A Complete Efficiency Ranking of Decision Making Units in DEA:
with an Empirical Study

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Abstract

The efficiency measures provided by DEA can be used for ranking Decision Making Units (DMU's), but this ranking cannot be applied to efficient units. Anderson and Peterson have proposed a modified efficiency measure for efficient units which can be used for ranking, but this ranking breaks down in case of units with at least one input of zero. This paper proposes an alternative efficiency measure that removes this problem. The model is illustrated by an application to the University for Teacher Education, for which the Anderson - Peterson model was not able to give a ranking for two units, which were ranked successfully by the proposed model.

(Data Envelopment Analysis, Efficiency, Ranking)
Introduction.

Data Envelopment Analysis (DEA) provides a measure of efficiency of a Decision Making Unit relative to other such units producing the same outputs with the same inputs. This technique, developed by Charnes, Cooper, and Rhodes (1978), and extended by Banker, Charnes, and Cooper (BCC) (1984), is a linear programming model for an analysis of inputs and outputs. The model does not require prior weights on inputs and outputs.

The standard DEA method assigns an efficiency score of less than one to inefficient DMU's, from which improvement can be derived. However, efficient DMU's receive a score of 1, so that for these units no improvement can be given. A model for ranking efficient DMU's was proposed by Andersen and Petersen (1993). Their model was called Extended-DEA, and used in this study for the University for Teacher Education (UTE). However, this model breaks down inefficient units with at least one zero input.

In this paper, a new definition of efficiency is proposed that can be extended for ranking efficient DMU's. The extended method is applied to data for UTE.

The role of zeros in data has been considered by Charnes, Cooper and Thrall (1991) and Thompson-Dharmpalaha and Thrall (1993) but this paper deals with the problem of ranking the efficient DMU's involving zeros in input data.

The paper unfolds as follows. Section 2 represents the Andersen and Petersen model. Section 3 presents a model based on a definition of efficiency in production possibility set (PPS). In section 4, the two models are compared, using two illustrative examples. Section 5 applies the two models to the UTE data. Summary is given in section 6.

2 The Andersen-Petersen Model.

The standard DEA method assigns an efficiency score of less than one to inefficient units. A score less than one means that a linear combination of the other units could produce at least the same vector of output using a smaller vector of inputs. This score can be used to rank inefficient units. Andersen and Petersen (1993) developed a similar model for ranking efficient DMU's, which in the standard DEA method have a score of 1. The basic idea in their model is to compare the unit under evaluation with a linear combination of all other units, i.e., all units excluding the unit itself. In this case, an efficiency score above 1 is obtained for efficient units. This score reflects the radial distance from the unit under evaluation to the production frontier estimated with the exclusion of that unit, i.e., the maximum proportional increase in inputs producing at least the same outputs.

The Andersen-Petersen model (AP-Model) is identical with the CCR method, except that the unit under evaluation is not included in the combination. Therefore the $p^{th}$ DMU can be evaluated as follows:

$$r_p^* = \min_{r} \left[ r - \varepsilon \left[ \sum_{i=1}^{m} s_i + \sum_{r=1}^{s_r} s'_r \right] \right]$$

subject to:

$$\sum_{j=1}^{n} \lambda_j X_{ij} + s_i = r X_{ip}, \quad i = 1, \ldots, m,$$

$$\sum_{j=1}^{n} \lambda_j Y_{rj} - s'_r = Y_{rp}, \quad r = 1, \ldots, s,$$

$$\lambda_j, s_i, s'_r \geq 0, \quad \forall j, i, r,$$

where $Y_{rj}$ is the $r^{th}$ output and $X_{ij}$ is the $i^{th}$ input for the $j^{th}$ DMU, $r_p$ is a scalar defining the share of $p^{th}$ DMU input vector which is required in order to produce the output vector of $p^{th}$ DMU, $\lambda_j$ denotes the intensity of the $j^{th}$ DMU, and $\varepsilon$ is an non-Archimedean infinitesimal.
3 Efficiency Analysis by an Alternative Measure.

There are \( n \) DMU's to be evaluated, each consumes varying amounts of \( m \) different inputs to produce \( s \) different outputs.

In the model formulation, \( X_p \) and \( Y_p \) denote, respectively, the nonnegative vectors of input and output values for DMU\(_p\).

**Definition.** The production possibility set (PFS) \( T \) is the set \( \{(X_t, Y_t) \mid \text{the outputs } Y_t \text{ can be produced with the inputs } X_t \} \).

The set of \( n \) DMU's of actual production possibility \( (X_j, Y_j), j = 1, \ldots, n \) is considered. Our focus is on the empirically defined production possibility set \( T \) with constant returns assumption that is specified by the following four postulates:

- **Postulate 1** (Ray Unboundedness). If \( (X_t, Y_t) \in T \) then \( (\lambda X_t, \lambda Y_t) \in T \) for all \( \lambda \geq 0 \).

- **Postulate 2** (Convexity). If \( (X_t, Y_t) \in T \) and \( (X_u, Y_u) \in T \), then \( (\lambda X_t + (1 - \lambda)X_u, \lambda Y_t + (1 - \lambda)Y_u) \in T \) for all \( \lambda \in [0, 1] \).

- **Postulate 3** (Monotonicity). If \( (X_t, Y_t) \in T \) and \( X_u \geq X_t, Y_u \leq Y_t \) then \( (X_u, Y_u) \in T \).

- **Postulate 4** (Inclusion of Observations). The observed \( (X_j, Y_j) \in T \) for all \( j = 1, \ldots, n \).

- **Postulate 5** (Minimum extrapolation). If a production possibility set \( T' \) satisfies Postulates 1, 2, 3, and, 4 then \( T \subseteq T' \).

The unique production possibility set with constant returns assumption determined by the above postulates is given by:

\[
T = \{(X_t, Y_t) \mid X_t \geq \sum_{j=1}^{n} \lambda_j X_j, Y_t \leq \sum_{j=1}^{n} \lambda_j Y_j, \\
\lambda_j \geq 0, \ j = 1, \ldots, n \}.
\]

The boundary of this convex set consists of a straight line, plane, or hyperplane through the origin, as \( T \) is a convex cone that contains all of DMU's, see Figure 1 for the simplest case of single input and single output.

![Production Possibility Set](https://via.placeholder.com/150)

**Figure 1: Production Possibility Set.**

For efficiency evaluation relative to the set \( T \), we have the following two linear programming problems:

\[
\begin{align*}
r_p^* &= \min r_p \\
&\text{subject to:} \\
(r_p X_p, Y_p) &\in T,
\end{align*}
\]

\[
\begin{align*}
w_p &= \min w_p \\
&\text{subject to:} \\
(X_p + w_p e, Y_p) &\in T,
\end{align*}
\]

which give the CCR-Model and our formulation respectively as follows:

\[
\begin{align*}
r_p^* &= \min r_p \\
&\text{subject to:} \\
\sum_{j=1}^{n} \lambda_j X_j &\leq r_p X_p, \\
\sum_{j=1}^{n} \lambda_j Y_j &\geq Y_p, \\
\lambda_j &\geq 0, \ j = 1, \ldots, n.
\end{align*}
\]
dependent upon the units of measurement of input data, \( X_j \), \( j = 1, \ldots, n \). However, it is possible to obtain unit independence by normalization, as discussed later.

The unique production possibility set with variable returns assumption determined by postulates 2, 3, 4, and 5 is given by:

\[
T = \{ (X_i, Y_i) \mid X_i \geq \sum_{j=1}^n \lambda_j X_j, \ Y_i \leq \sum_{j=1}^n \lambda_j Y_j, \ \sum_{j=1}^n \lambda_j = 1, \ \lambda_j \geq 0, \ \lambda_j \geq 0, \ \lambda_j \geq 0, \ j = 1, \ldots, n \}.
\]

A discussion similar to the constant returns assumption leads to the BCC-Model and our second formulation, and extension of our second formulation for ranking efficient units can be as follow:

\[
x_p^* = \min x_p - \epsilon \left[ \sum_{i=1}^m s_i + \sum_{r=1}^s s_r' \right]
\]

subject to:

\[
\begin{align*}
\sum_{j \in P} \lambda_j x_{ij} + s_i &= x_{ip} + x_{ip}^* \quad i = 1, \ldots, m, \\
\sum_{j \notin P} \lambda_j y_{ij} - s_i' &= y_{ip} - y_{ip}^* \quad r = 1, \ldots, s,
\end{align*}
\]

where \( s_i, s_i' \geq 0, \ \forall j, i, r \).

4 The Comparison of the Two Models.

Two models for ranking the efficient DMU's were discussed in section 2 and 3. Section 2 represented the AP-Model and section 3 represented the JAM-Model. This section compares these two models using two illustrative examples.

In an actual set of data, it is possible that one or more of the data inputs and outputs are zero. It is also possible that some data inputs and outputs are small in comparison with other inputs and outputs. In these cases, AP-Model, can not correctly evaluate
the efficiency of the DMU's. If the DMU under evaluation has at least one input equal to zero, the AP-Model will be infeasible and if the DMU has at least one input which is small in comparison with other inputs, AP-Model will measure this DMU without stability.

The measure given by JAM-Model successfully evaluates the above cases, so that meaningful scores are obtained for all data.

In order to make usual scores, the scores in JAM-Model may be rescaled from \([-1, +1]\) to \([0\%, 200\%]\) so that of 0 is rescaled to 100%. A score less than 100% means that the corresponding DMU is inefficient and greater than or equal to 100% means that the corresponding DMU is efficient.

### 4.1 Illustrative Example 1:

Table 1 gives an example of the above cases.

<table>
<thead>
<tr>
<th></th>
<th>A₁</th>
<th>A₂</th>
<th>A₃</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>input1</td>
<td>2</td>
<td>0</td>
<td>.1</td>
<td>5</td>
<td>10</td>
<td>10</td>
<td>2</td>
</tr>
<tr>
<td>input2</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>5</td>
<td>4</td>
<td>6</td>
<td>12</td>
</tr>
<tr>
<td>output1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>output2</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 1: Comparison Test Data.

There are 5 DMU's (A, B, C, D and E) each consume two inputs to produce two outputs with constant returns assumptions. In order to remove the effects of changes in units of measurement from the input data, the data must be normalized before applying the method. It can be done by dividing inputs data by the maximum input (for each input).

In this example DMUₐ₁, DMUₐ₂ and DMUₐ₃ are compared with all other DMU's (B, C, D and E) in the following three paragraphs by CCR-Model, AP-Model and JAM-Model.

- DMUₐ₁ is evaluated to be efficient by CCR-Model, and is evaluated as efficient by AP-Model with efficiency score equal to 147%. It is evaluated as efficient by JAM-Model with efficiency score equal to +0.276, which rescales to 100(1 + 0.276) = 127.6%. In this case, there is no problem.

- Consider now DMUₐ₁, which has an input equal to zero. DMUₐ₂ is evaluated to be efficient by CCR-Model, but it can not be evaluated by AP-Model. However, it is evaluated as efficient by JAM-Model with efficiency score equal to +0.310 which rescales to 131.0%.

- Consider now DMUₐ₃, which has an input equal to 0.1. DMUₐ₄ is evaluated as efficient by CCR-Model, and can be evaluated as efficient by AP-Model with efficiency score equal to 200% which is unstable. It is evaluated as efficient by JAM-Model with efficiency score equal to +0.309 which rescales to 130.9%.

### 4.2 Illustrative Example 2:

A comparison of these two procedures for ranking DMU's is illustrated on the Farrell frontier. Consider the DMU's of Figure 2, each produces one output using two inputs with constant returns assumption. DMUₐ₅ is efficient and it can be evaluated by AP-Model with efficiency score of \((100\% \times C)\) and evaluated by JAM-Model with efficiency score \(w_{C} \) which rescales to \(100(1 + w_{C})\).
DMU_A has small value for input 1
(see Figure 4):

In this figure, AP-Model evaluates DMU_A with efficiency score of $(100 \frac{O_A}{O_A})\%$ that is much greater than 100%, where is unstable, but this DMU is evaluated with efficiency score of $100(1 + w_A)\%$ by JAM-Model.

DMU_A and DMU_B are similar units that have small values for input 1 (see Figure 5):

Figure 2: Farrell Efficiency Measurements.

DMU_A has zero for input 1 (see Figure 3):

Figure 3: DMU_A has zero for input 1.

This figure AP-Model evaluates DMU_A with efficiency score of $(100 \frac{O_A}{O_A})\%$ that is so but this DMU is evaluated with efficiency score of $100(1 + w_A)\%$ by JAM-Model.
In this figure, AP-Model evaluates $DMU_A$ with efficiency score of much greater than 100%, while $DMU_B$ is evaluated with efficiency score of about 100%, and it is obvious that these results are unstable, but these DMU’s are evaluated with efficiency scores of $100(1 + w_A)^\%$ and about 100% by JAM-Model, respectively.

- $DMU_A$ has zero for input 1 and $DMU_B$ has small value for input 1 (see Figure 6):

![Figure 6: DMU_A has zero for input 1 and DMU_B has small value for input 1.](image)

AP-Model evaluates $DMU_A$ with efficiency score of $(100 \frac{OA'}{OA})^\%$ that is $\infty$ while $DMU_B$ is evaluated with efficiency score of about 100% and unstepability is observed, but these DMU’s are evaluated with efficiency scores of $100(1 + w_A)^\%$ and 100% by JAM-Model, respectively.

5 An Empirical Study.

In Jahanshahloo and Alierezace (1995), the evaluation of teaching in the UTE was considered. Teaching inputs were expressed in teacher hours and classified in terms of two outputs, course enrollment in undergraduate and graduate studies (see appendix).

Table 2 gives the AP-Model results, efficiency scores and reference sets, with two inputs and two outputs. Six units were found to be efficient. The AP-Model assigns infinity values to the Department of Women’s Physical Education, the 9th DMU, and Institute of Mathematics, the 19th DMU, which are indicated by an asterisk.

The academic units at the UTE may be evaluated using JAM-Model. Table 3 gives the efficiency scores and reference sets obtained by this model. The ranking is approximately the same as that of Table 2, except that DMU’s 9 and 19 now have an explicit ranking.

Extended model, JAM-Model, successfully evaluated all efficient academic units at the UTE, in contrast to AP-Model.

In the application of AP-Model and JAM-Model on real data-set of the UTE, computational DEA issues of Ali (1994), Ali (1993), and Ali and Seiford (1993) have been considered.

6 Summary.

If an efficient DMU has at least one input equal to zero the Andersen-Petersen model gives an infinite result for this DMU, and if the DMU has at least one input with small value in comparison with other inputs, this model measures this DMU without stability. These cases are successfully evaluated and ranked by the new model proposed in this paper.

Computational difficulties of the Andersen-Petersen model were observed in evaluating efficient academic units at the UTE. This model could not evaluate the Department of Women’s Physical Education and the Institute of Mathematics. These academic units were successfully evaluated by the new model.
<table>
<thead>
<tr>
<th>DMU</th>
<th>Eff.</th>
<th>Ref. Sets</th>
<th>( (\epsilon = 0.33 \times 10^{-6}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>*</td>
<td></td>
<td>( \lambda_2 = 0.492 \quad \lambda_7 = 1.173 \quad \lambda_{10} = 0.114 )</td>
</tr>
<tr>
<td>19</td>
<td>*</td>
<td></td>
<td>( \lambda_2 = 0.938 \quad \lambda_{10} = 2.064 )</td>
</tr>
<tr>
<td>2</td>
<td>174%</td>
<td>( \lambda_2 = 0.492 \quad \lambda_7 = 0.479 \quad \lambda_{10} = 0.353 )</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>133%</td>
<td>( \lambda_1 = 1.060 \quad \lambda_2 = 0.603 )</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>130%</td>
<td>( \lambda_1 = 0.585 \quad \lambda_2 = 0.073 )</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>115%</td>
<td>( \lambda_2 = 0.375 \quad \lambda_5 = 0.091 \quad \lambda_{10} = 0.338 )</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>97%</td>
<td>( \lambda_2 = 0.978 \quad \lambda_5 = 0.191 \quad \lambda_{10} = 0.186 )</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>96%</td>
<td>( \lambda_1 = 0.542 \quad \lambda_2 = 0.156 )</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>95%</td>
<td>( \lambda_1 = 0.564 \quad \lambda_2 = 0.285 \quad \lambda_{10} = 0.392 )</td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>89%</td>
<td>( \lambda_1 = 0.131 \quad \lambda_2 = 0.274 )</td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>85%</td>
<td>( \lambda_1 = 0.231 \quad \lambda_2 = 0.582 )</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>71%</td>
<td>( \lambda_1 = 0.726 \quad \lambda_2 = 0.232 )</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>66%</td>
<td>( \lambda_1 = 1.210 \quad \lambda_2 = 0.099 )</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>63%</td>
<td>( \lambda_1 = 0.048 \quad \lambda_2 = 0.528 \quad \lambda_{10} = 0.504 )</td>
<td></td>
</tr>
</tbody>
</table>

Table 2: AP-Model Efficiency Scores for 19 Academic Units of the UTE.
<table>
<thead>
<tr>
<th>DMU</th>
<th>Eff.</th>
<th>Rescaled</th>
<th>Ref. Sets</th>
<th>$\epsilon = 0.55 \times 10^{-6}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>19</td>
<td>+0.281</td>
<td>128%</td>
<td>$\lambda_{15} = 0.579$</td>
<td>$\lambda_{17} = 0.850$</td>
</tr>
<tr>
<td>5</td>
<td>+0.104</td>
<td>110%</td>
<td>$\lambda_{2} = 0.033$</td>
<td>$\lambda_{6} = 0.831$</td>
</tr>
<tr>
<td>2</td>
<td>+0.092</td>
<td>169%</td>
<td>$\lambda_{2} = 0.575$</td>
<td>$\lambda_{7} = 0.047$</td>
</tr>
<tr>
<td>15</td>
<td>+0.065</td>
<td>166%</td>
<td>$\lambda_{1} = 0.938$</td>
<td>$\lambda_{19} = 0.094$</td>
</tr>
<tr>
<td>1</td>
<td>+0.047</td>
<td>105%</td>
<td>$\lambda_{2} = 0.575$</td>
<td>$\lambda_{19} = 1.491$</td>
</tr>
<tr>
<td>9</td>
<td>+0.043</td>
<td>104%</td>
<td>$\lambda_{2} = 0.576$</td>
<td>$\lambda_{10} = 0.177$</td>
</tr>
<tr>
<td>8</td>
<td>-0.010</td>
<td>99%</td>
<td>$\lambda_{2} = 0.228$</td>
<td>$\lambda_{8} = 0.648$</td>
</tr>
<tr>
<td>3</td>
<td>-0.011</td>
<td>99%</td>
<td>$\lambda_{1} = 0.590$</td>
<td>$\lambda_{9} = 0.701$</td>
</tr>
<tr>
<td>7</td>
<td>-0.020</td>
<td>98%</td>
<td>$\lambda_{2} = 0.310$</td>
<td>$\lambda_{9} = 0.428$</td>
</tr>
<tr>
<td>10</td>
<td>-0.021</td>
<td>98%</td>
<td>$\lambda_{1} = 0.655$</td>
<td>$\lambda_{2} = 0.609$</td>
</tr>
<tr>
<td>17</td>
<td>-0.022</td>
<td>98%</td>
<td>$\lambda_{2} = 0.376$</td>
<td>$\lambda_{6} = 0.091$</td>
</tr>
<tr>
<td>18</td>
<td>-0.051</td>
<td>95%</td>
<td>$\lambda_{2} = 0.870$</td>
<td>$\lambda_{5} = 0.265$</td>
</tr>
<tr>
<td>6</td>
<td>-0.070</td>
<td>93%</td>
<td>$\lambda_{1} = 0.044$</td>
<td>$\lambda_{19} = 0.094$</td>
</tr>
<tr>
<td>4</td>
<td>-0.118</td>
<td>88%</td>
<td>$\lambda_{1} = 0.561$</td>
<td>$\lambda_{2} = 0.128$</td>
</tr>
<tr>
<td>16</td>
<td>-0.141</td>
<td>86%</td>
<td>$\lambda_{1} = 0.007$</td>
<td>$\lambda_{2} = 0.903$</td>
</tr>
<tr>
<td>12</td>
<td>-0.153</td>
<td>85%</td>
<td>$\lambda_{1} = 0.717$</td>
<td>$\lambda_{15} = 0.049$</td>
</tr>
<tr>
<td>14</td>
<td>-0.235</td>
<td>77%</td>
<td>$\lambda_{1} = 0.752$</td>
<td>$\lambda_{19} = 0.278$</td>
</tr>
<tr>
<td>11</td>
<td>-0.250</td>
<td>75%</td>
<td>$\lambda_{1} = 0.379$</td>
<td>$\lambda_{2} = 0.054$</td>
</tr>
<tr>
<td>13</td>
<td>-0.457</td>
<td>54%</td>
<td>$\lambda_{1} = 1.151$</td>
<td>$\lambda_{15} = 0.137$</td>
</tr>
</tbody>
</table>

Table 3: JAM-Model Efficiency Scores for 19 Academic Units of the UTE.
Useful comments from Dr. D. S. Hearn, Professor of Economics, the University of Victoria, Canada, Dr. R. M. Thrall, Professor of Administration, Jones Graduate School of Business, Rice University, and anonymous referee are gratefully acknowledged.


<table>
<thead>
<tr>
<th>No.</th>
<th>Department/Institute</th>
<th>I1</th>
<th>I2</th>
<th>O1</th>
<th>O2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Faculty of Literature</td>
<td>81.0</td>
<td>87.6</td>
<td>5191</td>
<td>205</td>
</tr>
<tr>
<td>2</td>
<td>Persian Literature</td>
<td>85.0</td>
<td>12.8</td>
<td>3629</td>
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<tr>
<td>3</td>
<td>Theology and Islamic Culture</td>
<td>56.7</td>
<td>55.2</td>
<td>3302</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>History</td>
<td>91.0</td>
<td>78.8</td>
<td>3379</td>
<td>8</td>
</tr>
<tr>
<td>5</td>
<td>Geography</td>
<td>216.0</td>
<td>72.0</td>
<td>5368</td>
<td>639</td>
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<td>6</td>
<td>Foreign Languages</td>
<td>58.0</td>
<td>25.6</td>
<td>1674</td>
<td>0</td>
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<td>Arabic Language and Literature</td>
<td>112.2</td>
<td>8.8</td>
<td>2350</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>Faculty of Physical Ed.</td>
<td>203.2</td>
<td>52.0</td>
<td>6315</td>
<td>414</td>
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<td>0</td>
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<td>105.2</td>
<td>7689</td>
<td>66</td>
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<td>127.0</td>
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APPENDIX: Inputs and Outputs for 19 Academic Units of the UTE in the First Semester, 1993-94.
References:

Galois Theory, Joseph Rotman, Publication 1990.


شکل زیر میدانهای $Q(\sqrt{n}, \sqrt{n}, \sqrt{1})$.

شکل زیرگروه‌های $Gal_Q(f)$.
برهان: بنابر حکمی که از قضیه 2 اگر \( \gamma \) آنگاه \( \gamma = \sqrt{p} + \sqrt{q} + \sqrt{t} \) \( \in \mathbb{Q}(\gamma) \) که \( \mathbb{Q}(\gamma) \) دو چندجمله‌ای مینیمال \( \gamma \) و در نتیجه چندجمله‌ای مینیمال \( \gamma \) روی \( \mathbb{Q} \) از درجه 8 می‌باشد. لذا اگر \( \deg(h(x)) \geq 8 \) آنگاه \( \gamma = h(\gamma) \). چندجمله‌ای ناگرفته باشد به فرمی که

\[
\begin{align*}
\gamma & = \sqrt{p} + \sqrt{q} + \sqrt{t}, \\
\gamma & = \sqrt{p} - \sqrt{q}, \\
\gamma & = p + q + t, \\
\end{align*}
\]

بنابراین 7 صفر \( f(x) \) است. از این که چندجمله‌ای مینیمال \( \gamma \) روی \( \mathbb{Q} \) تحول پایدار است. چگونه در غیراین صورت 7 صفر یک چندجمله‌ای از درجه کوچکتر از 8 می‌باشد که به نظر می‌رسد

با فرض \( 0 \leq i \leq 15 \), \( K_i = \phi(H_i) \) خواهیم داشت

\[
\begin{align*}
K_i & = \mathbb{Q}(\sqrt{p}, \sqrt{q}, \sqrt{t}), \\
K_{\frac{1}{2}} & = \mathbb{Q}(\sqrt{q}, \sqrt{t}), \\
K_{\frac{1}{3}} & = \mathbb{Q}(\sqrt{p}, \sqrt{t}), \\
K_{\frac{1}{4}} & = \mathbb{Q}(\sqrt{p}, \sqrt{q}), \\
K_{\frac{1}{5}} & = \mathbb{Q}(\sqrt{q}, \sqrt{p}), \\
K_{\frac{1}{6}} & = \mathbb{Q}(\sqrt{p}, \sqrt{q}), \\
K_{\frac{1}{7}} & = \mathbb{Q}(\sqrt{p}, \sqrt{t}), \\
K_{\frac{1}{8}} & = \mathbb{Q}(\sqrt{p}, \sqrt{q}, \sqrt{t}).
\end{align*}
\]

در صفحه بعد شکل زیرگرهای \( \mathbb{Q}(\gamma) \) و شکل زیرگرهای \( \gamma \) را دیده می‌شود.
\[ [Q(\sqrt{p}, \sqrt{q}, \sqrt{t}) : Q] = |\text{Gal}_Q(f)| = 8. \]

لذا (در این زیرگروه به شرح زیر است:

\[ H_0 = \{ \sigma_0 \}, \quad H_1 = \{ \sigma_0, \sigma_1 \}, \quad H_2 = \{ \sigma_0, \sigma_1, \sigma_2 \} \]
\[ H_3 = \{ \sigma_0, \sigma_1, \sigma_2, \sigma_3 \}, \quad H_4 = \{ \sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4 \} \]
\[ H_5 = \{ \sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5 \}, \quad H_6 = \{ \sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6 \} \]
\[ H_7 = \{ \sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7 \}, \quad H_8 = \{ \sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8 \} \]
\[ H_9 = \text{Gal}_Q(f). \]

\[ B = \{ \mathcal{K} Q \leq K \leq Q(\sqrt{p}, \sqrt{q}, \sqrt{t}) \} \quad \text{و} \quad A = \{ H_i \} \quad \text{یا} \quad i \leq 15. \]

\[ \psi : A \longrightarrow B, \quad H_i \sim \phi(H_i) \]

در این صورت \( Q(\alpha) \) زیرگروهی از \( Q(\sqrt{p}, \sqrt{q}, \sqrt{t}) \) است و اعداد \( c, b, a \) و \( c, b, a \) باشد. \( H_i \) عضوی دارد که \( \alpha = a \sqrt{p} + b \sqrt{q} + c \sqrt{t} \) را را روي خودش \( \psi \) را \( \phi(H_i) \) به \( \phi(H_i) \) منتقل نماید. بنابراین اگر \( Q(\alpha) \neq \phi(H_i) \) (\( 1 \leq i \leq 15 \)) در نتیجه \( Q(\alpha) : Q = [Q(\sqrt{p}, \sqrt{q}, \sqrt{t}) : Q] = 8. \]

\[ g(x) \in Q[x] \] یک چندجمله‌ای ناصع روي \( Q \) از درجه \( 8 \) می‌باشد. در نتیجه اگر \( g(\alpha) \) باشد به فرمی که \( g(\alpha) = \pm g(\alpha) \) تغییری در \( g(x) \) دارد.

\[ \deg(g(x)) \geq 8 \quad \text{و} \quad \deg(g(\alpha)) = 1 \]

\[ f(x) = x^8 \pm (p + q + t)x^7 + [(p + q + t)x^6 + \cdots + \pm g(x) \pm g_4(t)]x^5 \]
\[ - \pm (p + q + t)(p^4 + q^4 + t^4 - 4pq - 4pt - 4qt) + \pm g_4(t) \pm g_4(t)], \]

روی \( Q \) تحویل ناشی از دست.
برهان:

\[ Q(i, \sqrt{m}) = Q(i + \sqrt{m}) = Q \]

بنابراین جنگله‌ای مایه‌ی می‌تواند اگر \( x \in Q[x] \) باشد، در نتیجه اگر \( x + \sqrt{m} \) ریوی \( Q \) از درجه یک باشد، در نتیجه \( \deg(s(x)) \geq 1 \) آنگاه \( s(x) = 0 \) باشد. البته دقیق‌کن یکا \( \alpha \) باشد.

\[ \alpha = i + \sqrt{m}, \]

\[ \alpha^2 = -1 + m + 2i\sqrt{m}, \]

\[ \alpha^3 + (1 - m)^2 + 2(1 - m)\alpha = -4m, \]

\[ \alpha^4 + 2(1 - m)\alpha^2 + (m + 1)^2 = 0. \]

بنابراین \( \alpha \) صفر جنگله‌ای (می‌باشد. در نتیجه \( Q \) هر \( t(x) \) تجهیز این اسیر چه در غیر ایرانی‌ها

\[ f = (x^2 - p)(x^2 - q)(x^2 - t) \] یک جنگله‌ای از درجه حداکثر \( 3 \) باشد. البته دقیق‌کن یکا \( \alpha \) باشد.

\[ Q(\sqrt{p}, \sqrt{q}, \sqrt{t})/Q \] یک توزیع میدان تجزیه‌ای روهای \( f \) است. 

\[ \{a + b\sqrt{p} + c\sqrt{q} + d\sqrt{t} \mid a, b, c, d \in Q \} , \]

\[ Q(\sqrt{p}, \sqrt{q}, \sqrt{t})/Q \]

\[ \{a, b, c, d \} . \]

اگر \( q \in Q \) 

\[ \sigma(q) = q \] 

\[ \sigma(\sqrt{p}) = \pm \sqrt{p} \]

\[ \sigma(\sqrt{q}) = \pm \sqrt{q} \]

\[ \sigma(\sqrt{t}) = \pm \sqrt{t} \]

\[ \sigma(\sqrt{p}) = \pm \sqrt{p} \]

\[ \sigma(\sqrt{q}) = \pm \sqrt{q} \]

\[ \sigma(\sqrt{t}) = \pm \sqrt{t} \]

\[ \sigma(\sqrt{p}) = \pm \sqrt{p} \]

\[ \sigma(\sqrt{q}) = \pm \sqrt{q} \]

\[ \sigma(\sqrt{t}) = \pm \sqrt{t} \]

\[ \sigma(\sqrt{p}) = \pm \sqrt{p} \]

\[ \sigma(\sqrt{q}) = \pm \sqrt{q} \]

\[ \sigma(\sqrt{t}) = \pm \sqrt{t} \]

\[ \sigma(\sqrt{p}) = \pm \sqrt{p} \]

\[ \sigma(\sqrt{q}) = \pm \sqrt{q} \]

\[ \sigma(\sqrt{t}) = \pm \sqrt{t} \]
برهان: چون $a$ به هیچکاری از ریشه‌های $Q(\sqrt[p]{q})$, $Q(\sqrt[q]{p})$, $Q(\sqrt[p]{q})$ تعلق ندارد، و $Q(\sqrt[p]{q}) = Q(\sqrt[q]{p})$ است که با هیچکاری از ریشه‌های گروهی $Q(\sqrt[p]{q})$ باید نیستد.

قضیه 2: اگر $a$ عدد صحیح و ثابت و خالی از مریخ باشد به قسمی که $a \not\equiv q \pmod{p}$ و $a \not\equiv p \pmod{q}$، نیم‌تکرار باشد اگر $Q(\sqrt[p]{q})$ و $Q(\sqrt[q]{p})$ باشند.

برهان: با فرض داشته باشیم $g(x) = Q(\sqrt[p]{q})$. بنابراین $\alpha = a\sqrt[p]{q} + b\sqrt[q]{p}$. بنابراین $Q(\alpha) = Q(\sqrt[p]{q})$. در تئوری $Q(\sqrt[p]{q})$ می‌توانیم $Q$ را از درجه 2 می‌باشد. اگر $\deg(g(x)) \geq 4$ باشد، $g(x) \in Q[x]$ یک چندجمله‌ای ناصفر باشد به قسمی که

$$\deg(g(x)) \geq 4 
\alpha, g(\alpha) = \alpha.$$

(2)

نتیجه یکم: اگر $q$, $p$ عدد خالی از مریخ باشد به قسمی که $Q$ رخي که $Q$ تئوری $Q(\sqrt[p]{q})$ و $Q(\sqrt[q]{p})$ باشند.

نتیجه دوم: اگر $q$, $p$ عدد دیگری باشند به قسمی که $Q$ تئوری $Q$ باشد.

برهان: با فرض داشته باشیم $\alpha = 1$. در تئوری $Q(\sqrt[p]{q})$ باشد.

قضیه 3: اگر $m$ عدد صحیح و ثابت و مریخ کامل باشد به قسمی که $Q$ رخید دارد که $Q$ رخید.

(3)

نتیجه چهارم: اگر $Q$ رخید باشد به قسمی که $\alpha = b$.
یک زیرمیدان $E$ می‌باشد که شامل $F$ است. به عکس اگر $E$ زیرمیدانی از $F$ شامل باشد، در اینصورت $\{\sigma \in \text{Gal}_F(f) | \forall k \in K(\sigma(k) = k)\}$

$$A = \{K | K\text{ است }F \text{ که شامل } K\},$$

$$B = \{H | H\text{ است }\text{Gal}(f)\}.$$

آنگاه تابع $\psi : B \to A$ با ضریب $\psi$ پیک‌تک، $\psi(H) = \phi(H)(P)$، که در صورت که $q \neq p$ و $q \neq p$, $\phi$ که توزیع بر می‌یابد.

$$Q(\sqrt{p}, \sqrt{q}) = [a_r + a_0 \sqrt{p} + a_1 \sqrt{q} + \cdots + a_n \sqrt{pq} | a_i \in Q]$$

$$|\text{Gal}_Q(f)| = [Q(\sqrt{p}, \sqrt{q}) : Q] = 4.$$

با فرض $Q(\sqrt{p}, \sqrt{q})$ که در عضو $Q(R)$ را نام بگذاریم. $\phi$ جذب کرده‌اند $\phi$ می‌تواند $Q(\sqrt{p}, \sqrt{q})$ کافی است.

$$\begin{align*}
\sqrt{p} & \to \sqrt{p} \\
\sqrt{q} & \to -\sqrt{q}
\end{align*}$$

بنابراین $Q(\sqrt{p}, \sqrt{q})$ خاصیت $\phi$ می‌باشد.

$$\begin{align*}
\phi(a) = Q(\sqrt{p}, \sqrt{q})
\phi(H_r) = Q(\sqrt{p}, \sqrt{q})
\phi(H_t) = Q(\sqrt{p}, \sqrt{q})
\phi(H_t) = Q(\sqrt{pq})
H_r = \{p, q \neq p, q \neq p \}.
\end{align*}$$

در نتیجه $Q(\sqrt{p}, \sqrt{q})$ متناظر $H$ ها با برای $\phi$ که نام $Q(\sqrt{p}, \sqrt{q})$ نیز می‌باشد به شرح زیر می‌باشد.

$$Q(\sqrt{p}, \sqrt{q}) = Q(\sqrt{p}, \sqrt{q})$$

$$\phi(H_r) = Q(\sqrt{p}, \sqrt{q})$$

$$\phi(H_t) = Q(\sqrt{p}, \sqrt{q})$$

$$\phi(H_t) = Q(\sqrt{pq})$$

در اینجا $
\phi(Q(\sqrt{p}, \sqrt{q})) = Q(\sqrt{pq})$. اگر $a, b$ عدد صحیح باشند، $Q(\sqrt{p}, \sqrt{q})$ که نام $Q(\sqrt{p}, \sqrt{q})$ آنگاه $\alpha = a \sqrt{p} + b \sqrt{q}$