A Complete Efficiency Ranking of Decision Making Units in DEA: with an Empirical Study

G. R. JAHANSHAHLOO  
and  
S. MEHRABIAN  

Institute of Mathematics,  
University for Teacher Education.

Abstract

The efficiency measures provided by DEA can be used for ranking Decision Making Units (DMU's), but this ranking cannot be applied to efficient units. Anderson and Peterson have proposed a modified efficiency measure for efficient units which can be used for ranking, but this ranking breaks down in case of units with at least one input of zero. This paper proposes an alternative efficiency measure that removes this problem. The model is illustrated by an application to the University for Teacher Education, for which the Anderson - Peterson model was not able to give a ranking for two units, which were ranked successfully by the proposed model.

(Data Envelopment Analysis, Efficiency, Ranking)
Introduction.

Development Analysis (DEA) provides a measure of efficiency of a Decision Making Unit relative to other such units producing the same outputs with the same inputs. This technique, developed by Charnes, Cooper, and Rhodes (1978), and extended by Banker, Charnes, and Cooper (BCC) (1984), is a linear programming approach for an analysis of inputs and outputs. The method does not require prior weights on inputs and outputs.

The standard DEA method assigns an efficiency score of less than one to inefficient DMU's, from which improvements can be derived. However, efficient DMU's are assigned an efficiency of 1, so that for these units no improvement can be given. A model for ranking efficient DMU's was proposed by Andersen and Petersen (1993). Their model was called Extended-DEA, and is used in this study for the University for Teacher Education (UTE). However, this model breaks down efficient units with at least one zero input.

In this paper, a new definition of efficiency is proposed that can be extended for ranking efficient DMU's. The extended method is applied to data for the UTE.

The role of zeros in data has been considered by Charnes, Cooper and Thrall (1991) and Thompson, Dharmapala, and Thrall (1993) but this paper deals with the problem of ranking the efficient DMU's involving zeros in input data.

The paper unfolds as follows. Section 2 represents the Andersen and Petersen model. Section 3 presents a model based on a definition of efficiency in projection possibility set (PPS). In section 4, the two models are compared, using two illustrative examples. Section 5 applies the two models to the UTE data. A summary is given in section 6.

2 The Andersen-Petersen Model.

The standard DEA method assigns an efficiency score of less than one to inefficient units. A score less than one means that a linear combination of the other units could produce at least the same vector of output using a smaller vector of inputs. This score can be used to rank inefficient units. Andersen and Petersen (1993) developed a similar model for ranking efficient DMU's, which in the standard DEA method have a score of 1. The basic idea in their model is to compare the unit under evaluation with a linear combination of all other units, i.e., all units excluding the unit itself. In this case, an efficiency score above 1 is obtained for efficient units. This score reflects the radial distance from the unit under evaluation to the production frontier estimated with the exclusion of that unit, i.e., the maximum proportional increase in inputs producing at least the same outputs.

The Andersen-Petersen model (AP-Model) is identical with the CCR method, except that the unit under evaluation is not included in the combination. Therefore the $p^{th}$ DMU can be evaluated as follows:

$$r_p^* = \min \{ r_p - \epsilon \left[ \sum_{i=1}^{m} s_i + \sum_{r=1}^{s} s_r' \right] \}$$

subject to:

$$\sum_{j=1}^{n} \lambda_j X_{ij} + s_i = r_p X_{ip}, \quad i = 1, \ldots, m,$$

$$\sum_{j=1}^{n} \lambda_j Y_{rj} - s_r' = Y_{rp}, \quad r = 1, \ldots, s,$$

$$\lambda_j, s_i, s_r' \geq 0, \quad \forall j, i, r,$$

where $Y_{rj}$ is the $r^{th}$ output and $X_{ij}$ is the $i^{th}$ input for the $j^{th}$ DMU, $r_p$ is a scalar defining the share of $p^{th}$ DMU input vector which is required in order to produce the output vector of $p^{th}$ DMU, $\lambda_j$ denotes the intensity of the $j^{th}$ DMU, and $\epsilon$ is an non-Archimedean infinitesimal.
3 Efficiency Analysis by an Alternative Measure.

There are $n$ DMU's to be evaluated, each consumes varying amounts of $m$ different inputs to produce $s$ different outputs.

In the model formulation, $X_p$ and $Y_p$ denote, respectively, the nonnegative vectors of input and output values for DMU$_p$.

**Definition.** The production possibility set (PPS) $T$ is the set $\{(X_t, Y_t) \mid$ the outputs $Y_t$ can be produced with the inputs $X_t\}$.

The set of $n$ DMU's of actual production possibility ($X_j$, $Y_j$), $j = 1, \ldots, n$ is considered. Our focus is on the empirically defined production possibility set $T$ with constant returns assumption that is specified by the following four postulates:

- **Postulate 1 (Ray Unboundedness).** If $(X_t, Y_t) \in T$ then $(\lambda X_t, \lambda Y_t) \in T$ for all $\lambda \geq 0$.

- **Postulate 2 (Convexity).** If $(X_t, Y_t) \in T$ and $(X_u, Y_u) \in T$, then $(\lambda X_t + (1 - \lambda)X_u, \lambda Y_t + (1 - \lambda)Y_u) \in T$ for all $\lambda \in [0, 1]$.

- **Postulate 3 (Monotonicity).** If $(X_t, Y_t) \in T$ and $X_u \geq X_t$, $Y_u \leq Y_t$ then $(X_u, Y_u) \in T$.

- **Postulate 4 (Inclusion of Observations).** The observed $(X_j, Y_j) \in T$ for all $j = 1, \ldots, n$.

- **Postulate 5 (Minimum extrapolation).** If a production possibility set $T'$ satisfies Postulates 1, 2, 3, and, 4 then $T \subseteq T'$.

The unique production possibility set with constant returns assumption determined by the above postulates is given by:

$$T = \{(X_t, Y_t) \mid X_t \geq \sum_{j=1}^{m} \lambda_j X_j, Y_t \leq \sum_{j=1}^{n} \lambda_j Y_j, \lambda_j \geq 0, j = 1, \ldots, n\}.$$  

The boundary of this convex set consists of a straight line, plane, or hyperplane through the origin, as $T$ is a convex cone that contains all of DMU's, see Figure 1 for the simplest case of single input and single output.

![Figure 1: Production Possibility Set.](image)

For efficiency evaluation relative to the set $T$, we have the following two linear programming problems:

$$r_p^* = \min_r r_p$$

subject to:

$$(r_p X_p, Y_p) \in T,$$  

$$X_p + w_p e, Y_p \in T,$$  

which give the CCR-Model and our formulation respectively as follows:

$$r_p^* = \min_r r_p$$

subject to:

$$\sum_{j=1}^{m} \lambda_j X_j \leq r_p X_p,$$  

$$\sum_{j=1}^{n} \lambda_j Y_j \geq Y_p,$$  

$$\lambda_j \geq 0, j = 1, \ldots, n.$$  

*33*
dependent upon the units of measurement of input data, \( X_j \), \( j = 1, \ldots, n \). However, it is possible to obtain unit independence by normalization, as discussed later.

The unique production possibility set with variable returns assumption determined by postulates 2, 3, 4, and 5 is given by:

\[
T = \{(X_i, Y_i) | X_i \geq \sum_{j=1}^{n} \lambda_j X_j, \ Y_i \leq \sum_{j=1}^{n} \lambda_j Y_j, \ \sum_{j=1}^{n} \lambda_j = 1, \ \lambda_j \geq 0, \ j = 1, \ldots, n_i, \}
\]

A discussion similar to the constant returns assumption leads to the BCC-Model and our second formulation, and extension of our second formulation for ranking efficient units can be as follow:

\[
x^*_p = \min x_p - \epsilon \left[ \sum_{i=1}^{n} s_i + \sum_{r=1}^{s} s'_r \right]
\]

subject to:

\[
\sum_{j=1}^{n} \lambda_j x_{ij} + s_i = x_{ip} + x^*_p, \ i = 1, \ldots, m,
\]

\[
\sum_{j=1}^{n} \lambda_j y_{rj} - s'_r = y_{rp} - x^*_p, \ r = 1, \ldots, s,
\]

\[
\lambda_j, s_i, s'_r \geq 0, \ \forall j, i, r.
\]

4 The Comparison of the Two Models.

Two models for ranking the efficient DMU's were discussed in section 2 and 3. Section 2 represented the AP-Model and section 3 represented the JAM-Model. This section compares these two models using two illustrative examples.

In an actual set of data, it is possible that one or more of the data inputs and outputs are zero. It is also possible that some data inputs and outputs are small in comparison with other inputs and outputs.

In these cases, AP-Model, can not correctly evaluate
the efficiency of the DMU’s. If the DMU under evaluation has at least one input equal to zero, the AP-Model will be infeasible and if the DMU has at least one input which is small in comparison with other inputs, AP-Model will measure this DMU without stability.

The measure given by JAM-Model successfully evaluates the above cases, so that meaningful scores are obtained for all data.

In order to make usual scores, the scores in JAM-Model may be rescaled from [-1, 1] to [0%, 200%] so that 0 is rescaled to 100%. A score less than 100% means that the corresponding DMU is inefficient and greater than or equal to 100% means that the corresponding DMU is efficient.

4.1 Illustrative Example 1:

Table 1 gives an example of the above cases.

<table>
<thead>
<tr>
<th></th>
<th>A₁</th>
<th>A₂</th>
<th>A₃</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>input1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>5</td>
<td>10</td>
<td>10</td>
<td>2</td>
</tr>
<tr>
<td>input2</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>5</td>
<td>4</td>
<td>6</td>
<td>12</td>
</tr>
<tr>
<td>output1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>output2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 1: Comparison Test Data.

There are 5 DMU’s (A, B, C, D and E) each consume two inputs to produce two outputs with constant returns assumption. In order to remove the effects of changes in units of measurement from the input data, the data must be normalized before applying the method. It can be done by dividing inputs data by the maximum input (for each input).

In this example DMUₐ₁, DMUₐ₂ and DMUₐ₄ are compared with all other DMU’s (B, C, D and E) in the following three paragraphs by CCR-Model, AP-Model and JAM-Model.

- DMUₐ₁ is evaluated to be efficient by CCR-Model, and is evaluated as efficient by AP-Model with efficiency score equal to 100%. It is evaluated as efficient by JAM-Model with efficiency score equal to +0.276, which rescales to 100(1 + 0.276) = 127.6%. In this case, there is no problem.

- Consider now DMUₐ₄, which has an input equal to zero. DMUₐ₄ is evaluated to be efficient by CCR-Model, but it can not be evaluated by AP-Model. However, it is evaluated as efficient by JAM-Model with efficiency score equal to +0.310 which rescales to 131.0%.

- Consider now DMUₐ₃, which has an input equal to 0.1. DMUₐ₃ is evaluated as efficient by CCR-Model, and can be evaluated as efficient by AP-Model with efficiency score equal to 2000% which is unstable. It is evaluated as efficient by JAM-Model with efficiency score equal to +0.309 which rescales to 130.9%.

4.2 Illustrative Example 2:

A comparison of these two procedures for ranking DMU’s is illustrated on the Farrell frontier. Consider the DMU’s of Figure 2, each produces one output using two inputs with constant returns assumption. DMUₐ₃ is efficient and it can be evaluated by AP-Model with efficiency score of (100 uₐ₃/C) and evaluated by JAM-Model with efficiency score uₐ₃ which rescales to 100(1 + uₐ₃/C).
In this figure, AP-Model evaluates DMU_A with efficiency score of 
100\(\frac{DA'}{DA}\)% that is so but this DMU is evaluated with
efficiency score of 100(1 + w_A)% by JAM-Model.

- DMU_A has small value for input 1
  (see Figure 4):

- DMU_A and DMU_B are similar units that have
  small values for input 1 (see Figure 5):
In this figure, AP-Model evaluates DMU_A with efficiency score of much greater than 100%, while DMU_B is evaluated with efficiency score of about 100%, and it is obvious that these results are unstable, but these DMU's are evaluated with efficiency scores of 100(1 + w_A)% and about 100% by JAM-Model, respectively.

- DMU_A has zero for input 1 and DMU_B has small value for input 1 (see Figure 6):

![Figure 6: DMU_A has zero for input 1 and DMU_B has small value for input 1.](image)

AP-Model evaluates DMU_A with efficiency score of \( \frac{100(1 + w_A)}{w_A} \)% that is \( \infty \) while DMU_B is evaluated with efficiency score of about 100% and unstability is observed, but these DMU's are evaluated with efficiency scores of 100(1 + w_A)% and 100% by JAM-Model, respectively.

5 An Empirical Study.

In Jahnazahoo and Ali Ezaee (1995), the evaluation of teaching in the UTE was considered. Teaching inputs were expressed in teacher hours and classified in terms of two outputs, course enrollment in undergraduate and graduate studies (see appendix).

Table 2 gives the AP-Model results, efficiency scores and reference sets, with two inputs and two outputs. Six units were found to be efficient. The AP-Model assigns infinity values to the Department of Women's Physical Education, the 9th DMU, and Institute of Mathematics, the 19th DMU, which are indicated by an asterisk.

The academic units at the UTE may be evaluated using JAM-Model. Table 3 gives the efficiency scores and reference sets obtained by this model. The ranking is approximately the same as that of Table 2, except that DMU's 9 and 19 now have an explicit ranking.

Extended model, JAM-Model, successfully evaluated all efficient academic units at the UTE, in contrast to AP-Model.

In the application of AP-Model and JAM-Model on real data-set of the UTE, computational DEA issues of Ali (1994), Ali (1993), and Ali and Seiford (1993) have been considered.

6 Summary.

If an efficient DMU has at least one input equal to zero the Andersen-Petersen model gives an infinite result for this DMU, and if the DMU has at least one input with small value in comparison with other inputs, this model measures this DMU without stability. These cases are successfully evaluated and ranked by the new model proposed in this paper.

Computational difficulties of the Andersen-Petersen model were observed in evaluating efficient academic units at the UTE. This model could not evaluate the Department of Women's Physical Education and the Institute of Mathematics. These academic units were successfully evaluated by the new model.
<table>
<thead>
<tr>
<th>DMU</th>
<th>Eff.</th>
<th>Ref. Sets</th>
<th>( (\epsilon = 0.33 \times 10^{-6}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>*</td>
<td></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>*</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>174%</td>
<td>( \lambda_2 = 0.492 )</td>
<td>( \lambda_7 = 1.173 )</td>
</tr>
<tr>
<td>15</td>
<td>133%</td>
<td>( \lambda_1 = 0.938 )</td>
<td>( \lambda_{19} = 2.064 )</td>
</tr>
<tr>
<td>5</td>
<td>130%</td>
<td>( \lambda_8 = 0.956 )</td>
<td>( \lambda_{19} = 0.479 )</td>
</tr>
<tr>
<td>1</td>
<td>115%</td>
<td>( \lambda_2 = 0.492 )</td>
<td>( \lambda_{19} = 0.220 )</td>
</tr>
<tr>
<td>8</td>
<td>97%</td>
<td>( \lambda_2 = 0.276 )</td>
<td>( \lambda_8 = 0.548 )</td>
</tr>
<tr>
<td>10</td>
<td>96%</td>
<td>( \lambda_1 = 1.060 )</td>
<td>( \lambda_2 = 0.603 )</td>
</tr>
<tr>
<td>3</td>
<td>95%</td>
<td>( \lambda_1 = 0.585 )</td>
<td>( \lambda_2 = 0.073 )</td>
</tr>
<tr>
<td>17</td>
<td>89%</td>
<td>( \lambda_2 = 0.375 )</td>
<td>( \lambda_5 = 0.091 )</td>
</tr>
<tr>
<td>18</td>
<td>85%</td>
<td>( \lambda_2 = 0.978 )</td>
<td>( \lambda_5 = 0.191 )</td>
</tr>
<tr>
<td>7</td>
<td>71%</td>
<td>( \lambda_2 = 0.487 )</td>
<td>( \lambda_9 = 0.204 )</td>
</tr>
<tr>
<td>12</td>
<td>66%</td>
<td>( \lambda_1 = 0.564 )</td>
<td>( \lambda_2 = 0.285 )</td>
</tr>
<tr>
<td>4</td>
<td>63%</td>
<td>( \lambda_1 = 0.542 )</td>
<td>( \lambda_2 = 0.156 )</td>
</tr>
<tr>
<td>6</td>
<td>58%</td>
<td>( \lambda_1 = 0.131 )</td>
<td>( \lambda_2 = 0.274 )</td>
</tr>
<tr>
<td>16</td>
<td>57%</td>
<td>( \lambda_1 = 0.231 )</td>
<td>( \lambda_2 = 0.582 )</td>
</tr>
<tr>
<td>14</td>
<td>54%</td>
<td>( \lambda_1 = 0.726 )</td>
<td>( \lambda_2 = 0.232 )</td>
</tr>
<tr>
<td>13</td>
<td>45%</td>
<td>( \lambda_1 = 1.210 )</td>
<td>( \lambda_2 = 0.099 )</td>
</tr>
<tr>
<td>11</td>
<td>45%</td>
<td>( \lambda_1 = 0.048 )</td>
<td>( \lambda_3 = 0.528 )</td>
</tr>
</tbody>
</table>

Table 2: AP-Model Efficiency Scores for 19 Academic Units of the UTE.
<table>
<thead>
<tr>
<th>DMU</th>
<th>Eff.</th>
<th>Rescaled</th>
<th>Ref. Sets</th>
<th>Ref. Sets (ε = 0.55 × 10^{-6})</th>
</tr>
</thead>
<tbody>
<tr>
<td>19</td>
<td>+0.281</td>
<td>128%</td>
<td>λ₁₅ = 0.579</td>
<td>λ₁₉ = 0.094</td>
</tr>
<tr>
<td>5</td>
<td>+0.104</td>
<td>110%</td>
<td>λ₂ = 0.033</td>
<td>λ₁₉ = 0.580</td>
</tr>
<tr>
<td>2</td>
<td>+0.092</td>
<td>109%</td>
<td>λ₂ = 0.575</td>
<td>λ₁₉ = 0.274</td>
</tr>
<tr>
<td>15</td>
<td>+0.065</td>
<td>106%</td>
<td>λ₁ = 0.938</td>
<td>λ₁₉ = 1.491</td>
</tr>
<tr>
<td>1</td>
<td>+0.047</td>
<td>105%</td>
<td>λ₂ = 0.575</td>
<td>λ₁₀ = 0.358</td>
</tr>
<tr>
<td>9</td>
<td>+0.043</td>
<td>104%</td>
<td>λ₂ = 0.789</td>
<td>λ₁₀ = 0.701</td>
</tr>
<tr>
<td>8</td>
<td>-0.010</td>
<td>99%</td>
<td>λ₂ = 0.228</td>
<td>λ₈ = 0.648</td>
</tr>
<tr>
<td>3</td>
<td>-0.011</td>
<td>99%</td>
<td>λ₁ = 0.590</td>
<td>λ₉ = 0.701</td>
</tr>
<tr>
<td>7</td>
<td>-0.020</td>
<td>98%</td>
<td>λ₂ = 0.310</td>
<td>λ₉ = 0.428</td>
</tr>
<tr>
<td>10</td>
<td>-0.021</td>
<td>98%</td>
<td>λ₁ = 1.055</td>
<td>λ₂ = 0.609</td>
</tr>
<tr>
<td>17</td>
<td>-0.022</td>
<td>98%</td>
<td>λ₂ = 0.376</td>
<td>λ₆ = 0.091</td>
</tr>
<tr>
<td>18</td>
<td>-0.051</td>
<td>95%</td>
<td>λ₂ = 0.870</td>
<td>λ₁₈ = 0.338</td>
</tr>
<tr>
<td>6</td>
<td>-0.070</td>
<td>93%</td>
<td>λ₁ = 0.044</td>
<td>λ₁₉ = 0.094</td>
</tr>
<tr>
<td>4</td>
<td>-0.118</td>
<td>88%</td>
<td>λ₁ = 0.561</td>
<td>λ₂ = 0.128</td>
</tr>
<tr>
<td>16</td>
<td>-0.141</td>
<td>86%</td>
<td>λ₁ = 0.007</td>
<td>λ₂ = 0.903</td>
</tr>
<tr>
<td>12</td>
<td>-0.153</td>
<td>85%</td>
<td>λ₁ = 0.717</td>
<td>λ₁₉ = 0.029</td>
</tr>
<tr>
<td>14</td>
<td>-0.235</td>
<td>77%</td>
<td>λ₁ = 0.752</td>
<td>λ₁₉ = 0.278</td>
</tr>
<tr>
<td>11</td>
<td>-0.250</td>
<td>75%</td>
<td>λ₁ = 0.379</td>
<td>λ₂ = 0.054</td>
</tr>
<tr>
<td>13</td>
<td>-0.457</td>
<td>54%</td>
<td>λ₁ = 1.151</td>
<td>λ₁₅ = 0.137</td>
</tr>
</tbody>
</table>

Table 3: JAM-Model Efficiency Scores for 19 Academic Units of the UTE.


<table>
<thead>
<tr>
<th>No.</th>
<th>Department/Institute</th>
<th>I1</th>
<th>I2</th>
<th>O1</th>
<th>O2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Persian Literature</td>
<td>81.0</td>
<td>87.6</td>
<td>5191</td>
<td>205</td>
</tr>
<tr>
<td>2</td>
<td>Theology and Islamic Culture</td>
<td>85.0</td>
<td>12.8</td>
<td>3629</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>History</td>
<td>56.7</td>
<td>55.2</td>
<td>3302</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>Geography</td>
<td>91.0</td>
<td>78.8</td>
<td>3379</td>
<td>8</td>
</tr>
<tr>
<td>5</td>
<td>Foreign Languages</td>
<td>216.0</td>
<td>72.0</td>
<td>5368</td>
<td>639</td>
</tr>
<tr>
<td>6</td>
<td>Arabic Language and Literature</td>
<td>58.0</td>
<td>25.6</td>
<td>1674</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>Social Sciences</td>
<td>112.2</td>
<td>8.8</td>
<td>2350</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>Faculty of Physical Ed.</td>
<td>203.2</td>
<td>52.0</td>
<td>6315</td>
<td>414</td>
</tr>
<tr>
<td>9</td>
<td>Men Physical Education</td>
<td>186.6</td>
<td>0.0</td>
<td>2865</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>Mathematics</td>
<td>143.4</td>
<td>105.2</td>
<td>7689</td>
<td>66</td>
</tr>
<tr>
<td>11</td>
<td>Geology</td>
<td>108.7</td>
<td>127.0</td>
<td>2165</td>
<td>266</td>
</tr>
<tr>
<td>12</td>
<td>Biology</td>
<td>105.7</td>
<td>134.4</td>
<td>3963</td>
<td>315</td>
</tr>
<tr>
<td>13</td>
<td>Chemistry</td>
<td>235.0</td>
<td>236.8</td>
<td>6643</td>
<td>236</td>
</tr>
<tr>
<td>14</td>
<td>Physics</td>
<td>146.3</td>
<td>124.0</td>
<td>4611</td>
<td>128</td>
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<tr>
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<td>203.0</td>
<td>4869</td>
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</tr>
<tr>
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<td>Foundations of Education</td>
<td>118.7</td>
<td>48.2</td>
<td>3313</td>
<td>16</td>
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<tr>
<td>17</td>
<td>Instructional Technology</td>
<td>186.6</td>
<td>47.4</td>
<td>1853</td>
<td>230</td>
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<tr>
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<td>Psychology</td>
<td>146.0</td>
<td>50.8</td>
<td>4578</td>
<td>217</td>
</tr>
<tr>
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<td>Guidance and Counseling</td>
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<td>91.3</td>
<td>0</td>
<td>508</td>
</tr>
</tbody>
</table>

APPENDIX: Inputs and Outputs for 19 Academic Units of the UTE in the First Semester, 1993-94.
ferences:

Galois Theory, Joseph Rotman, Publication 1990.


برهان: بنابر بحث‌های قبل از قسمت 4 اگر $\gamma$ آگاهی $\gamma = \sqrt{p} + \sqrt{q} + \sqrt{t}$ باشد، آنگاه $\gamma$ در نتیجه چندجمله‌ای $Q_\gamma$ روي $Q$ از درجه 8 می‌باشد. لذا اگر $\gamma$ $\in Q[x]$ باشد به فرمی که $\deg(h(x)) \geq 8$ آگاهی $h(x)$ به قسمی که $\gamma = \sqrt{p} + \sqrt{q} + \sqrt{t}$ را از دید می‌گیرد.

بنابراین $\gamma$ صفر $f(x)$ است. از این که چندجمله‌ای $Q_\gamma$ روي $Q$ تحولی نباید است. چه در غیراین صورت $\sqrt{p}$ چندجمله‌ای از درجه کوچکتر از 8 می‌باشد که در صفحه بعد شکله زیر نشان داده می‌شود.

با فرض $0 \leq i \leq 15$, $K_i = \phi(H_i)$ سوابق داشته

$K_0 = Q(\sqrt{p}, \sqrt{q}, \sqrt{t})$, $K_1 = Q(\sqrt{p}, \sqrt{t})$, $K_2 = Q(\sqrt{q}, \sqrt{t})$, $K_3 = Q(\sqrt{p}, \sqrt{q})$

$K_4 = Q(\sqrt{q}, \sqrt{t})$, $K_5 = Q(\sqrt{p}, \sqrt{q}, \sqrt{t})$, $K_6 = Q(\sqrt{p}, \sqrt{q}, \sqrt{t})$, $K_7 = Q(\sqrt{p}, \sqrt{q})$

$K_8 = Q(\sqrt{q}, \sqrt{t})$, $K_9 = Q(\sqrt{p}, \sqrt{q}, \sqrt{t})$, $K_{10} = Q(\sqrt{p}, \sqrt{q}, \sqrt{t})$

$K_{11} = Q(\sqrt{p}, \sqrt{q}, \sqrt{t})$, $K_{12} = Q(\sqrt{p}, \sqrt{q}, \sqrt{t})$, $K_{13} = Q(\sqrt{p}, \sqrt{q}, \sqrt{t})$

$K_{14} = Q(\sqrt{p}, \sqrt{q}, \sqrt{t})$, $K_{15} = Q(\sqrt{p}, \sqrt{q}, \sqrt{t})$

در صفحه بعد شکله زیر نشان داده می‌شود. $\Gal_Q(f)$ آگاهی $Q(\sqrt{p}, \sqrt{q}, \sqrt{t})$ به فرمی که $\Gal_Q(f)$ را جهت مقایسه نشان می‌ده.
[Q(\sqrt{p}, \sqrt{q}, \sqrt{r}) : Q] = |\text{Gal}_Q(f)| = 8.

لذا دارای 16 زیرگروه به شرح زیر است:

\[ H_0 = \{\sigma_0\}, \quad H_1 = \{\sigma_0, \sigma_1\}, \quad H_2 = \{\sigma_0, \sigma_1, \sigma_2\} \]
\[ H_3 = \{\sigma_0, \sigma_3\}, \quad H_4 = \{\sigma_0, \sigma_4\}, \quad H_5 = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3\} \]
\[ H_6 = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4\}, \quad H_7 = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\} \]
\[ H_8 = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6\} \]
\[ H_9 = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7\} \]
\[ H_{10} = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8\} \]

\[ B = \{K : Q \leq K \leq Q(\sqrt{p}, \sqrt{q}, \sqrt{r})\} \quad \text{و} \quad A = \{H_i\} \quad 0 \leq i \leq 15 \]

\[ \psi : A \rightarrow B, \quad H_i \sim \phi(H_i) \]

\[ Q(\sqrt{p}, \sqrt{q}, \sqrt{r}) \]

در این صورت \( Q(\alpha) \) زیرگروهی از \( Q(\sqrt{p}, \sqrt{q}, \sqrt{r}) \) است و برای هر \( 1 \leq i \leq 15 \) اعداد \( a, b, c \) و \( d \) باید باشند. \( \alpha = a\sqrt{p} + b\sqrt{q} + c\sqrt{r} \) و \( d \) باشد.

\[ Q(\alpha) = Q(\sqrt{p}, \sqrt{q}, \sqrt{r}) \]

در نتیجه اینکه \( Q(\alpha) \neq \phi(H_i) \) برای \( 1 \leq i \leq 15 \) نیک. بنابراین، \( Q(\alpha) \) دارای 16 زیرگروه است.

\[ [Q(\alpha) : Q] = [Q(\sqrt{p}, \sqrt{q}, \sqrt{r}) : Q] = 8. \]

\[ g(\alpha) \in Q[x] \]

\[ \deg(g(\alpha)) \geq 8 \quad \text{آنگاه} \quad g(\alpha) = 5 \]

\[ \text{ قضیه 3: اگر} \quad f(x) = x^5 - t(p + q + t)x^4 + \frac{1}{2}(p + q + t)^5 + \frac{1}{2}(p' + q' + t')x^5 \]
\[ - t(p + q + t)(p' + q' + t' - pq - tpt - qt^2) + \frac{1}{2}pqtx^5 \]
\[ + (p' + q' + t' - pq - tpt - qt^2), \]

\[ \text{روی} \quad Q \text{ تبدیل نپذیر است.} \]
بزندا، بنابراین \(Q(i, \sqrt{m}) = Q(i + \sqrt{m})\) به دست آمده است. 

بنابراین، چندجمله‌ای مینیمال ریوی \(Q(i + \sqrt{m})\) به دست آمده است. 

\[
\begin{align*}
\alpha &= i + \sqrt{m}, \\
\alpha^7 &= -1 + m + 3i \sqrt{m}, \\
\alpha^7 + (1 - m)^7 + 2(1 - m)\alpha^3 &= -2m, \\
\alpha^7 + (1 - m)^7 + 2(1 - m)\alpha^3 &= -2m, \\
\alpha^7 + 2(1 - m)\alpha^3 + (m + 1)^7 &= 0.
\end{align*}
\]

بنابراین، \(\alpha\) صفر چندجمله‌ای \(Q(i, \sqrt{m})\) می‌باشد. در نتیجه، ریوی \(Q(i, \sqrt{m})\) به دست آمده است. 

اگر \(\alpha\) صفر چندجمله‌ای \(Q(i, \sqrt{m})\) می‌باشد، در نتیجه، ریوی \(Q(i, \sqrt{m})\) به دست آمده است. 

\(f = (x^7 - p)(x^7 - q)(x^7 - t)\) ریوی \(Q(\sqrt{p}, \sqrt{q}, \sqrt{t})/Q\) است. 

\[\begin{align*}
\sigma_0 : & \begin{cases} 
\sqrt{p} \rightarrow \sqrt{p}, & \sigma_1 : \sqrt{q} \rightarrow -\sqrt{q}, & \sigma_2 : \sqrt{t} \rightarrow \sqrt{t} \\
\sqrt{q} \rightarrow \sqrt{q}, & \sigma_1 : \sqrt{q} \rightarrow \sqrt{q}, & \sigma_2 : \sqrt{t} \rightarrow \sqrt{t} \\
\sqrt{t} \rightarrow \sqrt{t}, & \sigma_1 : \sqrt{p} \rightarrow \sqrt{p}, & \sigma_2 : \sqrt{p} \rightarrow \sqrt{p} \\
\sqrt{p} \rightarrow -\sqrt{p}, & \sigma_1 : \sqrt{q} \rightarrow \sqrt{q}, & \sigma_2 : \sqrt{t} \rightarrow -\sqrt{t} \\
\sqrt{q} \rightarrow -\sqrt{q}, & \sigma_1 : \sqrt{p} \rightarrow \sqrt{p}, & \sigma_2 : \sqrt{p} \rightarrow -\sqrt{p} \\
\sqrt{t} \rightarrow -\sqrt{t}, & \sigma_1 : \sqrt{q} \rightarrow \sqrt{q}, & \sigma_2 : \sqrt{t} \rightarrow -\sqrt{t} \\
\sqrt{p} \rightarrow -\sqrt{p}, & \sigma_1 : \sqrt{q} \rightarrow \sqrt{q}, & \sigma_2 : \sqrt{t} \rightarrow -\sqrt{t} \\
\sqrt{q} \rightarrow -\sqrt{q}, & \sigma_1 : \sqrt{p} \rightarrow \sqrt{p}, & \sigma_2 : \sqrt{p} \rightarrow \sqrt{p} \\
\sqrt{t} \rightarrow -\sqrt{t}, & \sigma_1 : \sqrt{q} \rightarrow -\sqrt{q}, & \sigma_2 : \sqrt{t} \rightarrow -\sqrt{t} \\
\sqrt{p} \rightarrow -\sqrt{p}, & \sigma_1 : \sqrt{q} \rightarrow -\sqrt{q}, & \sigma_2 : \sqrt{t} \rightarrow -\sqrt{t}.
\end{cases}
\]
برهان: چون $\alpha$ به هیچیک از زیرمجموعه‌های $(\sqrt{p}, q)$، $Q(\sqrt{p})$, $Q(q)$، $Q(\sqrt{p}, q)$ تعادل ندارد، و $Q(\sqrt{p})$، $Q(\sqrt{q})$، $Q(\sqrt{p}, q)$ است که با هیچیک از این چهار زیرمجموعه پیشین است $Q(\sqrt{p}, q)$. 

قضیه 2: آگر عدد صحیح و مثبت و خالی از مربع باشد به قسمی که $a \neq 0$ اعداد $b, a$ و $q$ از $(\sqrt{p}, q)$ نااصف باشند، آنگاه $Q(\sqrt{p}, q)$ تحلیل ناپذیر است.

برهان: با فرضیه $\alpha = a + b\sqrt{q}$. بنابراین $a$ از $(\sqrt{p})$ می‌باشد. از اینکه $Q(\sqrt{p}, q)$ تحلیل ناپذیر است، ساری‌الاحمد $Q$ روی $Q(\sqrt{p}, q)$ از $2$ می‌باشد. لذا $Q(\sqrt{p}, q)$ تحلیل ناپذیر است.

قضیه 3: آگر عدد صحیح و مثبت و مربع کامل باشد به قسمی که $a \neq 0$ اعداد $b, a$ و $q$ از $(\sqrt{p}, q)$ نااصف باشند، آنگاه $Q(\sqrt{p}, q)$ تحلیل ناپذیر است.

برهان: با فرضیه $\alpha = b + a\sqrt{q}$.
یک زیرمیدان $E$ می‌باشد که شامل $F$ است. به عکس اگر زیرمیدانی از $E$ شامل $F$ است، به عهده $Gal_F(E)$ زیرگروهی از $Gal_K(E)$ وجود داشته باشد که شامل $E$ است. بنابراین:

$$A = \{K : F, E \}$$

$$B = \{H : Gal_F(H), E \}$$

آنگاه نتیجه (1) $\psi(H) = \phi(H)$ با ضعیت $\psi : B \rightarrow A$ می‌باشد.

فرض کنید $p, q$ و $q \neq (p, q) \neq q$ با عدد خالی از مرزی باشد به قسمی که

$$Q(\sqrt{p}, \sqrt{q})/Q$$

یک توزیع میدان تجزیه‌ای $f$ است.

$$Q(\sqrt{p}, \sqrt{q}) = \{a + a_1\sqrt{p} + a_2\sqrt{q} + a_3\sqrt{pq} : a_i \in Q\}$$

$$|Gal_Q(f)| = [Q(\sqrt{p}, \sqrt{q}) : Q] = 4.$$ 

زاپسی: $0 \leq i \leq 3$ (2) $\sigma_0$، $\sigma_1$، $\sigma_2$، $\sigma_3$ به فرض $Q(\sqrt{p}, \sqrt{q})/Q$ که به عضو $\sigma$ گفته می‌شود که $\sigma(\sqrt{p}) = \pm \sqrt{p}$ و $\sigma(\sqrt{q}) = \pm \sqrt{q}$ توجه به $\psi$ که $\psi$ به فرض $f$ است.

$\sqrt{p} \rightarrow \sqrt{p}$,

$\sqrt{q} \rightarrow \sqrt{q}$,

$\sigma_1 : \sqrt{p} \rightarrow \sqrt{p}$,

$\sqrt{q} \rightarrow \sqrt{q}$,

$\sigma_2 : \sqrt{p} \rightarrow \sqrt{p}$,

$\sqrt{q} \rightarrow \sqrt{q}$,

$\sigma_3 : \sqrt{p} \rightarrow \sqrt{p}$,

$\sqrt{q} \rightarrow \sqrt{q}$.

بنابراین $Gal_Q(f)$ زیرگروهی $Gal_Q(f)$ دارد.

$H = \{\sigma_0\}$,

$H_1 = \{\sigma_0, \sigma_1\}$,

$H_2 = Gal_Q(f)$,

$H_3 = \{\sigma_0, \sigma_3\}$,

$H_4 = \{\sigma_0, \sigma_2\}$.

در نتیجه زیرمیدان‌های متناهی $H$، $H_1$، $H_2$، $H_3$, $H_4$, $H_5$, $H_6$, $H_7$ به شرح زیر می‌باشند:

$\phi(H_2) = Q(\sqrt{p}, \sqrt{q})$

$\phi(H_3) = Q(\sqrt{p})$,

$\phi(H_4) = Q(\sqrt{q})$,

$\phi(H_5) = Q(\sqrt{p})$,

$\phi(H_6) = Q(\sqrt{q})$.

ولی اگر $a, b \in b, a, a \neq (p, q) \neq (q, p)$ در عدد خالی از مرزی باشد به قسمی که

$Q(\alpha) = Q(\sqrt{p}, \sqrt{q})$ آنگاه

$$\alpha = a\sqrt{p} + b\sqrt{q}.$$