Characterization of Filters Preserving Reciprocality

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ABSTRACT

In this paper we characterize the system function of a linear filter that its output will be a reciprocal process whenever its input is a reciprocal one.

Introduction

Let $X = \{X(t), -\infty < t < \infty\}$ be a process defined on some complete probability space $(\Omega, \mathcal{F}, P)$. The notion of reciprocity was first defined by Jamison [1], and studied in some extend by pasha [2] and [3]. The process $X$ has reciprocal property on $(-\infty, \infty)$ if for each $n \in \mathbb{N}$, and for each reals $u < v$, and for each reals $t_1, \ldots, t_n$ in the complement of interval $(u, v)$, and finally for each $t \in (u, v)$, the conditional distribution of $X_t$ given $X_u, X_{u+}, X_{u-}, \ldots, X_{t_n}$ is the same as the conditional distribution of $X_t$ given $X_u$ and $X_r$.

In [2] a martingale representation of Gaussian stationary reciprocal processes is given. In [3] the notion of reciprocity is generalized. Jamison [1] proved that the covariance function of Gaussian stationary reciprocal processes with zero mean is of the following form

$$C_X(t) = E(X(s)X(t+s)) = be^{-a|t|} \quad t \in \mathbb{R},$$

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for some positive numbers a and b. It is clear that $\sigma^2(X(t)) = b$.

In this paper we make the following assumptions:

Assumption A: we assume that the process $X$ satisfies the following conditions:

(i) $X$ is Gaussian,

(ii) $X$ is stationary,

(iii) The mean of $X_t$ is zero

(iv) The covariance function of $X_t$ is continuous,

(v) $X$ has reciprocal property on $(-\infty, \infty)$.

Linear filters

Let $X$ be the input of a linear filter with quasi system function $h$, i.e.

$$h(t) = 0, \quad t \leq 0.$$ 

Let $Y = \{y(t), -\infty < t < \infty\}$ be the output of the system, i.e.

$$Y(t) = \int_{-\infty}^{\infty} h(t)X(t-s)ds.$$ 

It is well known that if the process $X$ is Gaussian and stationary then the output process $Y$ also is Gaussian and stationary. In the following we want to determine the function $h$ so that if $X$ satisfies assumption A, then $Y$ satisfies the assumption A, specifically it has reciprocal property.

If $X$ is stationary then the covariance function of $Y$ is given by

$$C_Y(t) = E(y(t+s)y(s)) = \int_{-\infty}^{\infty} h(s)C_X(s+t)ds = C_X(t) * h(-s)$$

where $\ast$ stands for the convolution of the function $C_X(t)$ and $h_1(t) = h(-t)$.

We will use the following notions in the sequel:

$$C_X(t) = E(X(t+s)X(s))$$

$$C_Y(t) = E(Y(t+s)Y(s))$$

$$C_{XY}(t) = E(X(t+s)Y(s))$$

$$S_X(w) = \int_{-\infty}^{\infty} e^{-iwt}C_X(t)dt,$$

$$H(w) = \int_{0}^{\infty} e^{-iwt}h(t)dt.$$ 

$S_Y(w), S_{XY}(w)$ will be defined similarly.

Lemma. Let $X$ satisfies assumption A (i)-(iv), then

$$S_{XY}(w) = S_X(w)H(-w)$$

$$S_Y(w) = S_{XY}(w)H(w).$$

Proof. We have

$$C_{XY} = \int_{0}^{\infty} h(s)C_X(s+t)ds = \int_{-\infty}^{\infty} h(s)C_X(s+t)ds = \int_{-\infty}^{\infty} h(-s)C_X(t-s)ds = (C_X \ast h_1)(t)$$

where $h_1(t) = h(-t)$. Therefore by taking the Fourier transform we will have

$$S_{XY}(w) = S_X(w)H_1(w) = S_X(w)H(-w).$$

Where $H_1(w)$ is the fourier transform of $h_1$, which is equal to $H(-w)$. A similar computation will prove the second equality.

Now we have the following theorem.
Theorem 1. Let \( X \) satisfies assumption \( A ((i)-(v)) \) and \( C_X(t) = be^{-a|t|} \). Let \( Y \) be the output of the linear quasi system with system function \( h \). Then \( Y \) is reciprocal if and only if
\[
H(w)H(-w) = \frac{a'b'(a^2 + w^2)}{ab(a'^2 + w^2)}.
\]
for some positive numbers \( a, b, a', b' \).

Proof. Assume that the input and output of the system satisfies assumption \( A \). From
\[
C_X(t) = be^{-a|t|}
\]
we get
\[
S_X(w) = \frac{2ab}{a^2 + w^2}.
\]
Similarly, for some \( a' > 0, b' > 0 \), we have
\[
C_Y(t) = b'e^{-a'|t|}
\]
Therefore
\[
S_Y(w) = \frac{2a'b'}{a'^2 + w^2}.
\]
But, from lemma 1, we have
\[
S_Y(w) = S_{XY}(w)H(w)
= S_{XY}(w).H(-w)H(w)
\]
Therefore
\[
\frac{2a'b'}{a'^2 + w^2} = \frac{2ab}{a^2 + w^2}H(w)H(-w)
\]
From here we get:
\[
H(w)H(-w) = \frac{a'b'(a^2 + w^2)}{ab(a'^2 + w^2)}.
\]
Now assume that \( H \) satisfies the above relation and the input process satisfies assumption \( A ((i)-(v)) \), we prove that \( Y \) satisfies assumption \( A ((i)-(v)) \). The only property that we have to prove is the reciprocal property of \( Y \). From lemma 1 and the given condition on \( H \) we have
\[
S_Y(w) = S_X(w)H(w)H(-w)
= \frac{2ab}{a^2 + w^2} \frac{a'b'(a^2 + w^2)}{ab(a'^2 + w^2)}.
\]
Thus,
\[
S_Y(w) = \frac{2ab'}{a'^2 + w^2}.
\]
This is the Fourier transform of a function of the following form
\[
C_Y(t) = b'e^{-a'|t|}
\]
Now from the Jamison result in [1] we conclude that \( Y \) has reciprocal property.

Example: An example of this kind of filters is
\[
h(t) = \frac{b'}{bt^2 + b^2}.
\]
The Fourier transform of \( h \) is
\[
H(w) = \sqrt{\frac{b'}{b}} e^{-\pi w^2} \sqrt{b}
\]
Therefore, for any \( a > 0 \) we have
\[
H(w)H(-w) = \frac{b'}{b}
= \frac{b'a(a^2 + w^2)}{ba(a'^2 + w^2)}.
\]
This filter will take an input with covariance function
\[
C_X(t) = be^{-a|t|}
\]
to an output with covariance function
\[
C_Y(t) = b'e^{-a'|t|}.
\]
This filter gives more weight to the most recent input than to the most far inputs.

References

