A Complete Efficiency Ranking of Decision Making Units in DEA: with an Empirical Study

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Abstract

The efficiency measures provided by DEA can be used for ranking Decision Making Units (DMU's), but this ranking cannot be applied to efficient units. Anderson and Peterson have proposed a modified efficiency measure for efficient units which can be used for ranking, but this ranking breaks down in case of units with at least one input of zero. This paper proposes an alternative efficiency measure that removes this problem. The model is illustrated by an application to the University for Teacher Education, for which the Anderson - Peterson model was not able to give a ranking for two units, which were ranked successfully by the proposed model.

(Data Envelopment Analysis, Efficiency, Ranking)
2 The Andersen-Petersen Model.

The standard DEA method assigns an efficiency score of less than one to inefficient DMU’s, from which one can be derived. However, efficient DMU’s have an efficiency of 1, so that for these units nothing can be given. A model for ranking efficient units was proposed by Andersen and Petersen. Their model was called Extended-DEA, and later in this study for the University for Teacher Education (UTE). However, this model breaks down inefficient units with at least one zero input.

In this paper, a new definition of efficiency is proposed that can be extended for ranking efficient units. The extended model is applied to data for UTE.

The role of zeros in data has been considered by Cooper, Thrall, and Thrall (1993) and also by Thrall and Thrall (1993) but this paper solves the problem of ranking the efficient units involving zeros in input data.

The paper unfolds as follows. Section 2 represents the Andersen and Petersen model. Section 3 presents a model based on a definition of efficiency in production possibility set (PPS). In section 4, the two models are compared, using two illustrative examples. In section 5 applies the two models to the UTE data. A summary is given in section 6.
3 Efficiency Analysis by an Alternative Measure.

There are $n$ DMU’s to be evaluated, each consumes varying amounts of $m$ different inputs to produce $s$ different outputs.

In the model formulation, $X_p$ and $Y_p$ denote, respectively, the nonnegative vectors of input and output values for DMU$_p$.

**Definition.** The production possibility set (PPS) $T$ is the set $\{(X_t, Y_t) |$ the outputs $Y_t$ can be produced with the inputs $X_t \}.$

The set of $n$ DMU’s of actual production possibility $(X_j, Y_j)$, $j = 1, \ldots, n$ is considered. Our focus is on the empirically defined production possibility set $T$ with constant returns assumption that is specified by the following four postulates:

- **Postulate 1 (Ray Unboundedness).** If $(X_t, Y_t) \in T$ then $(\lambda X_t, \lambda Y_t) \in T$ for all $\lambda \geq 0$.

- **Postulate 2 (Convexity).** If $(X_t, Y_t) \in T$ and $(X_u, Y_u) \in T$, then $(\lambda X_t + (1 - \lambda) X_u, \lambda Y_t + (1 - \lambda) Y_u) \in T$ for all $\lambda \in [0, 1]$.

- **Postulate 3 (Monotonicity).** If $(X_t, Y_t) \in T$ and $X_u \geq X_t$, $Y_u \leq Y_t$ then $(X_u, Y_u) \in T$.

- **Postulate 4 (Inclusion of Observations).** The observed $(X_j, Y_j) \in T$ for all $j = 1, \ldots, n$.

- **Postulate 5 (Minimum extrapolation).** If a production possibility set $T'$ satisfies Postulates 1, 2, 3, and 4 then $T \subseteq T'$.

The unique production possibility set with constant returns assumption determined by the above postulates is given by:

$$T = \{(X_t, Y_t) | X_t \geq \sum_{j=1}^{n} \lambda_j X_j, \quad Y_t \leq \sum_{j=1}^{n} \lambda_j Y_j, \quad \lambda_j \geq 0, \quad j = 1, \ldots, n \}.$$ 

The boundary of this convex set consists of a straight line, plane, or hyperplane through the origin, as $T$ is a convex cone that contains all of DMU’s, see Figure 1 for the simplest case of single input and single output.

![Figure 1: Production Possibility Set.](image)

For efficiency evaluation relative to the set $T$, we have the following two linear programming problems:

$$r_p^* = \min r_p \quad \text{subject to:} \quad (r_p X_p, Y_p) \in T,$$

$$w_p^* = \min w_p \quad \text{subject to:} \quad (X_p + w_p e, Y_p) \in T,$$

which give the CCR-Model and our formulation respectively as follows:

$$r_p^* = \min r_p \quad \text{subject to:} \quad \sum_{j=1}^{n} \lambda_j X_j \leq r_p X_p,$$

$$\sum_{j=1}^{n} \lambda_j Y_j \geq Y_p,$$

$$\lambda_j \geq 0, \quad j = 1, \ldots, n,$$
dependent upon the units of measurement of input
data, \( X_j, j = 1, \ldots, n \). However, it is possible to
obtain unit independence by normalization, as dis-
cussed later.

The unique production possibility set with variable
returns assumption determined by postulates 2, 3, 4,
and 5 is given by:

\[
T = \{(X_i, Y_i) \mid X_i \geq \sum_{j=1}^{n} \lambda_j X_j, Y_i \leq \sum_{j=1}^{n} \lambda_j Y_j, \\
\sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0, \quad j = 1, \ldots, n \}
\]

A discussion similar to the constant returns as-
sumption leads to the BCC-Model and our second
formulation, and extension of our second formulation
for ranking efficient units can be as follow:

\[
x_p^* = \min x_p - \varepsilon \left[ \sum_{i=1}^{m} s_i + \sum_{r=1}^{s} s'_r \right]
\]

subject to:

\[
\sum_{j \in J} \lambda_j X_{ij} + s_i = X_{ip} + w_p, \quad i = 1, \ldots, m,
\]

\[
\sum_{j \notin J} \lambda_j Y_{ij} - s'_r = Y_{rp}, \quad r = 1, \ldots, s,
\]

\[
\lambda_j \geq 0, \quad j = 1, \ldots, n,
\]

\[
s_i, s'_r \geq 0, \quad \forall j, i, r.
\]

4 The Comparison of the Two Models.

Two models for ranking the efficient DMU’s were
discussed in section 2 and 3. Section 2 represented
the AP-Model and section 3 represented the JAM-
Model. This section compares these two models using
two illustrative examples.

In an actual set of data, it is possible that one or
more of the data inputs and outputs are zero. It is
also possible that some data inputs and outputs are
small in comparison with other inputs and outputs.

In these cases, AP-Model, can not correctly evaluate
the efficiency of the DMU's. If the DMU under evaluation has at least one input equal to zero, the AP-Model will be infeasible and if the DMU has at least one input which is small in comparison with other inputs, AP-Model will measure this DMU without stability.

The measure given by JAM-Model successfully evaluates the above cases, so that meaningful scores are obtained for all data.

In order to make usual scores, the scores in JAM-Model may be rescaled from \([-1, +1]\) to [0%, 200%] so that of 0 is rescaled to 100%. A score less than 100% means that the corresponding DMU is inefficient and greater than or equal to 100% means that the corresponding DMU is efficient.

### 4.1 Illustrative Example 1:

Table 1 gives an example of the above cases.

<table>
<thead>
<tr>
<th></th>
<th>A₁</th>
<th>A₂</th>
<th>A₃</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>input1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>5</td>
<td>10</td>
<td>10</td>
<td>2</td>
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<tr>
<td>input2</td>
<td>8</td>
<td>1</td>
<td>8</td>
<td>5</td>
<td>4</td>
<td>6</td>
<td>12</td>
</tr>
<tr>
<td>output1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>output2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

**Table 1: Comparison Test Data.**

There are 5 DMU’s (A, B, C, D and E) each consume two inputs to produce two outputs with constant returns assumption. In order to remove the effects of changes in units of measurement from the input data, the data must be normalized before applying the method. It can be done by dividing inputs data by the maximum input (for each input).

In this example DMUₐ₁, DMUₐ₂ and DMUₐ₃ are compared with all other DMU’s (B, C, D and E) in the following three paragraphs by CCR-Model, AP-Model and JAM-Model.

- **DMUₐ₁**, is evaluated to be efficient by CCR-Model, and is evaluated as efficient by AP-Model with efficiency score equal to 147%. It is evaluated as efficient by JAM-Model with efficiency score equal to +0.276, which rescales to \(100(1 + 0.276) = 127.6\%\). In this case, there is no problem.

- **Consider now DMUₐ₂**, which has an input equal to zero. DMUₐ₂ is evaluated to be efficient by CCR-Model, but it can not be evaluated by AP-Model. However, it is evaluated as efficient by JAM-Model with efficiency score equal to +0.310 which rescales to 131.0%.

- **Consider now DMUₐ₃**, which has an input equal to 0.1. DMUₐ₃ is evaluated as efficient by CCR-Model, and can be evaluated as efficient by AP-Model with efficiency score equal to 2000% which is unstable. It is evaluated as efficient by JAM-Model with efficiency score equal to +0.309 which rescales to 130.9%.

### 4.2 Illustrative Example 2:

A comparison of these two procedures for ranking DMU’s is illustrated on the Farrell frontier. Consider the DMU’s of Figure 2, each produces one output using two inputs with constant returns assumption. DMUₐ is efficient and it can be evaluated by AP-Model with efficiency score of \(100\varphi_C\) and evaluated by JAM-Model with efficiency score \(\omega_C\) which rescales to \(100(1 + \omega_C)\).
• DMU_A has small value for input 1
(see Figure 4):

In this figure, AP-Model evaluates DMU_A with efficiency score of \((100 \frac{A'_A}{B'_A})\)% that is much greater than 100%, where is unstable, but this DMU is evaluated with efficiency score of \(100(1+w_A)\)% by JAM-Model.

• DMU_A and DMU_B are similar units that have small values for input 1 (see Figure 5):
In this figure, AP-Model evaluates $DMU_A$ with efficiency score of much greater than 100%, while $DMU_B$ is evaluated with efficiency score of about 100%, and it is obvious that these results are unstable, but these DMU's are evaluated with efficiency scores of $100(1 + w_A)^\%$ and about 100% by JAM-Model, respectively.

- $DMU_A$ has zero for input 1 and $DMU_B$ has small value for input 1 (see Figure 6):

![Figure 6: DMU_A has zero for input 1 and DMU_B has small value for input 1.](image)

AP-Model evaluates $DMU_A$ with efficiency score of $100\frac{OA'}{OA}$% that is infinite while $DMU_B$ is evaluated with efficiency score of about 100% and instability is observed, but these DMU's are evaluated with efficiency scores of $100(1 + w_A)^\%$ and 100% by JAM-Model, respectively.

5 An Empirical Study.

In Jahanolahloo and Aliemazzadeh (1995), the evaluation of teaching in the UTE was considered. Teaching inputs were expressed in teacher hours and classified in terms of two outputs, course enrollment in undergraduate and graduate studies (see appendix).

Table 2 gives the AP-Model results, efficiency scores and reference sets, with two inputs and two outputs. Six units were found to be efficient. The AP-Model assigns infinity values to the Department of Women's Physical Education, the 9th DMU, and Institute of Mathematics, the 19th DMU, which are indicated by an asterisk.

The academic units at the UTE may be evaluated using JAM-Model. Table 3 gives the efficiency scores and reference sets obtained by this model. The ranking is approximately the same as that of Table 2, except that DMU's 9 and 19 now have an explicit ranking.

Extended model, JAM-Model, successfully evaluated all efficient academic units at the UTE, in contrast to AP-Model.

In the application of AP-Model and JAM-Model on real data-set of the UTE, computational DEA issues of Ali (1994), Ali (1993), and Ali and Seiford (1993) have been considered.

6 Summary.

If an efficient DMU has at least one input equal to zero the Andersen-Petersen model gives an infinite result for this DMU, and if the DMU has at least one input with small value in comparison with other inputs, this model measures this DMU without stability. These cases are successfully evaluated and ranked by the new model proposed in this paper.

Computational difficulties of the Andersen-Petersen model were observed in evaluating efficient academic units at the UTE. This model could not evaluate the Department of Women's Physical Education and the Institute of Mathematics. These academic units were successfully evaluated by the new model.
<table>
<thead>
<tr>
<th>DMU</th>
<th>Eff.</th>
<th>Ref. Sets</th>
<th>($\epsilon = 0.33 \times 10^{-6}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>174%</td>
<td>$\lambda_2 = 0.492$</td>
<td>$\lambda_7 = 1.173$</td>
</tr>
<tr>
<td>15</td>
<td>133%</td>
<td>$\lambda_1 = 0.938$</td>
<td>$\lambda_{19} = 2.064$</td>
</tr>
<tr>
<td>5</td>
<td>130%</td>
<td>$\lambda_8 = 0.956$</td>
<td>$\lambda_{19} = 0.479$</td>
</tr>
<tr>
<td>1</td>
<td>115%</td>
<td>$\lambda_2 = 0.492$</td>
<td>$\lambda_{10} = 0.220$</td>
</tr>
<tr>
<td>8</td>
<td>97%</td>
<td>$\lambda_2 = 0.276$</td>
<td>$\lambda_5 = 0.548$</td>
</tr>
<tr>
<td>10</td>
<td>96%</td>
<td>$\lambda_1 = 1.060$</td>
<td>$\lambda_2 = 0.503$</td>
</tr>
<tr>
<td>3</td>
<td>95%</td>
<td>$\lambda_1 = 0.585$</td>
<td>$\lambda_2 = 0.073$</td>
</tr>
<tr>
<td>17</td>
<td>89%</td>
<td>$\lambda_2 = 0.375$</td>
<td>$\lambda_5 = 0.091$</td>
</tr>
<tr>
<td>18</td>
<td>85%</td>
<td>$\lambda_2 = 0.978$</td>
<td>$\lambda_5 = 0.191$</td>
</tr>
<tr>
<td>7</td>
<td>71%</td>
<td>$\lambda_2 = 0.487$</td>
<td>$\lambda_9 = 0.204$</td>
</tr>
<tr>
<td>12</td>
<td>66%</td>
<td>$\lambda_1 = 0.564$</td>
<td>$\lambda_2 = 0.285$</td>
</tr>
<tr>
<td>4</td>
<td>63%</td>
<td>$\lambda_1 = 0.542$</td>
<td>$\lambda_2 = 0.156$</td>
</tr>
<tr>
<td>6</td>
<td>58%</td>
<td>$\lambda_1 = 0.131$</td>
<td>$\lambda_2 = 0.274$</td>
</tr>
<tr>
<td>16</td>
<td>57%</td>
<td>$\lambda_1 = 0.231$</td>
<td>$\lambda_2 = 0.582$</td>
</tr>
<tr>
<td>14</td>
<td>54%</td>
<td>$\lambda_1 = 0.726$</td>
<td>$\lambda_2 = 0.232$</td>
</tr>
<tr>
<td>13</td>
<td>45%</td>
<td>$\lambda_1 = 1.210$</td>
<td>$\lambda_2 = 0.099$</td>
</tr>
<tr>
<td>11</td>
<td>45%</td>
<td>$\lambda_1 = 0.048$</td>
<td>$\lambda_2 = 0.528$</td>
</tr>
</tbody>
</table>

Table 2: AP-Model Efficiency Scores for 19 Academic Units of the UTE.
<table>
<thead>
<tr>
<th>DMU</th>
<th>Eff.</th>
<th>Rescaled</th>
<th>Ref. Sets</th>
<th>( \epsilon = 0.55 \times 10^{-6} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>19</td>
<td>+0.281</td>
<td>128%</td>
<td>( \lambda_{15} = 0.579 )</td>
<td>( \lambda_{19} = 0.094 )</td>
</tr>
<tr>
<td>5</td>
<td>+0.104</td>
<td>110%</td>
<td>( \lambda_{2} = 0.033 )</td>
<td>( \lambda_{9} = 0.580 )</td>
</tr>
<tr>
<td>2</td>
<td>+0.092</td>
<td>109%</td>
<td>( \lambda_{2} = 0.575 )</td>
<td>( \lambda_{17} = 0.850 )</td>
</tr>
<tr>
<td>15</td>
<td>+0.065</td>
<td>106%</td>
<td>( \lambda_{1} = 0.938 )</td>
<td>( \lambda_{19} = 1.491 )</td>
</tr>
<tr>
<td>1</td>
<td>+0.047</td>
<td>105%</td>
<td>( \lambda_{2} = 0.575 )</td>
<td>( \lambda_{10} = 0.274 )</td>
</tr>
<tr>
<td>9</td>
<td>-0.043</td>
<td>104%</td>
<td>( \lambda_{2} = 0.789 )</td>
<td>( \lambda_{15} = 0.358 )</td>
</tr>
<tr>
<td>8</td>
<td>-0.010</td>
<td>98%</td>
<td>( \lambda_{2} = 0.228 )</td>
<td>( \lambda_{9} = 0.701 )</td>
</tr>
<tr>
<td>3</td>
<td>-0.011</td>
<td>99%</td>
<td>( \lambda_{1} = 0.590 )</td>
<td>( \lambda_{9} = 0.648 )</td>
</tr>
<tr>
<td>7</td>
<td>-0.020</td>
<td>98%</td>
<td>( \lambda_{2} = 0.310 )</td>
<td>( \lambda_{9} = 0.701 )</td>
</tr>
<tr>
<td>10</td>
<td>-0.021</td>
<td>98%</td>
<td>( \lambda_{1} = 1.055 )</td>
<td>( \lambda_{9} = 0.806 )</td>
</tr>
<tr>
<td>17</td>
<td>-0.022</td>
<td>98%</td>
<td>( \lambda_{2} = 0.376 )</td>
<td>( \lambda_{9} = 0.609 )</td>
</tr>
<tr>
<td>18</td>
<td>-0.051</td>
<td>95%</td>
<td>( \lambda_{2} = 0.870 )</td>
<td>( \lambda_{18} = 0.338 )</td>
</tr>
<tr>
<td>6</td>
<td>-0.070</td>
<td>93%</td>
<td>( \lambda_{1} = 0.044 )</td>
<td>( \lambda_{19} = 0.094 )</td>
</tr>
<tr>
<td>4</td>
<td>-0.118</td>
<td>88%</td>
<td>( \lambda_{2} = 0.551 )</td>
<td>( \lambda_{19} = 0.128 )</td>
</tr>
<tr>
<td>16</td>
<td>-0.141</td>
<td>86%</td>
<td>( \lambda_{1} = 0.007 )</td>
<td>( \lambda_{19} = 0.029 )</td>
</tr>
<tr>
<td>12</td>
<td>-0.153</td>
<td>85%</td>
<td>( \lambda_{1} = 0.717 )</td>
<td>( \lambda_{15} = 0.903 )</td>
</tr>
<tr>
<td>14</td>
<td>-0.235</td>
<td>77%</td>
<td>( \lambda_{1} = 0.752 )</td>
<td>( \lambda_{19} = 0.278 )</td>
</tr>
<tr>
<td>11</td>
<td>-0.250</td>
<td>75%</td>
<td>( \lambda_{1} = 0.379 )</td>
<td>( \lambda_{19} = 0.371 )</td>
</tr>
<tr>
<td>13</td>
<td>-0.457</td>
<td>54%</td>
<td>( \lambda_{1} = 1.151 )</td>
<td>( \lambda_{15} = 0.137 )</td>
</tr>
</tbody>
</table>

Table 3: JAM-Model Efficiency Scores for 19 Academic Units of the UTE.


<table>
<thead>
<tr>
<th>No.</th>
<th>Department/Institute</th>
<th>I1</th>
<th>I2</th>
<th>O1</th>
<th>O2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Faculty of Literature</td>
<td>81.0</td>
<td>87.6</td>
<td>5191</td>
<td>205</td>
</tr>
<tr>
<td>2</td>
<td>Persian Literature</td>
<td>85.0</td>
<td>12.8</td>
<td>3629</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>Theology and Islamic Culture</td>
<td>56.7</td>
<td>55.2</td>
<td>3302</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>History</td>
<td>91.0</td>
<td>78.8</td>
<td>3379</td>
<td>8</td>
</tr>
<tr>
<td>5</td>
<td>Geography</td>
<td>216.0</td>
<td>72.0</td>
<td>5368</td>
<td>639</td>
</tr>
<tr>
<td>6</td>
<td>Foreign Languages</td>
<td>58.0</td>
<td>25.6</td>
<td>1674</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>Social Sciences</td>
<td>112.2</td>
<td>8.8</td>
<td>2350</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>Faculty of Physical Ed.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>Men Physical Education</td>
<td>203.2</td>
<td>52.0</td>
<td>6315</td>
<td>414</td>
</tr>
<tr>
<td>10</td>
<td>Women Physical Education</td>
<td>186.6</td>
<td>0.0</td>
<td>2865</td>
<td>0</td>
</tr>
<tr>
<td>11</td>
<td>Faculty of Sciences</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>Mathematics</td>
<td>143.4</td>
<td>105.2</td>
<td>7689</td>
<td>66</td>
</tr>
<tr>
<td>13</td>
<td>Geology</td>
<td>108.7</td>
<td>127.0</td>
<td>2165</td>
<td>266</td>
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<tr>
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<td>Biology</td>
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<td>134.4</td>
<td>3963</td>
<td>315</td>
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<td>Chemistry</td>
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<td>236.8</td>
<td>6643</td>
<td>236</td>
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<td>Physics</td>
<td>146.3</td>
<td>124.0</td>
<td>4611</td>
<td>128</td>
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<td>17</td>
<td>Faculty of Education</td>
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<td>50.8</td>
<td>4578</td>
<td>217</td>
</tr>
<tr>
<td>22</td>
<td>Institute of Mathematics</td>
<td>0.0</td>
<td>91.3</td>
<td>0</td>
<td>508</td>
</tr>
</tbody>
</table>

APPENDIX: Inputs and Outputs for 19 Academic Units of the UTE in the First Semester, 1993-94.
References:

Galois Theory, Joseph Rotman, Publication 1990.


شبکه زیر میدانهای \( Q(\sqrt{p}, \sqrt{q}, \sqrt{r}) \)

شبکه زیر گروه‌های \( \text{Gal}_Q(p) \)
برهان: بنابر بحث‌های قبل از قضیه ۴ اگر \( \gamma \) انگاه \( \gamma = \sqrt{p} + \sqrt{q} + \sqrt{t} \) باشد خواهیم داشت \( Q(\gamma) = Q(\sqrt{p}, \sqrt{q}, \sqrt{t}) \in Q[\gamma] \). با این حال نتیجه چندجمله‌ای مینیمال \( \gamma \) روی \( Q \) درجه ۸ می‌باشد. لذا اگر \( \deg(h(x)) \geq 8 \) انگاه \( h(\gamma) = 0 \) خواهد بود.

بنابراین صفر \( f(x) \) است. از این که چندجمله‌ای مینیمال \( \gamma \) روی \( Q \) تحول یافته است. چه در غیر این صورت \( \gamma \) درجه ۸ چندجمله‌ای از درجه کوچکتر از ۸ می‌باشد که کنارگیری می‌گردد.

با فرض \( 0 \leq i \leq 15 \), \( K_i = \phi(H_i) \) طویل داشته:\n
\[
\begin{align*}
K_0 &= Q(\sqrt{p}, \sqrt{q}, \sqrt{t}), \quad K_1 = Q(\sqrt{q}, \sqrt{t}), \quad K_{15} = Q(\sqrt{p}, \sqrt{q}) \\
K_5 &= Q(\sqrt{t}, \sqrt{pq}), \quad K_6 = Q(\sqrt{q}, \sqrt{pt}), \quad K_7 = Q(\sqrt{p}, \sqrt{qt}), \quad K_{16} = Q(\sqrt{pt}, \sqrt{qt}) \\
K_8 &= Q(\sqrt{t}), \quad K_9 = Q(\sqrt{q}), \quad K_10 = Q(\sqrt{pt}), \quad K_{11} = Q(\sqrt{pq}), \quad K_{12} = Q(\sqrt{qt}) \\
K_13 &= Q(\sqrt{pq}), \quad K_{14} = Q(\sqrt{pt}), \quad K_{17} = Q.
\end{align*}
\]

در صفحه بعد شکل زیرگره‌های \( \text{Gal}_Q(f) \) و شکل زیرپدیدهای \( Q(\sqrt{p}, \sqrt{q}, \sqrt{t}) \) را جهت مقایسه نشان می‌دهم.
[\mathcal{Q}(\sqrt{p}, \sqrt{q}, \sqrt{t}) : \mathcal{Q}] = |\text{Gal}_\mathcal{Q}(f)| = \mathcal{V}.

لذا (دارای 16 زیرگروه به شرح زیر است:)

\[ H_0 = \{\sigma_0\}, \quad H_1 = \{\sigma_0, \sigma_1\}, \quad H_2 = \{\sigma_0, \sigma_1, \sigma_2\}, \quad H_3 = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3\} \]
\[ H_4 = \{\sigma_0, \sigma_3\}, \quad H_5 = \{\sigma_0, \sigma_3, \sigma_4\}, \quad H_6 = \{\sigma_0, \sigma_3, \sigma_4, \sigma_5\} \]
\[ H_7 = \{\sigma_0, \sigma_4\}, \quad H_8 = \{\sigma_0, \sigma_4, \sigma_5\}, \quad H_9 = \{\sigma_0, \sigma_4, \sigma_5, \sigma_6\} \]
\[ H_{10} = \{\sigma_1, \sigma_2\}, \quad H_{11} = \{\sigma_1, \sigma_2, \sigma_3\}, \quad H_{12} = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\} \]
\[ H_{13} = \text{Gal}_\mathcal{Q}(f). \]

فرض کنید \( K \leq \mathcal{Q}(\sqrt{p}, \sqrt{q}, \sqrt{t}) \) و \( A = \{H_i | 0 \leq i \leq 15\} \)

\[ \psi : A \rightarrow B, \]

\[ H_i \sim \phi(H_i) \]

\[ \mathcal{Q}(\sqrt{p}, \sqrt{q}, \sqrt{t}) : \mathcal{Q} \]

\( Q \) زوج اعداد جبری باشد و \( Q(\sqrt[p]{p}) \)

\[ \alpha = u \sqrt[p]{p} + b \sqrt[q]{q} + c \sqrt[t]{t} \]

\( \alpha \notin \phi(H_i) \)

\( Q(\alpha) = 0 \) و \( Q(\alpha) \neq \phi(H_i), \quad 0 \leq i \leq 15 \)

\( Q(\alpha) = 0 \) و \( Q(\alpha) \neq \phi(H_i), \quad 0 \leq i \leq 15 \)

\[ \text{تصویر نمی‌گردد. بنابراین (15) پیمان‌های} \]

\[ \text{بانابراین زنجیره‌ای می‌باشد. در نتیجه} \]

\[ \text{از درجه} \quad 8 \text{می‌باشد، در نتیجه} \quad \alpha \text{از درجه} \quad 8 \text{می‌باشد، در نتیجه} \]

\[ \deg(g(x)) \geq 8 \text{آنها} \]

\[ g(\alpha) = \text{آنها} \]

\[ \text{بانابراین دو عدد باشند اکثر باشند آنها} \]

\[ f(x) = x^8 - \frac{1}{2}(p + q + t)x^6 + \frac{1}{2}[(p + q + t)^2 + (p^2 + q^2 + t^2)]x^4 \]
\[ - \frac{1}{2}[(p + q + t)(p^2 + q^2 + t^2) - 2pq - 2pt - 2qt + 14pq]x^2 \]
\[ + (p^2 + q^2 + t^2 - 2pq - 2pt - 2qt)^2, \]

\[ \text{روی} \quad \mathcal{Q} \text{تحويل باید ایست.} \]
برهان: \[ Q(i, \sqrt{m}) = Q(i + \sqrt{m}) : Q = 4 \]

بنابراین، ریشه‌ی درجه ۴ از یک چندجمله‌ای محاسبه‌شده در جبر اگر در جبر اتوبال می‌باشد، باشد باشد به‌قسمت که آگاهی از آن باشد، باشد پل‌پوش باشد در جبر اتوبال در جبر اتوبال می‌باشد.

\[ \alpha = i + \sqrt{m}, \]
\[ \alpha^2 = -1 + m + 2i\sqrt{m}, \]
\[ \alpha^3 + (1 - m)^2 + 2(1 - m)\alpha = -4m, \]
\[ \alpha^4 + 2(1 - m)\alpha^2 + (1 + m)^2 = 0. \]

بنابراین، حاصل ترکیب ترکیب باشد باشد در جبر اتوبال می‌باشد، در جبر اتوبال می‌باشد.

اگر کدی \( f = (x^2 - p)(x^2 - q)(x^2 - t) \) باشد، \( q \) و \( t \) باشد، لازم می‌باشد، باشد باشد پل‌پوش باشد در جبر اتوبال می‌باشد، باشد باشد پل‌پوش باشد در جبر اتوبال می‌باشد.

\[ Q(\sqrt{p}, \sqrt{q}, \sqrt{t}) / Q \]

فرزند کدی \( f = (x^2 - p)(x^2 - q)(x^2 - t) \) باشد، لازم می‌باشد، باشد باشد پل‌پوش باشد در جبر اتوبال می‌باشد.

\[ \{ a + b\sqrt{p} + c\sqrt{q} + d\sqrt{t} : a, b, c, d \in Q \}, \]

\[ q = \{ \sigma, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7 \}. \]

اگر گروه متشکل از \( \sigma \in \text{Gal}(f) \) باشد، خلاصه ای در جبر اتوبال می‌باشد، باشد باشد پل‌پوش باشد در جبر اتوبال می‌باشد.

\[ \sigma(\sqrt{q}) = \pm \sqrt{q} \text{ و } \sigma(\sqrt{t}) = \pm \sqrt{t} \]
برهان: چون $\alpha$ به هیچیک از قویمیانهای $Q(a), Q(\sqrt{pq}), Q(\sqrt{q})$، $Q(\sqrt{p})$ ثابت که با هیچیک از ریاضیات به نسبت $Q(\sqrt{p}, \sqrt{q})$ تعادل ندارد، و $Q(\alpha) = Q(\sqrt{p}, \sqrt{q})$ است که $\alpha = a\sqrt{p} + b\sqrt{q}$. اگر عدد صحیح و مثبت $p$ و $q$، $p \neq q$ و $a$ و $b$ اعداد گویایی باشند، به همراه با عدد صحیح $x$ در $Q$ نتیجه گرفته که $f(x) = x^3 - 3(p + q)x^2 + (p^2 + q^2) = 0$ و $\deg(g(x)) \geq 4$، آنگاه $g(a) = 0$.

نتیجه یک: اگر $p$, $q$، $p \neq q$، عدد صحیح و مثبت باشند، به همراه با عدد صحیح $x$ در $Q$ نتیجه گرفته که $f(x) = x^3 - 3(p + q)x^2 + (p^2 + q^2) = 0$ و $\deg(g(x)) \geq 4$، آنگاه $g(a) = 0$.

نتیجه یک: اگر $p$, $q$، $p \neq q$، عدد صحیح و مثبت باشند، به همراه با عدد صحیح $x$ در $Q$ نتیجه گرفته که $f(x) = x^3 - 3(p + q)x^2 + (p^2 + q^2) = 0$ و $\deg(g(x)) \geq 4$، آنگاه $g(a) = 0$.

نتیجه یک: اگر $p$, $q$، $p \neq q$، عدد صحیح و مثبت باشند، به همراه با عدد صحیح $x$ در $Q$ نتیجه گرفته که $f(x) = x^3 - 3(p + q)x^2 + (p^2 + q^2) = 0$ و $\deg(g(x)) \geq 4$، آنگاه $g(a) = 0$.
یک اگر که شامل $E$ و شامل $F$ است. به علاکس اگر $K$ زیرمیدانی از $E$ که شامل $F$ است. جواب: $\{\sigma \in \text{Gal}_F(f) \mid \forall k \in K(\sigma(k) = k)\}$

$$A = \{K \mid \text{است} F \text{که شامل} E \text{و شامل} K\}$$

$$B = \{h \mid \text{است} \text{Gal}_H(f) \text{که شامل} F \text{و شامل} H\}$$

اثبات تابع $(1)$ با ضابطه $B \rightarrow A$

فرض کنید $p, q$ دو عدد خالی از مرز باشنده به قسمتی که $\sqrt{p}, \sqrt{q}$ این صورت $\frac{Q(\sqrt{p}, \sqrt{q})}{Q}$ یک توزیع میدان تجزیه‌ای $f$ است.

$$Q(\sqrt{p}, \sqrt{q}) = \{a_n + a_1 \sqrt{p} + a_2 \sqrt{q} + a_3 \sqrt{pq} \mid a_i \in Q\}$$

$$|\text{Gal}_f(f)| = [Q(\sqrt{p}, \sqrt{q}) : Q] = 4.$$

با ضابطه $Q(\sqrt{p}, \sqrt{q})$، $\text{Gal}_f(f) = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ که در عضو $Q$ را تابع زیر میدانی به شرح زیر می‌باشد:

$$\begin{align*}
\sqrt{p} &\rightarrow -\sqrt{p}, \\
\sqrt{q} &\rightarrow -\sqrt{q}, \\
\sqrt{pq} &\rightarrow -\sqrt{pq}
\end{align*}$$

BABAYEN (f) دارای زیرگروههای $\text{Gal}_f(f) = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$.

در نتیجه $\text{Zيرمیدانهای متناضر}$، $H$ به این رابطه $(1)$ تناش زیرمیدانهای $Q(\sqrt{p}, \sqrt{q})$ نیز می‌باشد به شرح زیر می‌باشد:

$$\phi(H_0) = Q(\sqrt{p}, \sqrt{q})$$

$$\phi(H_1) = Q(\sqrt{q}), \quad \phi(H_2) = Q(\sqrt{p}), \quad \phi(H_3) = Q(\sqrt{pq})$$

لم 1: اگر عدد $b, a$ انگه $a = a\sqrt{p} + b\sqrt{q}$ و $b \neq p, q \neq p$ در عدد خالی از مرز باشنده به قسمتی که $Q(\alpha) = Q(\sqrt{p}, \sqrt{q})$ آنگه $\alpha = a\sqrt{p} + b\sqrt{q}$.