A Complete Efficiency Ranking of Decision Making Units in DEA: with an Empirical Study

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Abstract

The efficiency measures provided by DEA can be used for ranking Decision Making Units (DMU's), but this ranking cannot be applied to efficient units. Anderson and Peterson have proposed a modified efficiency measure for efficient units which can be used for ranking, but this ranking breaks down in case of units with at least one input of zero. This paper proposes an alternative efficiency measure that removes this problem. The model is illustrated by an application to the University for Teacher Education, for which the Anderson - Peterson model was not able to give a ranking for two units, which were ranked successfully by the proposed model.

(Data Envelopment Analysis, Efficiency, Ranking)
2 The Andersen-Petersen Model.

The standard DEA method assigns an efficiency score of less than one to inefficient DMU's, from which inefficiency can be derived. However, efficient DMU's were assigned an efficiency of 1, so that for these units no score can be given. A model for ranking efficient DMU's was proposed by Andersen and Petersen (1993). Their model was called Extended-DEA, and used in this study for the University for Teacher Education (UTE). However, this model breaks down for efficient units with at least one zero input.

In this paper, a new definition of efficiency is proposed that can be extended for ranking efficient units. The extended method is applied to data for UTE.

The role of zeros in data has been considered by Charnes, Cooper and Thrall (1991) and Thomps-Dharmaphala, and Thrall (1993) but this paper deals with the problem of ranking the efficient units involving zeros in input data.

The paper unfolds as follows. Section 2 represents the Andersen and Petersen model. Section 3 presents a model based on a definition of efficiency in a payoff possibility set (PPS). In section 4, the two models are compared, using two illustrative examples. Section 5 applies the two models to the UTE data. Summary is given in section 6.
3 Efficiency Analysis by an Alternative Measure.

There are $n$ DMU's to be evaluated, each consumes varying amounts of $m$ different inputs to produce $s$ different outputs.

In the model formulation, $X_i$ and $Y_i$ denote, respectively, the nonnegative vectors of input and output values for DMU$_i$.

**Definition.** The production possibility set (PPS) $T$ is the set $\{(X_i, Y_i) | \text{the outputs } Y_i \text{ can be produced with the inputs } X_i\}$.

The set of $n$ DMU's of actual production possibility $(X_j, Y_j)$, $j = 1, \ldots, n$ is considered. Our focus is on the empirically defined production possibility set $T$ with constant returns assumption that is specified by the following four postulates:

- **Postulate 1 (Ray Unboundedness).** If $(X_i, Y_i) \in T$ then $(\lambda X_i, \lambda Y_i) \in T$ for all $\lambda \geq 0$.
- **Postulate 2 (Convexity).** If $(X_i, Y_i) \in T$ and $(X_u, Y_u) \in T$, then $(\lambda X_i + (1 - \lambda)X_u, \lambda Y_i + (1 - \lambda)Y_u) \in T$ for all $\lambda \in [0, 1]$.
- **Postulate 3 (Monotonicity).** If $(X_i, Y_i) \in T$ and $X_u \geq X_i$, $Y_u \leq Y_i$ then $(X_u, Y_u) \in T$.
- **Postulate 4 (Inclusion of Observations).** The observed $(X_j, Y_j) \in T$ for all $j = 1, \ldots, n$.
- **Postulate 5 (Minimum Extrapolation).** If a production possibility set $T'$ satisfies Postulates 1, 2, 3, and, 4 then $T \subset T'$.

The unique production possibility set with constant returns assumption determined by the above postulates is given by:

$$T = \{(X_i, Y_i) | X_i \geq \sum_{j=1}^{n} \lambda_j X_j, Y_i \leq \sum_{j=1}^{n} \lambda_j Y_j, \lambda_j \geq 0, \ j = 1, \ldots, n\}.$$

The boundary of this convex set consists of a straight line, plane, or hyperplane through the origin, as $T$ is a convex cone that contains all of DMU's, see Figure 1 for the simplest case of single input and single output.

![Figure 1: Production Possibility Set.](image)

For efficiency evaluation relative to the set $T$, we have the following two linear programming problems:

- **CCR-Model**
  $$r_p^* = \min r_p \text{ subject to:}$$
  $$\sum_{j=1}^{m} \lambda_j X_j \leq r_p X_p, \quad \sum_{j=1}^{n} \lambda_j Y_j \geq Y_p, \quad \lambda_j \geq 0, \ j = 1, \ldots, n.$$
dependent upon the units of measurement of input data, $X_j$, $j = 1, \ldots, n$. However, it is possible to obtain unit independence by normalization, as discussed later.

The unique production possibility set with variable returns assumption determined by postulates 2, 3, 4, and 5 is given by:

$$T = \{(X_i, Y_i) \mid X_i \geq \sum_{j=1}^{n} \lambda_j X_j, \ Y_i \leq \sum_{j=1}^{n} \lambda_j Y_j, \ n \geq 1, \lambda_j \geq 0, \ j = 1, \ldots, n\}.$$  

A discussion similar to the constant returns assumption leads to the BCC-Model and our second formulation, and extension of our second formulation for ranking efficient units can be as follows:

$$x_p^* = \min x_p - \epsilon \left[ \sum_{i=1}^{m} s_i + \sum_{r=1}^{s} s_r' \right]$$

subject to:

$$\sum_{j \in P} \lambda_j X_{ij} + s_i = X_{ip} + w_p, \ i = 1, \ldots, m,$$

$$\sum_{j \in P} \lambda_j Y_{ij} - s_r' = Y_{ip}, \ r = 1, \ldots, s,$$

$$\lambda_j, s_i, s_r' \geq 0, \ \forall j, \ i, \ r.$$

4 The Comparison of the Two Models.

Two models for ranking the efficient DMU's were discussed in section 2 and 3. Section 2 represented the AP-Model and section 3 represented the JAM-Model. This section compares these two models using two illustrative examples.

In an actual set of data, it is possible that one or more of the data inputs and outputs are zero. It is also possible that some data inputs and outputs are small in comparison with other inputs and outputs. In these cases, AP-Model, can not correctly evaluate
the efficiency of the DMU's. If the DMU under evaluation has at least one input equal to zero, the AP-Model will be infeasible and if the DMU has at least one input which is small in comparison with other inputs, AP-Model will measure this DMU without stability.

The measure given by JAM-Model successfully evaluates the above cases, so that meaningful scores are obtained for all data.

In order to make usual scores, the scores in JAM-Model may be rescaled from \([-1, +1]\) to \([0\%, 200\%]\) so that 0 is rescaled to 100%. A score less than 100% means that the corresponding DMU is inefficient and greater than or equal to 100% means that the corresponding DMU is efficient.

### 4.1 Illustrative Example 1:

Table 1 gives an example of the above cases.

<table>
<thead>
<tr>
<th></th>
<th>(A_1)</th>
<th>(A_2)</th>
<th>(A_3)</th>
<th>(B)</th>
<th>(C)</th>
<th>(D)</th>
<th>(E)</th>
</tr>
</thead>
<tbody>
<tr>
<td>input1</td>
<td>2</td>
<td>0.1</td>
<td>5</td>
<td>10</td>
<td>10</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>input2</td>
<td>8</td>
<td>8</td>
<td>4</td>
<td>6</td>
<td>12</td>
<td></td>
<td></td>
</tr>
<tr>
<td>output1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>output2</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Comparison Test Data.

There are 5 DMU's (A, B, C, D and E) each consume two inputs to produce two outputs with constant returns assumption. In order to remove the effects of changes in units of measurement from the input data, the data must be normalized before applying the method. It can be done by dividing inputs data by the maximum input (for each input).

In this example DMU\(_{A_1}\), DMU\(_{A_2}\) and DMU\(_{A_3}\) are compared with all other DMU's (B, C, D and E) in the following three paragraphs by CCR-Model, AP-Model and JAM-Model.

- DMU\(_{A_1}\), is evaluated to be efficient by CCR-Model, and is evaluated as efficient by AP-Model with efficiency score equal to 147%. It is evaluated as efficient by JAM-Model with efficiency score equal to +0.276, which rescales to 100(1 + 0.276) = 127.6%. In this case, there is no problem.

- Consider now DMU\(_{A_2}\), which has an input equal to zero. DMU\(_{A_2}\) is evaluated to be efficient by CCR-Model, but it can not be evaluated by AP-Model. However, it is evaluated as efficient by JAM-Model with efficiency score equal to +0.310 which rescales to 131.0%.

- Consider now DMU\(_{A_3}\), which has an input equal to 0.1. DMU\(_{A_3}\) is evaluated as efficient by CCR-Model, and can be evaluated as efficient by AP-Model with efficiency score equal to 2000% which is unstable. It is evaluated as efficient by JAM-Model with efficiency score equal to +0.309 which rescales to 130.9%.

### 4.2 Illustrative Example 2:

A comparison of these two procedures for ranking DMU's is illustrated on the Farrell frontier. Consider the DMU's of Figure 2, each produces one output using two inputs with constant returns assumption. DMU\(_{C}\) is efficient and it can be evaluated by AP-Model with efficiency score of \(100\frac{w_{C}}{\bar{u}_{C}}\) and evaluated by JAM-Model with efficiency score \(w_{C}\) which rescales to \(100\frac{w_{C}}{\bar{u}_{C}}\).
• DMU_A has small value for input 1 (see Figure 4):

In this figure, AP-Model evaluates DMU_A with efficiency score of \(\frac{OA'}{OA}\)% that is much greater than 100%, where is unstable, but this DMU is evaluated with efficiency score of \(100(1 + w_A)\)% by JAM-Model.

• DMU_A and DMU_B are similar units that have small values for input 1 (see Figure 5):
In this figure, AP-Model evaluates DMU_A with efficiency score of much greater than 100%, while DMU_B is evaluated with efficiency score of about 100%, and it is obvious that these results are unstable, but these DMU's are evaluated with efficiency scores of 100(1 + \omega_A)% and about 100% by JAM-Model, respectively.

- DMU_A has zero for input 1 and DMU_B has small value for input 1 (see Figure 6):

![Figure 6: DMU_A has zero for input 1 and DMU_B has small value for input 1.]

AP-Model evaluates DMU_A with efficiency score of (100 - OA')% that is \infty while DMU_B is evaluated with efficiency score of about 100% and instability is observed, but these DMU's are evaluated with efficiency scores of 100(1 + \omega_A)% and 100% by JAM-Model, respectively.

5 An Empirical Study.

In Jahnahshioo and Alirezacee (1995), the evaluation of teaching in the UTE was considered. Teaching inputs were expressed in teacher hours and classified in terms of two outputs, course enrollment in undergraduate and graduate studies (see appendix).

Table 2 gives the AP-Model results, efficiency scores and reference sets, with two inputs and two outputs. Six units were found to be efficient. The AP-Model assigns infinity values to the Department of Women's Physical Education, the 9th DMU, and Institute of Mathematics, the 19th DMU, which are indicated by an asterisk.

The academic units at the UTE may be evaluated using JAM-Model. Table 3 gives the efficiency scores and reference sets obtained by this model. The ranking is approximately the same as that of Table 2, except that DMU's 9 and 19 now have an explicit ranking.

Extended model, JAM-Model, successfully evaluated all efficient academic units at the UTE, in contrast to AP-Model.

In the application of AP-Model and JAM-Model on real data-set of the UTE, computational DEA issues of Ali (1994), Ali (1993), and Ali and Seiford (1993) have been considered.

6 Summary.

If an efficient DMU has at least one input equal to zero the Andersen-Petersen model gives an infinite result for this DMU, and if the DMU has at least one input with small value in comparison with other inputs, this model measures this DMU without stability. These cases are successfully evaluated and ranked by the new model proposed in this paper.

Computational difficulties of the Andersen-Petersen model were observed in evaluating efficient academic units at the UTE. This model could not evaluate the Department of Women's Physical Education and the Institute of Mathematics. These academic units were successfully evaluated by the new model.
<table>
<thead>
<tr>
<th>DMU</th>
<th>Eff.</th>
<th>Ref. Sets</th>
<th>($\epsilon = 0.33 \times 10^{-6}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>*</td>
<td></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>*</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>174%</td>
<td>$\lambda_2 = 0.492$</td>
<td>$\lambda_7 = 1.173$</td>
</tr>
<tr>
<td>15</td>
<td>133%</td>
<td>$\lambda_1 = 0.938$</td>
<td>$\lambda_{19} = 2.064$</td>
</tr>
<tr>
<td>5</td>
<td>130%</td>
<td>$\lambda_8 = 0.956$</td>
<td>$\lambda_{19} = 0.479$</td>
</tr>
<tr>
<td>1</td>
<td>115%</td>
<td>$\lambda_2 = 0.492$</td>
<td>$\lambda_{10} = 0.220$</td>
</tr>
<tr>
<td>8</td>
<td>97%</td>
<td>$\lambda_2 = 0.276$</td>
<td>$\lambda_3 = 0.548$</td>
</tr>
<tr>
<td>10</td>
<td>96%</td>
<td>$\lambda_1 = 1.060$</td>
<td>$\lambda_2 = 0.503$</td>
</tr>
<tr>
<td>3</td>
<td>95%</td>
<td>$\lambda_1 = 0.585$</td>
<td>$\lambda_2 = 0.073$</td>
</tr>
<tr>
<td>17</td>
<td>89%</td>
<td>$\lambda_2 = 0.375$</td>
<td>$\lambda_5 = 0.091$</td>
</tr>
<tr>
<td>18</td>
<td>85%</td>
<td>$\lambda_2 = 0.978$</td>
<td>$\lambda_5 = 0.191$</td>
</tr>
<tr>
<td>7</td>
<td>71%</td>
<td>$\lambda_2 = 0.487$</td>
<td>$\lambda_9 = 0.204$</td>
</tr>
<tr>
<td>12</td>
<td>66%</td>
<td>$\lambda_1 = 0.564$</td>
<td>$\lambda_2 = 0.285$</td>
</tr>
<tr>
<td>4</td>
<td>63%</td>
<td>$\lambda_1 = 0.542$</td>
<td>$\lambda_2 = 0.156$</td>
</tr>
<tr>
<td>6</td>
<td>58%</td>
<td>$\lambda_1 = 0.131$</td>
<td>$\lambda_2 = 0.274$</td>
</tr>
<tr>
<td>16</td>
<td>57%</td>
<td>$\lambda_1 = 0.231$</td>
<td>$\lambda_2 = 0.582$</td>
</tr>
<tr>
<td>14</td>
<td>54%</td>
<td>$\lambda_1 = 0.726$</td>
<td>$\lambda_2 = 0.232$</td>
</tr>
<tr>
<td>13</td>
<td>45%</td>
<td>$\lambda_1 = 1.210$</td>
<td>$\lambda_3 = 0.099$</td>
</tr>
<tr>
<td>11</td>
<td>45%</td>
<td>$\lambda_1 = 0.048$</td>
<td>$\lambda_2 = 0.528$</td>
</tr>
</tbody>
</table>

Table 2: AP-Model Efficiency Scores for 19 Academic Units of the UTE.
<table>
<thead>
<tr>
<th>DMU</th>
<th>Eff.</th>
<th>Rescaled</th>
<th>Ref. Sets</th>
<th>$(\varepsilon = 0.55 \times 10^{-6})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>19</td>
<td>+0.281</td>
<td>128%</td>
<td>$\lambda_5 = 0.579$</td>
<td>$\lambda_{17} = 0.850$ $\lambda_{19} = 0.094$</td>
</tr>
<tr>
<td>5</td>
<td>+0.104</td>
<td>110%</td>
<td>$\lambda_2 = 0.033$</td>
<td>$\lambda_6 = 0.831$ $\lambda_{19} = 0.580$</td>
</tr>
<tr>
<td>2</td>
<td>+0.092</td>
<td>109%</td>
<td>$\lambda_2 = 0.575$</td>
<td>$\lambda_7 = 0.047$ $\lambda_{10} = 0.274$</td>
</tr>
<tr>
<td>15</td>
<td>+0.065</td>
<td>106%</td>
<td>$\lambda_1 = 0.938$</td>
<td>$\lambda_{19} = 1.491$</td>
</tr>
<tr>
<td>1</td>
<td>+0.047</td>
<td>105%</td>
<td>$\lambda_2 = 0.575$</td>
<td>$\lambda_{10} = 0.177$ $\lambda_{15} = 0.358$</td>
</tr>
<tr>
<td>9</td>
<td>-0.043</td>
<td>104%</td>
<td>$\lambda_2 = 0.789$</td>
<td>$\lambda_9 = 0.701$</td>
</tr>
<tr>
<td>8</td>
<td>-0.010</td>
<td>99%</td>
<td>$\lambda_2 = 0.228$</td>
<td>$\lambda_5 = 0.648$ $\lambda_9 = 0.701$</td>
</tr>
<tr>
<td>3</td>
<td>-0.011</td>
<td>99%</td>
<td>$\lambda_1 = 0.590$</td>
<td>$\lambda_2 = 0.066$</td>
</tr>
<tr>
<td>7</td>
<td>-0.020</td>
<td>98%</td>
<td>$\lambda_2 = 0.310$</td>
<td>$\lambda_9 = 0.428$</td>
</tr>
<tr>
<td>10</td>
<td>-0.021</td>
<td>98%</td>
<td>$\lambda_1 = 1.055$</td>
<td>$\lambda_2 = 0.609$</td>
</tr>
<tr>
<td>17</td>
<td>-0.022</td>
<td>98%</td>
<td>$\lambda_2 = 0.376$</td>
<td>$\lambda_5 = 0.091$ $\lambda_{18} = 0.338$</td>
</tr>
<tr>
<td>18</td>
<td>-0.051</td>
<td>95%</td>
<td>$\lambda_2 = 0.870$</td>
<td>$\lambda_5 = 0.265$ $\lambda_{19} = 0.094$</td>
</tr>
<tr>
<td>6</td>
<td>-0.070</td>
<td>93%</td>
<td>$\lambda_1 = 0.044$</td>
<td>$\lambda_2 = 0.398$</td>
</tr>
<tr>
<td>4</td>
<td>-0.118</td>
<td>88%</td>
<td>$\lambda_1 = 0.551$</td>
<td>$\lambda_2 = 0.128$</td>
</tr>
<tr>
<td>16</td>
<td>-0.141</td>
<td>86%</td>
<td>$\lambda_1 = 0.007$</td>
<td>$\lambda_2 = 0.903$ $\lambda_{19} = 0.029$</td>
</tr>
<tr>
<td>12</td>
<td>-0.153</td>
<td>85%</td>
<td>$\lambda_1 = 0.717$</td>
<td>$\lambda_{15} = 0.049$ $\lambda_{19} = 0.278$</td>
</tr>
<tr>
<td>14</td>
<td>-0.235</td>
<td>77%</td>
<td>$\lambda_1 = 0.752$</td>
<td>$\lambda_2 = 0.194$</td>
</tr>
<tr>
<td>11</td>
<td>-0.250</td>
<td>75%</td>
<td>$\lambda_1 = 0.379$</td>
<td>$\lambda_2 = 0.054$ $\lambda_{19} = 0.371$</td>
</tr>
<tr>
<td>13</td>
<td>-0.457</td>
<td>54%</td>
<td>$\lambda_1 = 1.151$</td>
<td>$\lambda_{15} = 0.137$</td>
</tr>
</tbody>
</table>

Table 3: JAM-Model Efficiency Scores for 19 Academic Units of the UTE.


<table>
<thead>
<tr>
<th>No.</th>
<th>Department/Institute</th>
<th>II</th>
<th>I2</th>
<th>O1</th>
<th>O2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Persian Literature</td>
<td>81.0</td>
<td>87.6</td>
<td>5191</td>
<td>205</td>
</tr>
<tr>
<td>2</td>
<td>Theology and Islamic Culture</td>
<td>85.0</td>
<td>12.8</td>
<td>3629</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>History</td>
<td>56.7</td>
<td>55.2</td>
<td>3302</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>Geography</td>
<td>91.0</td>
<td>78.8</td>
<td>3379</td>
<td>8</td>
</tr>
<tr>
<td>5</td>
<td>Foreign Languages</td>
<td>216.0</td>
<td>72.0</td>
<td>5368</td>
<td>639</td>
</tr>
<tr>
<td>6</td>
<td>Arabic Language and Literature</td>
<td>58.0</td>
<td>25.6</td>
<td>1674</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>Social Sciences</td>
<td>112.2</td>
<td>8.8</td>
<td>2350</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>Faculty of Physical Ed.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>Men Physical Education</td>
<td>203.2</td>
<td>52.0</td>
<td>6315</td>
<td>414</td>
</tr>
<tr>
<td>10</td>
<td>Women Physical Education</td>
<td>186.6</td>
<td>0.0</td>
<td>2865</td>
<td>0</td>
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<tr>
<td>11</td>
<td>Mathematics</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>12</td>
<td>Geology</td>
<td>108.7</td>
<td>127.0</td>
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<td>266</td>
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<td>Biology</td>
<td>105.7</td>
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<td>315</td>
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<td>236.8</td>
<td>6643</td>
<td>236</td>
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<td>Physics</td>
<td>146.3</td>
<td>124.0</td>
<td>4611</td>
<td>128</td>
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<td>16</td>
<td>Faculty of Education</td>
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<td></td>
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<td></td>
</tr>
<tr>
<td>17</td>
<td>Foundations of Education</td>
<td>57.0</td>
<td>203.0</td>
<td>4369</td>
<td>540</td>
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<td>18</td>
<td>Instructional Technology</td>
<td>118.7</td>
<td>48.2</td>
<td>3313</td>
<td>16</td>
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<td>19</td>
<td>Psychology</td>
<td>58.0</td>
<td>47.4</td>
<td>1853</td>
<td>230</td>
</tr>
<tr>
<td>19</td>
<td>Guidance and Counseling</td>
<td>146.0</td>
<td>50.8</td>
<td>4578</td>
<td>217</td>
</tr>
</tbody>
</table>

APPENDIX: Inputs and Outputs for 19 Academic Units of the UTE in the First Semester, 1993-94.
References:

Galois Theory, Joseph Rotman, Publication 1990.


شبکه زیر میدان‌های $Q(\sqrt{p}, \sqrt{q}, \sqrt{r})$

شبکه زیرگروه‌های $Gal_Q(f)$
برهان: بنابر بهتکه قبل از پاس 4 اگر $\gamma$ آنگاه $\gamma = \sqrt{p} + \sqrt{q} + \sqrt{t}$.

$[Q(\gamma) : Q] = 8$ در نتیجه $\gamma$ درجه $8$ مستقل. لذا اگر $[Q(x) : Q] = 8$ درجه $8$ مستقل باشد به فرمی که $h(\gamma) = a(x) + b(x)\gamma$.

$$\deg(h(x)) \geq 8$$

$$h(\gamma) = a(x) + b(x)\gamma$$

بنابراین $\gamma$ صفر ($f(x)$) است. از این که $\gamma$ درجه $8$ مستقل باشد نتیجه می‌شود که $\gamma$ صفر ($f(x)$) است. این نتیجه $\gamma$ درجه $8$ مستقل باشد اگر $f(x)$ صفر باشد که $\gamma$ درجه $8$ مستقل باشد که $f(x)$ صفر باشد.

با فرض $(\circ \leq i \leq 15)$، $K_i = \phi(H_i)$، 

\begin{align*}
K_i &= Q(\sqrt{p}, \sqrt{q}, \sqrt{t}), \quad K_{1} = Q(\sqrt{q}, \sqrt{t}), \quad K_{6} = Q(\sqrt{p}, \sqrt{t}), \quad K_{7} = Q(\sqrt{p}, \sqrt{q})
\end{align*}

\begin{align*}
K_{10} &= Q(\sqrt{p}, \sqrt{q}, \sqrt{t}), \quad K_{11} = Q(\sqrt{q}, \sqrt{t}), \quad K_{12} = Q(\sqrt{p}, \sqrt{q}), 
\end{align*}

\begin{align*}
K_{15} &= Q(\sqrt{p}, \sqrt{q}, \sqrt{t}), \quad K_{16} = Q(\sqrt{p}, \sqrt{q}, \sqrt{t}) 
\end{align*}

در صفحه بعد شکله برگونهای $Gal_Q(f)$ و شکله زیرپدیدهای $Q(\sqrt{p}, \sqrt{q}, \sqrt{t})$ را جهت مقایسه نشان دهیم.
لذا (دارای ۶۴ زیرگروه به شرح زیر است):

\[ H_0 = \{ \sigma \}, \quad H_1 = \{ \sigma, \sigma_1 \}, \quad H_2 = \{ \sigma, \sigma_1, \sigma_2 \}, \quad H_3 = \{ \sigma, \sigma_1, \sigma_2, \sigma_3 \} \]

\[ H_4 = \{ \sigma, \sigma_1, \sigma_2, \sigma_3, \sigma_4 \}, \quad H_5 = \{ \sigma, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5 \} \]

\[ H_6 = \{ \sigma, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6 \}, \quad H_7 = \{ \sigma, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7 \} \]

\[ H_8 = \{ \sigma, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8 \} \]

\[ H_9 = \text{Gal}_Q(f). \]

\[ B = \{ K | Q \leq K \leq Q(\sqrt[3]{p}, \sqrt[3]{q}, \sqrt[3]{t}) \} \quad \text{و} \quad A = \{ H_i \mid 0 \leq i \leq 15 \}. \]

\[ \psi : A \rightarrow B, \]

\[ H_i \sim \phi(H_i) \]

\[ Q(\sqrt[3]{p}, \sqrt[3]{q}, \sqrt[3]{t}) \]

در این صورت، \( Q(\alpha) \) مجموعه‌ای از \( Q(p, q, t) \) باشد که \( \alpha = u\sqrt[3]{p} + v\sqrt[3]{q} + w\sqrt[3]{t} \) و \( u, v, w \) اعداد گویایی باشند و \( 1 \leq u, v, w \leq 15 \)

\[ \psi(\alpha) = \phi(\alpha) \quad \text{و} \quad Q(\alpha) = Q(\sqrt[3]{p}, \sqrt[3]{q}, \sqrt[3]{t}) \]

\[ Q(\alpha) \neq \phi(\alpha) \quad \text{در نتیجه} \]

\[ Q(\alpha) : Q = Q(\sqrt[3]{p}, \sqrt[3]{q}, \sqrt[3]{t}) : Q = 8. \]

\[ g(x) \in Q[x] \]

بنابراین گروه‌ها از درجه ۸ باشند، در نتیجه اگر \( Q \) روش گروه‌های از درجه ۸ باشند، در نتیجه اگر \( Q \) باشد به فرمول \( g(\alpha) = \) آنگاه \( g(\alpha) = 8 \)

 تصمیم گیری:

\[ f(x) = x^8 - \chi(p + q + t)x^5 + \chi[(p + q + t)^2 + \chi(p + q + t^2)]x^8 \]

\[ - \chi[(p + q + t)(p^2 + q^2 + t^2) - \chi pq - \chi pt - \chi qt^2] + \chi pq(t^2)x^8 \]

\[ + (p^2 + q^2 + t^2 - \chi pq - \chi pt - \chi qt^2). \]

روی \( Q \) تحویل باقی‌می‌آید.
برهان: \[ Q(i, \sqrt{m}) = Q(i + \sqrt{m}) : Q = 4 \]

بنابراین چندجمله‌ای مینیمال \( Q \) ریز دارای \( i + \sqrt{m} \) باشد. در نتیجه اگر \( \deg(s(x)) \geq 4 \) آگاهی \( s(i + \sqrt{m}) = 0 \) باشد به قسمی که \( i + \sqrt{m} \) ریز \( Q \) باشد.

\[ \alpha = i + \sqrt{m}, \]
\[ \alpha^2 = -1 + m + \sqrt{m}, \]
\[ \alpha^2 + (1 - m)\alpha^2 + 2(1 - m)\alpha = -\sqrt{m}, \]
\[ \alpha^3 + (1 - m)\alpha^2 + 2(1 - m)\alpha = -m, \]
\[ \alpha^4 + 2(1 - m)\alpha^3 + (m + 1)\alpha = 0. \]

بنابراین \( \alpha \) ضریب چندجمله‌ای \( Q \) می‌باشد. در نتیجه \( t(x) \) ریز \( Q \) می‌باشد. در نتیجه اگر \( t(x) \) تابعی باشد که در غیر ایرادی

فرنگی کنده \( f(p, q, t) \) و \( t \) و \( p, q \) یک عدد اول دو پردازه است. به درستی \( Q(\sqrt{p}, \sqrt{q}, \sqrt{t})/Q \)

\[ t = \{ a + b\sqrt{p} + c\sqrt{q} + d\sqrt{t} | a, b, c, d \in Q \}, \]

\[ a, b, c, d \in Q \} \).

اگر \( \sigma(q) = q \) برای هر \( q \in Q \).

با بررسی مشخص کردن یک خاکی \( Q(\sqrt{p}, \sqrt{q}, \sqrt{t}) \rightarrow Q(\sqrt{p}, \sqrt{q}, \sqrt{t}) \)

\[ \left\{ \begin{array}{c}
\sqrt{p} \rightarrow \sqrt{p} \\
\sqrt{q} \rightarrow -\sqrt{q} \\
\sqrt{t} \rightarrow \sqrt{t}
\end{array} \right. \]

\[ \sigma_1 : \left\{ \begin{array}{c}
\sqrt{p} \rightarrow \sqrt{p} \\
\sqrt{q} \rightarrow -\sqrt{q} \\
\sqrt{t} \rightarrow \sqrt{t}
\end{array} \right. \]

\[ \sigma_2 : \left\{ \begin{array}{c}
\sqrt{p} \rightarrow -\sqrt{p} \\
\sqrt{q} \rightarrow -\sqrt{q} \\
\sqrt{t} \rightarrow \sqrt{t}
\end{array} \right. \]

\[ \sigma_3 : \left\{ \begin{array}{c}
\sqrt{p} \rightarrow -\sqrt{p} \\
\sqrt{q} \rightarrow \sqrt{q} \\
\sqrt{t} \rightarrow -\sqrt{t}
\end{array} \right. \]

\[ \sigma_4 : \left\{ \begin{array}{c}
\sqrt{p} \rightarrow -\sqrt{p} \\
\sqrt{q} \rightarrow \sqrt{q} \\
\sqrt{t} \rightarrow -\sqrt{t}
\end{array} \right. \]

\[ \sigma_5 : \left\{ \begin{array}{c}
\sqrt{p} \rightarrow \sqrt{p} \\
\sqrt{q} \rightarrow -\sqrt{q} \\
\sqrt{t} \rightarrow -\sqrt{t}
\end{array} \right. \]

\[ \sigma_6 : \left\{ \begin{array}{c}
\sqrt{p} \rightarrow -\sqrt{p} \\
\sqrt{q} \rightarrow -\sqrt{q} \\
\sqrt{t} \rightarrow -\sqrt{t}
\end{array} \right. \]

\[ \sigma_7 : \left\{ \begin{array}{c}
\sqrt{p} \rightarrow -\sqrt{p} \\
\sqrt{q} \rightarrow \sqrt{q} \\
\sqrt{t} \rightarrow -\sqrt{t}
\end{array} \right. \]

\[ \sigma_8 : \left\{ \begin{array}{c}
\sqrt{p} \rightarrow -\sqrt{p} \\
\sqrt{q} \rightarrow -\sqrt{q} \\
\sqrt{t} \rightarrow -\sqrt{t}
\end{array} \right. \]
برهان: چون \( \alpha \) هرهیچی از زیرمدانی از \( Q(\sqrt{pq}) \) است، که \( \alpha \) هرهیچی از این چهار زیرمدان با ابزار نیست بسی \( Q(\sqrt{p}, \sqrt{q}) \) استفاده نمی‌کند.

قضیه ۲: اگر \( p > q \) دو عدد صحیح و مثبت و خالی از مریخ باشد به قسمت که \( b, a \) و \( p, q \) دو عدد صحیح و مثبت و خالی از مریخ باشد به قسمت که \( Q(\sqrt{pq}) \) ناشی‌ترین اگزهای جنگ‌الماشی \( Q(\sqrt{p}, \sqrt{q}) \) باشد.

یک طرف داریم \( Q(\alpha) = Q(\sqrt{p}, \sqrt{q}) \) بنا بر است. \( Q(\alpha) = Q(\sqrt{p}, \sqrt{q}) \) بنا بر است. \( \alpha = a \sqrt{p} + b \sqrt{q} \) از این که \( Q(\sqrt{pq}) \) از \( Q(\sqrt{p}, \sqrt{q}) \) است، چه در غیر این صورت \( Q(\sqrt{pq}) \) از \( Q(\sqrt{p}, \sqrt{q}) \) است، چه در غیر این صورت \( Q(\sqrt{pq}) \) از \( Q(\sqrt{p}, \sqrt{q}) \) است.

نتیجه ۲: اگر \( q, p \) دو عدد خالی از مریخ باشد به قسمت که \( Q(\sqrt{pq}) \) ناشی‌ترین اگزهای جنگ‌الماشی \( Q(\sqrt{p}, \sqrt{q}) \) ناشی‌ترین اگزهای جنگ‌الماشی است.

برهان: با قراردادن \( a = b = 1 \) در قضیه ۲ نتیجه حاصل می‌شود.

نتیجه ۲: اگر \( q, p \) دو عدد طبیعی خالی از مریخ باشد به قسمت که \( Q(\sqrt{pq}) \) ناشی‌ترین اگزهای جنگ‌الماشی \( Q(\sqrt{p}, \sqrt{q}) \) ناشی‌ترین اگزهای جنگ‌الماشی است.

برهان: با قراردادن \( a = b \) در قضیه ۲ نتیجه حاصل می‌شود.

قضیه ۳: اگر \( m \) عددی صحیح و مثبت و مریخ کامل باشد به قسمت که \( m \) عددی اولیه و \( p > q \) عددی اولیه است، \( Q(\sqrt{pq}) \) ناشی‌ترین اگزهای جنگ‌الماشی \( Q(\sqrt{p}, \sqrt{q}) \) ناشی‌ترین اگزهای جنگ‌الماشی است.
\( \phi(H) = Q \),
\( \phi(H) = H \equiv \{a, b\} \),
\( H = \{a, b\} \).

\( \{v_a, v_b\} \)

\( H = \text{Galois}(f) \).

\[ \phi(H) = \{a + a, b + b\} = \{2a, 2b\} \leq Q \]

\( \phi(H) = \text{Galois}(f) \).

\( \text{Gal}(\overline{\mathbb{F}}_p) = \{1, \phi(H)\} \).

\( \phi(H) = \{2a, 2b\} \leq Q \)