and only if $R[x]$ is weak McCoy.

**Theorem 2.13.** If $R$ is a semi-commutative ring, then $R[x]$ is a weak McCoy ring if and only if $R$ is weak McCoy.

**Proof.** Suppose that $R$ is a weak McCoy ring. Let $F(t) = \sum_{i=0}^{m} f_it^i$, $G(t) = \sum_{j=0}^{n} g_it^j$ be non-zero polynomials in $R[x][t]$ such that $F(t)G(t) \in \text{nil}(R[x][t])$, where $f_i = \sum_{x=0}^{p} a_x x^i$, $g_j = \sum_{x=0}^{q} b_x x^j \in R[x]$. As in the proof of [1], let $k = \sum \deg f_i + \sum \deg g_j$, where the degree is as polynomial in $x$ and the degree of zero polynomial is taken to be 0. Then $F(x^k) = \sum_{i=0}^{m} f_i x^{ik}, G(x^k) = \sum_{j=0}^{n} g_j x^{jk} \in R[x]$, and the set of coefficients of the $F(x^k)$ is (respectively $G(x^k)$) equal to the set of all coefficients of $f_i$, $0 \leq i \leq m$ (respectively $g_j$, $0 \leq j \leq n$). Since $(F(t)G(t))^p = 0$, for some $p \geq 1$, and $x$ commutes with elements of $R$, $(F(x^k)G(x^k))^p = 0$. Since $R$ is weak McCoy, there is $0 \neq r \in R$ such that $F(x^k)r \in \text{nil}(R[x])$ and $a_x r \in \text{nil}(R)$, $f_i r \in \text{nil}(R[x])$ for $0 \leq i \leq m$, $0 \leq s \leq p$, by Lemma 2.11. Hence $F(t)r \in \text{nil}(R[x][t])$, by Lemma 2.12. Therefore $R[x]$ is weak McCoy.

Now suppose $R[x]$ is a weak McCoy ring and $f(t)g(t) \in \text{nil}(R[t]) \subseteq \text{nil}(R[x][t])$. Since $R[x]$ is weak McCoy, there exists $0 \neq h(x) \in R[x]$ such that $f(t)h(x) \in \text{nil}(R[x][t])$. Let $h(x) = a_0 + \ldots + a_n x^n \in R[x]$ ($a_0 \neq 0$). Then $f(t)a_0 \in \text{nil}(R[t])$, since $(f(t)h(x))^k = (f(t)a_0)^k + k_1 x + \ldots + k_m x^m$ with $k_1, \ldots, k_m \in R[t]$. Therefore $R$ is weak McCoy.

**Theorem 2.14.** Let $R$ be a ring and $\Delta$ a multiplicatively closed subset of $R$ consisting of central regular elements. Then $R$ is weak McCoy if and only if $\Delta^{-1}R$ is weak McCoy.
Proof. If \( R \) is is a weak McCoy ring, then by a similar way as used in Theorem 2.8, one can show that \( \Delta^{-1}R \) is weak McCoy.

Conversely, let \( \Delta^{-1}R \) be a weak McCoy ring. Let \( f(x) = \sum_{j=0}^{m} a_j x^j \) and \( g(x) = \sum_{j=0}^{n} b_j x^j \) be non-zero polynomials of \( R[x] \) such that \( f(x)g(x) \in \text{nil}(R[x]) \). Since \( \Delta^{-1}R \) is weak McCoy, \( f(x)(c\alpha^{-1}) \in \text{nil}((\Delta^{-1}R)[x]) \) for some non-zero \( c\alpha^{-1} \in \Delta^{-1}R \). Thus \( f(x)c \in \text{nil}(R[x]) \) and \( R \) is weak McCoy.

**Corollary 2.15.** Let \( R \) be a ring. Then \( R[x] \) is weak McCoy if and only if \( R[x,x^{-1}] \) is weak McCoy.

**Proof.** Clearly \( \Delta = \{1, x, x^2, \ldots, \} \) is a multiplicatively closed subset of \( R[x] \) consisting of central regular elements and \( \Delta^{-1}R[x] = R[x,x^{-1}] \). Hence the proof follows from Theorem 2.14.

**Theorem 2.16.** The classes of weak McCoy rings are closed under direct limits.

**Proof.** Let \( A = \{R_i, \alpha_{ij} \} \) be a direct system of weak McCoy rings \( R_i \) for \( i \in I \) and ring homomorphisms \( \alpha_{ij} : R_i \to R_j \) for each \( i \leq j \) with \( \alpha_{ii}(1) = 1 \), where \( I \) is a directed partially ordered set. Let \( R = \varinjlim R_i \) be the direct limit of \( A \) with \( \ell_i : R_i \to R \) and \( \ell_j \alpha_{ij} = \ell_i \). We show that \( R \) is weak McCoy ring. Let \( a, b \in R \). Then \( a = \ell_i(a_i), b = \ell_j(b_j) \) for some \( i, j \in I \) and there is \( k \in I \) such that \( i \leq k, j \leq k \).

Define \( a + b = \ell_k(\alpha_{ik}(a_i) + \alpha_{jk}(b_j)) \) and \( ab = \ell_k(\alpha_{ik}(a_i)\alpha_{jk}(b_j)) \), where \( \alpha_{ik}(a_i) \), \( \alpha_{jk}(b_j) \in R_k \). Then \( R \) forms a ring with \( 0 = \ell_i(0) \) and \( 1 = \ell_i(1) \). Let \( f, g \in R[x] \) be non-zero polynomials such that \( fg \in \text{nil}(R[x]) \). There is \( k \in I \) such that \( f, g \in R_k[x] \). Hence \( fg \in \text{nil}(R_k[x]) \). Since \( R_k \) is weak McCoy, there exists \( 0 \neq c_k \in R_k \) such that \( fc_k \in \text{nil}(R_k[x]) \). If \( c = \ell_k(c_k) \), then \( fc \in \text{nil}(R[x]) \) with non-zero \( c \). Therefore \( R \) is weak McCoy.

**Proposition 2.17.** (1) Let \( R \) be a ring. If there exists a non-zero ideal \( I \) of \( R \) such that \( I[x] \subseteq \text{nil}(R[x]) \), then \( R \) is weak McCoy.
(2) Every non-semiprime ring is weak McCoy.

(3) Let $R$ be a ring with a non-zero nilpotent ideal. Then $\text{Mat}_n(R)$ $(n \geq 2)$ is weak McCoy.

**Proof.** (1) Let $0 \neq f \in R[x]$. If $f \in I[x]$, then $fr \in \text{nil}(R[x])$ for all $r \in R$. If $f \notin I[x]$ then $fs \in I[x] \subseteq \text{nil}(R[x])$ for all non-zero $s \in I$. Thus $R$ is weak McCoy.

(2) Let $R$ be a ring with $N_+(R) \neq 0$. Since $0 \neq N_+(R)[x] = N_+(R[x]) \subseteq \text{nil}(R[x])$, $R$ is weak McCoy by (1).

(3) Since $\text{Mat}_n(R)$ is non-semiprime, hence by (1) $\text{Mat}_n(R)$ is weak McCoy.

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**References**


