A Complete Efficiency Ranking of Decision Making Units in DEA: with an Empirical Study

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Abstract

The efficiency measures provided by DEA can be used for ranking Decision Making Units (DMU's), but this ranking cannot be applied to efficient units. Anderson and Peterson have proposed a modified efficiency measure for efficient units which can be used for ranking, but this ranking breaks down in case of units with at least one input of zero. This paper proposes an alternative efficiency measure that removes this problem. The model is illustrated by an application to the University for Teacher Education, for which the Anderson - Peterson model was not able to give a ranking for two units, which were ranked successfully by the proposed model.

(Data Envelopment Analysis, Efficiency, Ranking)
Introduction.

Envelopment Analysis (DEA) provides a measure of efficiency of a Decision Making Unit relative to other such units producing the same outputs with the same inputs. This technique, developed by Charnes, Cooper, and Rhodes (1978), and extended by Banker, Charnes, and Cooper (BCC) (1984), is a linear programming technique for an analysis of inputs and outputs. The technique does not require prior weights on inputs and outputs.

The standard DEA method assigns an efficiency score of less than one to inefficient DMU's, from which inefficiency can be derived. However, efficient DMU's receive an efficiency of 1, so that for these units no inefficiency can be given. A model for ranking efficient DMU's was proposed by Andersen and Petersen (1993). Their model was called Extended-DEA, and is used in this study for the University for Teacher Education (UTE). However, this model breaks down efficient units with at least on zero input.

In this paper, a new definition of efficiency is proposed that can be extended for ranking efficient DMU's. The extended method is applied to data for UTE.

The role of zeros in data has been considered by Charnes, Cooper and Thrall (1991) and Thong-Dharmaphala, and Thrall (1993) but this paper deals with the problem of ranking the efficient DMU's involving zeros in input data.

The paper unfolds as follows. Section 2 represents the Andersen and Petersen model. Section 3 presents the model based on a definition of efficiency in projection possibility set (PPS). In section 4, the two models are compared, using two illustrative examples. Section 5 applies the two models to the UTE data. A summary is given in section 6.
3 Efficiency Analysis by an Alternative Measure.

There are $n$ DMU's to be evaluated, each consumes varying amounts of $m$ different inputs to produce $s$ different outputs.

In the model formulation, $X_p$ and $Y_p$ denote, respectively, the nonnegative vectors of input and output values for DMU$_p$.

Definition. The production possibility set (FPS) $T$ is the set $\{(X_t, Y_t) | \text{the outputs } Y_t \text{ can be produced with the inputs } X_t\}$.

The set of $n$ DMU's of actual production possibility $(X_j, Y_j)$, $j = 1, \ldots, n$ is considered. Our focus is on the empirically defined production possibility set $T$ with constant returns assumption that is specified by the following four postulates:

- **Postulate 1 (Ray Unboundedness).** If $(X_t, Y_t) \in T$ then $(\lambda X_t, \lambda Y_t) \in T$ for all $\lambda \geq 0$.

- **Postulate 2 (Convexity).** If $(X_t, Y_t) \in T$ and $(X_u, Y_u) \in T$, then $(\lambda X_t + (1 - \lambda)X_u, \lambda Y_t + (1 - \lambda)Y_u) \in T$ for all $\lambda \in [0, 1]$.

- **Postulate 3 (Monotonicity).** If $(X_t, Y_t) \in T$ and $X_u \geq X_t$, $Y_u \leq Y_t$ then $(X_u, Y_u) \in T$.

- **Postulate 4 (Inclusion of Observations).** The observed $(X_j, Y_j) \in T$ for all $j = 1, \ldots, n$.

- **Postulate 5 (Minimum extrapolation).** If a production possibility set $T'$ satisfies Postulates 1, 2, 3, and 4 then $T \subset T'$.

The unique production possibility set with constant returns assumption determined by the above postulates is given by:

$$T = \{(X_t, Y_t) | X_t \geq \sum_{j=1}^{n} \lambda_j X_j, Y_t \leq \sum_{j=1}^{n} \lambda_j Y_j, \lambda_j \geq 0, j = 1, \ldots, n\}.$$ 

The boundary of this convex set consists of a straight line, plane, or hyperplane through the origin, as $T$ is a convex cone that contains all of DMU's, see Figure 1 for the simplest case of single input and single output.

![Figure 1: Production Possibility Set.](image)

For efficiency evaluation relative to the set $T$, we have the following two linear programming problems:

The CCR-Model and our formulation respectively as follows:

$$r_p^* = \min r_p \quad \text{subject to} \quad (r_p X_p, Y_p) \in T,$$

$$w_p^* = \min w_p \quad \text{subject to} \quad (X_p + w_p e, Y_p) \in T,$$

which give the CCR-Model and our formulation respectively as follows:

$$r_p^* = \min r_p \quad \text{subject to} \quad \sum_{j=1}^{n} \lambda_j X_j \leq r_p X_p, \sum_{j=1}^{n} \lambda_j Y_j \geq Y_p, \lambda_j \geq 0, j = 1, \ldots, n.$$
dependent upon the units of measurement of input data, \( X_j, j = 1, \ldots, n \). However, it is possible to obtain unit independence by normalization, as discussed later.

The unique production possibility set with variable returns assumption determined by postulates 2, 3, 4, and 5 is given by:

\[
T = \{(X_i, Y_i) | X_i \geq \sum_{j=1}^{n} \lambda_j X_j, \quad Y_i \leq \sum_{j=1}^{n} \lambda_j Y_j, \quad \sum_{j=1}^{n} \lambda_j = 1, \quad \lambda_j \geq 0, \quad j = 1, \ldots, n \}
\]

A discussion similar to the constant returns assumption leads to the BCC-Model and our second formulation, and extension of our second formulation for ranking efficient units can be as follows:

\[
x_p^* = \min x_p - \epsilon \left[ \sum_{i=1}^{m} s_i^* + \sum_{r=1}^{s} s_r^* \right]
\]

subject to:

\[
\begin{align*}
\sum_{j \in P} \lambda_j X_{ij} + s_i^* &= X_{ip} + w_p, \quad i = 1, \ldots, m, \\
\sum_{j \in P} \lambda_j Y_{ij} - s_r^* &= Y_{rp}, \quad r = 1, \ldots, s, \\
\lambda_j &\geq 0, \quad j = 1, \ldots, n,
\end{align*}
\]

The Comparison of the Two Models.

Two models for ranking the efficient DMU's were discussed in section 2 and 3. Section 2 represented the AP-Model and section 3 represented the JAM-Model. This section compares these two models using two illustrative examples.

In an actual set of data, it is possible that one or more of the data inputs and outputs are zero. It is also possible that some data inputs and outputs are small in comparison with other inputs and outputs.

In these cases, AP-Model, can not correctly evaluate...
the efficiency of the DMU's. If the DMU under evaluation has at least one input equal to zero, the AP-Model will be infeasible and if the DMU has at least one input which is small in comparison with other inputs, AP-Model will measure this DMU without stability.

The measure given by JAM-Model successfully evaluates the above cases, so that meaningful scores are obtained for all data.

In order to make usual scores, the scores in JAM-Model may be rescaled from $[-1, +1]$ to $[0\%, 200\%]$ so that 0 is rescaled to 100%. A score less than 100% means that the corresponding DMU is inefficient and greater than or equal to 100% means that the corresponding DMU is efficient.

4.1 Illustrative Example 1:

Table 1 gives an example of the above cases.

<table>
<thead>
<tr>
<th></th>
<th>A₁</th>
<th>A₂</th>
<th>A₃</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>Input1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>5</td>
<td>10</td>
<td>10</td>
<td>2</td>
</tr>
<tr>
<td>Input2</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>5</td>
<td>4</td>
<td>6</td>
<td>12</td>
</tr>
<tr>
<td>Output1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Output2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 1: Comparison Test Data.

There are 5 DMU's (A, B, C, D and E) each consume two inputs to produce two outputs with constant returns assumption. In order to remove the effects of changes in units of measurement from the input data, the data must be normalized before applying the method. It can be done by dividing inputs data by the maximum input (for each input).

In this example DMUₐ₁, DMUₐ₂ and DMUₐ₃ are compared with all other DMU's (B, C, D and E) in the following three paragraphs by CCR-Model, AP-Model and JAM-Model.

- DMUₐ₁ is evaluated to be efficient by CCR-Model, and is evaluated as efficient by AP-Model with efficiency score equal to 147%. It is evaluated as efficient by JAM-Model with efficiency score equal to +0.276, which rescales to $100(1 + 0.276) = 127.6\%$. In this case, there is no problem.

- Consider now DMUₐ₂, which has an input equal to zero. DMUₐ₂ is evaluated to be efficient by CCR-Model, but it can not be evaluated by AP-Model. However, it is evaluated as efficient by JAM-Model with efficiency score equal to +0.310 which rescales to 131.0%.

- Consider now DMUₐ₃, which has an input equal to 0.1. DMUₐ₃ is evaluated as efficient by CCR-Model, and can be evaluated as efficient by AP-Model with efficiency score equal to 2000% which is unstable. It is evaluated as efficient by JAM-Model with efficiency score equal to +0.309 which rescales to 130.9%.

4.2 Illustrative Example 2:

A comparison of these two procedures for ranking DMU's is illustrated on the Farrell frontier. Consider the DMU's of Figure 2, each produces one output using two inputs with constant returns assumption. DMUₐ is efficient and it can be evaluated by AP-Model with efficiency score of (100%ₐ) and evaluated by JAM-Model with efficiency score $w$ₐ which rescales to $100(1 + w$ₐ).
In this figure, AP-Model evaluates DMU_A with efficiency score of \((100 \cdot \frac{a_1'}{a_1})\)% that is much greater than 100%, where is unstable, but this DMU is evaluated with efficiency score of \(100(1 + w_A)\)% by JAM-Model.

- DMU_A and DMU_B are similar units that have small values for input 1 (see Figure 5):
In this figure, AP-Model evaluates $DMU_A$ with efficiency score of much greater than 100%, while $DMU_B$ is evaluated with efficiency score of about 100%, and it is obvious that these results are unstable, but these $DMU$'s are evaluated with efficiency scores of $100(1 + w_A)\%$ and about 100% by JAM-Model, respectively.

- $DMU_A$ has zero for input 1 and $DMU_B$ has small value for input 1 (see Figure 6):

![Figure 6: $DMU_A$ has zero for input 1 and $DMU_B$ has small value for input 1.](image)

AP-Model evaluates $DMU_A$ with efficiency score of $(100 \frac{OA'}{OA})\%$ that is $\infty$ while $DMU_B$ is evaluated with efficiency score of about 100% and unstability is observed, but these $DMU$'s are evaluated with efficiency scores of $100(1 + w_A)\%$ and 100% by JAM-Model, respectively.

5 An Empirical Study.

In Jannahshahlooo and Alierezaee (1995), the evaluation of teaching in the UTE was considered. Teaching inputs were expressed in teacher hours and classified in terms of two inputs, professorial staff and instructors. Teaching outputs were expressed in student hours and classified in terms of two outputs, course enrollment in undergraduate and graduate studies (see appendix).

Table 2 gives the AP-Model results, efficiency scores and reference sets, with two inputs and two outputs. Six units were found to be efficient. The AP-Model assigns infinity values to the Department of Women's Physical Education, the 9th $DMU$, and Institute of Mathematics, the 19th $DMU$, which are indicated by an asterisk.

The academic units at the UTE may be evaluated using JAM-Model. Table 3 gives the efficiency scores and reference sets obtained by this model. The ranking is approximately the same as that of Table 2, except that $DMU$'s 9 and 19 now have an explicit ranking.

Extended model, JAM-Model, successfully evaluated all efficient academic units at the UTE, in contrast to AP-Model.

In the application of AP-Model and JAM-Model on real data-set of the UTE, computational DEA issues of Ali (1994), Ali (1993), and Ali and Seiford (1993) have been considered.

6 Summary.

If an efficient $DMU$ has at least one input equal to zero the Andersen-Petersen model gives an infinite result for this $DMU$, and if the $DMU$ has at least one input with small value in comparison with other inputs, this model measures this $DMU$ without stability. These cases are successfully evaluated and ranked by the new model proposed in this paper.

Computational difficulties of the Andersen-Petersen model were observed in evaluating efficient academic units at the UTE. This model could not evaluate the Department of Women's Physical Education and the Institute of Mathematics. These academic units were successfully evaluated by the new model.
<table>
<thead>
<tr>
<th>DMU</th>
<th>Eff.</th>
<th>Ref. Sets</th>
<th>((\epsilon = 0.33 \times 10^{-6}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>*</td>
<td></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>*</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>174%</td>
<td>(\lambda_2 = 0.492)</td>
<td>(\lambda_7 = 1.173)</td>
</tr>
<tr>
<td>15</td>
<td>133%</td>
<td>(\lambda_1 = 0.938)</td>
<td>(\lambda_{10} = 2.064)</td>
</tr>
<tr>
<td>5</td>
<td>130%</td>
<td>(\lambda_8 = 0.956)</td>
<td>(\lambda_{10} = 0.479)</td>
</tr>
<tr>
<td>1</td>
<td>115%</td>
<td>(\lambda_2 = 0.492)</td>
<td>(\lambda_{10} = 0.220)</td>
</tr>
<tr>
<td>8</td>
<td>97%</td>
<td>(\lambda_2 = 0.276)</td>
<td>(\lambda_5 = 0.548)</td>
</tr>
<tr>
<td>10</td>
<td>96%</td>
<td>(\lambda_1 = 1.060)</td>
<td>(\lambda_2 = 0.503)</td>
</tr>
<tr>
<td>3</td>
<td>95%</td>
<td>(\lambda_1 = 0.585)</td>
<td>(\lambda_2 = 0.073)</td>
</tr>
<tr>
<td>17</td>
<td>89%</td>
<td>(\lambda_2 = 0.375)</td>
<td>(\lambda_5 = 0.091)</td>
</tr>
<tr>
<td>18</td>
<td>85%</td>
<td>(\lambda_2 = 0.978)</td>
<td>(\lambda_5 = 0.191)</td>
</tr>
<tr>
<td>7</td>
<td>71%</td>
<td>(\lambda_2 = 0.487)</td>
<td>(\lambda_9 = 0.204)</td>
</tr>
<tr>
<td>12</td>
<td>66%</td>
<td>(\lambda_1 = 0.564)</td>
<td>(\lambda_2 = 0.285)</td>
</tr>
<tr>
<td>4</td>
<td>63%</td>
<td>(\lambda_1 = 0.542)</td>
<td>(\lambda_2 = 0.156)</td>
</tr>
<tr>
<td>6</td>
<td>58%</td>
<td>(\lambda_1 = 0.131)</td>
<td>(\lambda_2 = 0.274)</td>
</tr>
<tr>
<td>16</td>
<td>57%</td>
<td>(\lambda_1 = 0.231)</td>
<td>(\lambda_2 = 0.582)</td>
</tr>
<tr>
<td>14</td>
<td>54%</td>
<td>(\lambda_1 = 0.726)</td>
<td>(\lambda_2 = 0.232)</td>
</tr>
<tr>
<td>13</td>
<td>45%</td>
<td>(\lambda_1 = 1.210)</td>
<td>(\lambda_2 = 0.099)</td>
</tr>
<tr>
<td>11</td>
<td>45%</td>
<td>(\lambda_1 = 0.048)</td>
<td>(\lambda_2 = 0.528)</td>
</tr>
</tbody>
</table>

Table 2: AP-Model Efficiency Scores for 19 Academic Units of the UTE.
<table>
<thead>
<tr>
<th>DMU</th>
<th>Eff.</th>
<th>Rescaled</th>
<th>Ref. Sets</th>
<th>( (\epsilon = 0.55 \times 10^{-6}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>19</td>
<td>+0.281</td>
<td>128%</td>
<td>( \lambda_{15} = 0.579 ) ( \lambda_{17} = 0.850 ) ( \lambda_{19} = 0.094 )</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>+0.104</td>
<td>110%</td>
<td>( \lambda_{2} = 0.033 ) ( \lambda_{6} = 0.831 ) ( \lambda_{19} = 0.580 )</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>+0.092</td>
<td>109%</td>
<td>( \lambda_{2} = 0.575 ) ( \lambda_{7} = 0.047 ) ( \lambda_{10} = 0.274 )</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>+0.065</td>
<td>106%</td>
<td>( \lambda_{1} = 0.938 ) ( \lambda_{19} = 1.491 )</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>+0.047</td>
<td>105%</td>
<td>( \lambda_{2} = 0.575 ) ( \lambda_{10} = 0.177 ) ( \lambda_{15} = 0.358 )</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>+0.043</td>
<td>104%</td>
<td>( \lambda_{2} = 0.789 ) ( \lambda_{9} = 0.701 )</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>-0.010</td>
<td>99%</td>
<td>( \lambda_{2} = 0.228 ) ( \lambda_{9} = 0.648 ) ( \lambda_{9} = 0.701 )</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>-0.011</td>
<td>99%</td>
<td>( \lambda_{1} = 0.590 ) ( \lambda_{2} = 0.066 ) ( \lambda_{9} = 0.428 )</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>-0.020</td>
<td>98%</td>
<td>( \lambda_{2} = 0.310 ) ( \lambda_{9} = 0.909 )</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>-0.021</td>
<td>98%</td>
<td>( \lambda_{2} = 1.055 ) ( \lambda_{9} = 1.055 )</td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>-0.022</td>
<td>98%</td>
<td>( \lambda_{2} = 0.376 ) ( \lambda_{6} = 0.091 ) ( \lambda_{18} = 0.338 )</td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>-0.051</td>
<td>95%</td>
<td>( \lambda_{2} = 0.870 ) ( \lambda_{5} = 0.265 ) ( \lambda_{19} = 0.094 )</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>-0.070</td>
<td>93%</td>
<td>( \lambda_{1} = 0.044 ) ( \lambda_{2} = 0.398 )</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>-0.118</td>
<td>88%</td>
<td>( \lambda_{1} = 0.561 ) ( \lambda_{2} = 0.128 )</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>-0.141</td>
<td>86%</td>
<td>( \lambda_{1} = 0.007 ) ( \lambda_{2} = 0.903 ) ( \lambda_{19} = 0.029 )</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>-0.153</td>
<td>85%</td>
<td>( \lambda_{1} = 0.717 ) ( \lambda_{15} = 0.049 ) ( \lambda_{19} = 0.278 )</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>-0.235</td>
<td>77%</td>
<td>( \lambda_{1} = 0.752 ) ( \lambda_{2} = 0.194 )</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>-0.250</td>
<td>75%</td>
<td>( \lambda_{1} = 0.379 ) ( \lambda_{2} = 0.054 ) ( \lambda_{19} = 0.371 )</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>-0.457</td>
<td>54%</td>
<td>( \lambda_{1} = 1.151 ) ( \lambda_{15} = 0.137 )</td>
<td></td>
</tr>
</tbody>
</table>

Table 3: JAM-Model Efficiency Scores for 19 Academic Units of the UTE.


<table>
<thead>
<tr>
<th>No.</th>
<th>Department/Institute</th>
<th>I1</th>
<th>I2</th>
<th>O1</th>
<th>O2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Persian Literature</td>
<td>81.0</td>
<td>87.6</td>
<td>5191</td>
<td>205</td>
</tr>
<tr>
<td>2</td>
<td>Theology and Islamic Culture</td>
<td>85.0</td>
<td>12.8</td>
<td>3629</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>History</td>
<td>56.7</td>
<td>55.2</td>
<td>3302</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>Geography</td>
<td>91.0</td>
<td>78.8</td>
<td>3379</td>
<td>8</td>
</tr>
<tr>
<td>5</td>
<td>Foreign Languages</td>
<td>216.0</td>
<td>72.0</td>
<td>5368</td>
<td>639</td>
</tr>
<tr>
<td>6</td>
<td>Arabic Language and Literature</td>
<td>58.0</td>
<td>25.6</td>
<td>1674</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>Social Sciences</td>
<td>112.2</td>
<td>8.8</td>
<td>2350</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>Faculty of Physical Ed.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>Men Physical Education</td>
<td>203.2</td>
<td>52.0</td>
<td>6315</td>
<td>414</td>
</tr>
<tr>
<td>10</td>
<td>Women Physical Education</td>
<td>186.6</td>
<td>0.0</td>
<td>2865</td>
<td>0</td>
</tr>
<tr>
<td>11</td>
<td>Mathematics</td>
<td>143.4</td>
<td>105.2</td>
<td>7689</td>
<td>66</td>
</tr>
<tr>
<td>12</td>
<td>Geology</td>
<td>108.7</td>
<td>127.0</td>
<td>2165</td>
<td>266</td>
</tr>
<tr>
<td>13</td>
<td>Biology</td>
<td>105.7</td>
<td>134.4</td>
<td>3963</td>
<td>315</td>
</tr>
<tr>
<td>14</td>
<td>Chemistry</td>
<td>235.0</td>
<td>236.8</td>
<td>6643</td>
<td>236</td>
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<tr>
<td>15</td>
<td>Physics</td>
<td>146.3</td>
<td>124.0</td>
<td>4611</td>
<td>128</td>
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<td>16</td>
<td>Institute of Mathematics</td>
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<td></td>
<td></td>
<td></td>
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<td>Foundations of Education</td>
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<td>203.0</td>
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<td>540</td>
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<td>Instructional Technology</td>
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<td>48.2</td>
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<td>16</td>
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<td>19</td>
<td>Psychology</td>
<td>58.0</td>
<td>47.4</td>
<td>1853</td>
<td>230</td>
</tr>
<tr>
<td>20</td>
<td>Guidance and Counseling</td>
<td>146.0</td>
<td>50.8</td>
<td>4578</td>
<td>217</td>
</tr>
</tbody>
</table>

APPENDIX: Inputs and Outputs for 19 Academic Units of the UTE in the First Semester, 1993-94.
References:

Galois Theory, Joseph Rotman, Publication 1990.


شبکه زیر میدان‌های $Q(\sqrt{\eta}, \sqrt{\varepsilon}, \sqrt{i})$

شبکه زیرگروه‌های $Galo_0(j)$
برهان: یک بیانی می‌تواند اگر در قسمتی از مادیره پیدا شود، آنگاه $Q(\gamma) = \sqrt{p} + \sqrt{q} + \sqrt{t}$، بنابراین $\gamma = \sqrt{p} + \sqrt{q} + \sqrt{t}$، بررسی $\gamma = \sqrt{p}$ و در نتیجه $\gamma$ مینیما در راستای $Q$ در درجه 8 می‌باشد. بنابراین $\gamma = \sqrt{p} + \sqrt{q} + \sqrt{t}$، بررسی $\gamma = \sqrt{p}$ و در نتیجه $\gamma$ مینیما در راستای $Q$ در درجه 8 می‌باشد.

$\deg(h(x)) \geq 8$، بنابراین $h(\gamma) = 0$.

بنابراین 7 صفر $f(x)$ است. از این که $\gamma$ مینیموم $Q$ روي $Q$ تحویل یا یا انتین. چه در دریایی صورت 7 صفر $\gamma$ مینیموم $Q$ در درجه 8 می‌باشد که $\gamma$ است.

با فرض $(0 \leq i \leq 15)$، $K_i = \phi(H_i)$

$K_0 = Q(\sqrt{p}, \sqrt{q}, \sqrt{t})$, $K_1 = Q(\sqrt{q}, \sqrt{t})$, $K_2 = Q(\sqrt{p}, \sqrt{t})$, $K_3 = Q(\sqrt{p}, \sqrt{q})$

$K_4 = Q(\sqrt{q}, \sqrt{t})$, $K_5 = Q(\sqrt{p}, \sqrt{q})$, $K_6 = Q(\sqrt{p}, \sqrt{t})$, $K_7 = Q(\sqrt{p}, \sqrt{t})$

$K_8 = Q(\sqrt{q})$, $K_9 = Q(\sqrt{q})$, $K_{10} = Q(\sqrt{q})$, $K_{11} = Q(\sqrt{p})$, $K_{12} = Q(\sqrt{t})$

$K_{13} = Q(\sqrt{pq})$, $K_{14} = Q(\sqrt{pq})$, $K_{15} = Q$.

در صفحه بعد شکل زیر (1) و شکل زیر (14) $\Gamma$ و $\Gamma$ را جهت مقایسه نشان می‌دهد.
فرض کنید \( [Q(\sqrt[p]{b}, \sqrt[q]{q}, \sqrt[t]{t}) : Q] = |\text{Gal}_Q(f)| = 8 \). لذا (دارای ۱۶ زیرگروه به شرح زیر است: 

\[
\begin{align*}
H_\emptyset &= \{\cdot\} , \quad H_1 = \{\cdot, \sigma_1\} , \quad H_2 = \{\cdot, \sigma_2\} , \quad H_3 = \{\cdot, \sigma_3\} , \quad H_4 = \{\cdot, \sigma_4\} , \\
H_5 &= \{\cdot, \sigma_5\} , \quad H_6 = \{\cdot, \sigma_6\} , \quad H_7 = \{\cdot, \sigma_7\} , \quad H_8 = \{\cdot, \sigma_8\} , \\
H_9 &= \{\cdot, \sigma_9\} , \quad H_{10} = \{\cdot, \sigma_{10}\} , \quad H_{11} = \{\cdot, \sigma_{11}\} , \quad H_{12} = \{\cdot, \sigma_{12}\} , \\
H_{13} &= \{\cdot, \sigma_{13}\} , \quad H_{14} = \{\cdot, \sigma_{14}\} , \quad H_{15} = \{\cdot, \sigma_{15}\} , \quad H_{16} = \{\cdot, \sigma_{16}\} .
\end{align*}
\]

با توجه \( 1 \leq i \leq 15 \) است و برای هر (۱۵) \( \alpha \in \text{Gal}_Q(f) \) و در نتیجه \( Q(\alpha) \neq \phi(H_i) (1 \leq i \leq 15) \) تصویر نمی‌گردد. بنابراین (۱۵) \( Q(\alpha) \) از درجه ۸ می‌باشد. در نتیجه اگر \( g(x) \) درجه آنگاه

\[
\deg(g(x)) \geq 8 \quad \text{آنگاه} \quad g(\alpha) = 8
\]

بنابراین جنگل‌های مناسب روى \( Q \) از درجه ۸ می‌باشد. در نتیجه اگر \( g(x) \) درجه آنگاه

\[ f(x) = x^8 - \frac{1}{2}(p + q + t)x^7 + \frac{1}{2}(p - q + r)x^6 + \frac{1}{2}(p + q + t) - \frac{1}{2}(p - q - r)x^5 + \frac{1}{2}(p - q + r)x^4 + \frac{1}{2}(p + q - r)x^3 + \frac{1}{2}(p - q - r)x^2 + \frac{1}{2}(q + r)x + \frac{1}{2}(p - q - r) . \]

توجه نابینا برآور است.
النهاية: لذا 

\[ Q(i, \sqrt{m}) = Q(i + \sqrt{m}) : Q \]

بتاییان چندجمله‌ای مینیمال از درجه 4 می‌باشد. در نهایت اگر 

\[ s(x) \in Q[x] \]

از درجه 4 می‌باشد، در نتیجه اگر 

\[ \text{deg}(s(x)) \geq 4 \]

باشد به قسمی که 0، آنگاه \( s(i) + \sqrt{m} = 0 \).

\[ \alpha = i + \sqrt{m}, \]

\[ \alpha^2 = -1 + m + 2i\sqrt{m}, \]

\[ \alpha^3 + (1 - m)\alpha^2 + 2(1 - m)\alpha - 1 = 0, \]

\[ \alpha^4 + 2(1 - m)\alpha^3 + (m + 1)\alpha = 0. \]

بتاییان \( \alpha \) صفر چندجمله‌ای \( t(x) \) می‌باشد. در نتیجه \( t(x) \) تکمیل تابعی است که در خروجی اول در نتیجه \( t(x) \) روي Q می‌باشد. در نتیجه اگر 

\[ f = (x^t - p)(x^t - q)(x^t - r)(x^t - s), \]

فرصت‌های \( t \) و \( q \) و \( p \) و \( s \) عدد دوم به صورت نوعی باشد و 

\[ Q(\sqrt[p]{t}, \sqrt[q]{q}, \sqrt[t]{t})/Q \]

توضیح میدان نظریه‌ای روی \( t \) است. 

\[ t = \{ a + b\sqrt[p]{t} + c\sqrt[q]{q} + d\sqrt[t]{t} \}, \]

\[ Q(\sqrt[p]{t}, \sqrt[q]{q}, \sqrt[t]{t})/Q \]

\[ = \{ a, a_1, a_2, a_3, a_4, a_5, a_6, a_7 \}. \]

اگر \( \sigma \in Gal(Q(f)) \) آنگاه برای هر \( \sigma \) و \( q \) و \( \sqrt[p]{t} \) را معین کنیم.

\[ \sigma(\sqrt[p]{t}) = \pm \sqrt[p]{t} \] و \( \sigma(\sqrt[q]{q}) = \pm \sqrt[q]{q} \) و \( \sigma(\sqrt[t]{t}) = \pm \sqrt[t]{t} \) 

\[ \begin{array}{c}
\sqrt[p]{t} \rightarrow \sqrt[p]{t} \\
\sqrt[q]{q} \rightarrow -\sqrt[q]{q} \\
\sqrt[t]{t} \rightarrow \sqrt[t]{t}
\end{array} \]

\[ \begin{array}{c}
\sqrt[p]{t} \rightarrow -\sqrt[p]{t} \\
\sqrt[q]{q} \rightarrow \sqrt[q]{q} \\
\sqrt[t]{t} \rightarrow -\sqrt[t]{t}
\end{array} \]
برهان: چون $a$ به هیچیک از $Q(\sqrt{p}), Q(\sqrt{q}), Q(\sqrt{pq})$ تغییر ندارد و $Q(\sqrt{pq}) = Q(\sqrt{p}, \sqrt{q})$، است که با هیچیک از این چهار زیرمیدان باینری نیست $Q(\sqrt{p}, \sqrt{q})$.

قضیه 2: اگر $g(x)$ درجه 1 باشد و درجه 3 باشد و $Q(a, b, c)$ آزاد باشد، آنگاه هر دوی $a, b, c$ از این چهار زیرمیدان باینری نیست.

$Q(a) = Q(\sqrt{p}, \sqrt{q})$. بنابراین $a = \alpha \sqrt{p} + \beta \sqrt{q}$.

برهان: با فرض $Q(g(x)) = Q(\sqrt{p}, \sqrt{q})$.

در نتیجه هر چندجمله مینیمال $\alpha$ درجه 2 از 2 می باشد با این در نتیجه هر چندجمله مینیمال $\alpha$ درجه 2 از 2 می باشد با این در نتیجه هر چندجمله مینیمال $\alpha$ درجه 2 از 2 می باشد با این در نتیجه هر چندجمله مینیمال $\alpha$ درجه 2 از 2 می باشد.

$deg(g(x)) \geq 2$.

$
\alpha = a \sqrt{p} + b \sqrt{q},
\alpha' = a' \sqrt{p} + b' \sqrt{q},
\alpha'' = (a'' \sqrt{p} + b'' \sqrt{q})',
\alpha''' = (a''' \sqrt{p} + b''' \sqrt{q})''.
$

نتیجه 2: اگر $Q(x)$ درجه 1 باشد و درجه 3 باشد و $Q(a, b, c)$ آزاد باشد، آنگاه $a, b, c$ از این چهار زیرمیدان باینری نیست.

$Q(a) = Q(\sqrt{p}, \sqrt{q})$. بنابراین $a = \alpha \sqrt{p} + \beta \sqrt{q}$.

برهان: با فرض $Q(g(x)) = Q(\sqrt{p}, \sqrt{q})$.

در نتیجه هر چندجمله مینیمال $\alpha$ درجه 2 از 2 می باشد با این در نتیجه هر چندجمله مینیمال $\alpha$ درجه 2 از 2 می باشد با این در نتیجه هر چندجمله مینیمال $\alpha$ درجه 2 از 2 می باشد با این در نتیجه هر چندجمله مینیمال $\alpha$ درجه 2 از 2 می باشد.

$deg(g(x)) \geq 2$.

$
\alpha = a \sqrt{p} + b \sqrt{q},
\alpha' = a' \sqrt{p} + b' \sqrt{q},
\alpha'' = (a'' \sqrt{p} + b'' \sqrt{q})',
\alpha''' = (a''' \sqrt{p} + b''' \sqrt{q})''.
$

نتیجه 3: اگر $Q(x)$ درجه 1 باشد و درجه 3 باشد و $Q(a, b, c)$ آزاد باشد، آنگاه $a, b, c$ از این چهار زیرمیدان باینری نیست.

$Q(a) = Q(\sqrt{p}, \sqrt{q})$. بنابراین $a = \alpha \sqrt{p} + \beta \sqrt{q}$.

برهان: با فرض $Q(g(x)) = Q(\sqrt{p}, \sqrt{q})$.

در نتیجه هر چندجمله مینیمال $\alpha$ درجه 2 از 2 می باشد با این در نتیجه هر چندجمله مینیمال $\alpha$ درجه 2 از 2 می باشد با این در نتیجه هر چندجمله مینیمال $\alpha$ درجه 2 از 2 می باشد با این در نتیجه هر چندجمله مینیمال $\alpha$ درجه 2 از 2 می باشد.

$deg(g(x)) \geq 2$.

$
\alpha = a \sqrt{p} + b \sqrt{q},
\alpha' = a' \sqrt{p} + b' \sqrt{q},
\alpha'' = (a'' \sqrt{p} + b'' \sqrt{q})',
\alpha''' = (a''' \sqrt{p} + b''' \sqrt{q})''.
$
یک زیگورثمان E می‌باشد که شامل F است. به علاوه اگر Z بزرگ‌ترین E زیگورثمانی از E و شامل F است، به عنوان اگر \( \text{Gal}_F(f) \) زیگورثمانی از Z باشد. جمله نتایج:

\[ A = \{ K | K \) است \( Z \) \} \]

\[ B = \{ H | H \) است \( \text{Gal}_F(f) \} \]

\[ \phi(H) = \psi(B) \to A \]

انگل نتایج (1)‌ها:

- \( p \in \mathbb{P} \) فرض کنید \( p, q \) دو عدد خالی از مجموع باشند به قسمی که این صورت \( Q(\sqrt{p}, \sqrt{q})/Q \)

\[ Q(\sqrt{p}, \sqrt{q}) = \{ a + a\sqrt{p} + a\sqrt{q} + a\sqrt{pq} | a_i \in Q \} \]

\[ |\text{Gal}_Q(f)| = |Q(\sqrt{p}, \sqrt{q}) : Q| = 4. \]

به فرض \( Q(\sqrt{p}, \sqrt{q}) \) که هر عضو \( Q \) را نمایان می‌کند. در این صورت خود \( Q(\sqrt{p}, \sqrt{q}) \) کافی است

\[ Q(\sqrt{p}, \sqrt{q}) = Q(\sqrt{q}, \sqrt{p}) \]

\[ \phi(\sigma) = \begin{cases} \sqrt{p} \longrightarrow \sqrt{p} \\ \sqrt{q} \longrightarrow \sqrt{q} \end{cases}, \quad \phi(\sigma_1) = \begin{cases} \sqrt{p} \longrightarrow -\sqrt{p} \\ \sqrt{q} \longrightarrow -\sqrt{q} \end{cases}, \quad \phi(\sigma_2) = \begin{cases} \sqrt{p} \longrightarrow \sqrt{q} \\ \sqrt{q} \longrightarrow -\sqrt{q} \end{cases} \]

بنابراین (f) زیگورثمانی Gal \( f \) دارای Z بزرگ‌ترین

\[ H_\sigma = \{ \sigma \}, \quad H_\tau = \{ \sigma, \tau \}, \quad H_\rho = \text{Gal}_Q(f), \quad H_{\rho_1} = \{ \sigma, \tau \}, \quad H_{\rho_2} = \{ \sigma, \tau_\rho \}. \]

در نتیجه Z بزرگ‌ترین متناز، \( H \) طبقه‌بندی (1) که تناالمان

\[ Q(\sqrt{p}, \sqrt{q}) \]

\[ \phi(H_\sigma) = Q(\sqrt{p}, \sqrt{q}), \quad \phi(H_\tau) = Q(\sqrt{p}, \sqrt{q}), \quad \phi(H_{\rho_1}) = Q(\sqrt{p}, \sqrt{q}), \quad \phi(H_{\rho_2}) = Q(\sqrt{p}, \sqrt{q}). \]

\[ \langle \alpha = a + b\sqrt{p} + c\sqrt{q} | a, b, c \in \mathbb{Z} \rangle, \quad \langle \alpha = a + b\sqrt{p} + c\sqrt{q} | a, b, c \in \mathbb{Z} \rangle \]

\[ \alpha = a + b\sqrt{p} + c\sqrt{q}, \quad \langle \alpha = a + b\sqrt{p} + c\sqrt{q} | a, b, c \in \mathbb{Z} \rangle, \quad \langle \alpha = a + b\sqrt{p} + c\sqrt{q} | a, b, c \in \mathbb{Z} \rangle, \quad \langle \alpha = a + b\sqrt{p} + c\sqrt{q} | a, b, c \in \mathbb{Z} \rangle \]

\[ \alpha = a + b\sqrt{p} + c\sqrt{q}, \quad \langle \alpha = a + b\sqrt{p} + c\sqrt{q} | a, b, c \in \mathbb{Z} \rangle, \quad \langle \alpha = a + b\sqrt{p} + c\sqrt{q} | a, b, c \in \mathbb{Z} \rangle, \quad \langle \alpha = a + b\sqrt{p} + c\sqrt{q} | a, b, c \in \mathbb{Z} \rangle \]

\[ \alpha = a + b\sqrt{p} + c\sqrt{q}, \quad \langle \alpha = a + b\sqrt{p} + c\sqrt{q} | a, b, c \in \mathbb{Z} \rangle, \quad \langle \alpha = a + b\sqrt{p} + c\sqrt{q} | a, b, c \in \mathbb{Z} \rangle, \quad \langle \alpha = a + b\sqrt{p} + c\sqrt{q} | a, b, c \in \mathbb{Z} \rangle \]