Characterization of Filters Preserving Reciprocality

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ABSTRACT

In this paper we characterize the system function of a linear filter that its output will be a reciprocal process whenever its input is a reciprocal one.

Introduction

Let $X = \{X(t), -\infty < t < \infty\}$ be a process defined on some complete probability space $(\Omega, \mathcal{F}, P)$. The notion of reciprocity was first defined by Jamison [1], and studied in some extend by pasha [2] and [3]. The process $X$ has reciprocal property on $(-\infty, \infty)$ if for each $n \in \mathbb{N}$, and for each reals $u < v$, and for each reals $t_1, \ldots, t_n$ in the complement of interval $(u, v)$, and finally for each $t \in (u, v)$, the conditional distribution of $X_t$ given $X_{u}, X_{v}, X_{t_1}, \ldots, X_{t_n}$ is the same as the conditional distribution of $X_t$ given $X_u$ and $X_v$.

In [2] a martingale representation of Gaussian stationary reciprocal processes is given. In [3] the notion of reciprocity is generalized. Jamison [1] proved that the covariance function of Gaussian stationary reciprocal processes with zero mean is of the following form

$$C_X(t) = E(X(s)X(t+s)) = be^{-\alpha|t|} \quad t \in \mathbb{R},$$

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for some positive numbers a and b. It is clear that \( \sigma^2(X(t)) = b \).

In this paper we make the following assumptions:

Assumption A: we assume that the process \( X \) satisfies the following conditions:

(i) \( X \) is Gaussian,
(ii) \( X \) is stationary,
(iii) The mean of \( X_t \) is zero
(iv) The covariance function of \( X_t \) is continuous,
(v) \( X \) has reciprocal property on \( (-\infty, \infty) \).

**Linear filters**

Let \( X \) be the input of a linear filter with quasi system function \( h \), i.e.

\[
h(t) = 0, \quad t \leq 0.
\]

Let \( Y = \{y(t), -\infty < t < \infty \} \) be the output of the system, i.e.

\[
Y(t) = \int_0^\infty h(t)X(t-s)ds.
\]

It is well known that if the process \( X \) is Gaussian and stationary then the output process \( Y \) also is Gaussian and stationary. In the following we want to determine the function \( h \) so that if \( X \) satisfies assumption A, then \( Y \) satisfies the assumption A, specifically it has reciprocal property.

If \( X \) is stationary then the covariance function of \( Y \) is given by

\[
C_Y(t) = E(y(t+s)y(s)) = \int_0^\infty h(s)C_X(s+t)ds = C_X(t) * h(-s)
\]

where \( * \) stands for the convolution of the function \( C_X(t) \) and \( h_1(t) = h(-t) \).

We will use the following notions in the sequel:

\[
C_X(t) = E(X(t+s)X(s)),
C_Y(t) = E(Y(t+s)Y(s)),
C_{XY}(t) = E(X(t+s)Y(s)),
S_X(w) = \int_{-\infty}^{\infty} e^{-jwt}C_X(t)dt,
H(w) = \int_{0}^{\infty} e^{-jwt}h(t)dt.
\]

\( S_Y(w), S_{XY}(w) \) will be defined similarly.

**Lemma.** Let \( X \) satisfies assumption A ((i)-(iv)), then

\[
S_{XY}(w) = S_X(w)H(-w)
S_Y(w) = S_{XY}(w)H(w).
\]

**Proof.** We have

\[
C_{XY} = \int_0^\infty h(s)C_X(s+t)ds
= \int_{-\infty}^{\infty} h(s)C_X(s+t)ds
= \int_{-\infty}^{\infty} h(-s)C_X(t-s)ds
= (C_X * h_1)(t)
\]

where \( h_1(t) = h(-t) \). Therefore by taking the Fourier transform we will have

\[
S_{XY}(w) = S_X(w)H_1(w)
= S_X(w)H(-w).
\]

Where \( H_1(w) \) is the fourier transform of \( h_1 \), which is equal to \( H(-w) \). A similar computation will prove the second equality.

Now we have the following theorem.
Theorem 1. Let $X$ satisfies assumption $A ((i)-(v))$ and $C_X(t) = be^{-at}$. Let $Y$ be the output of the linear quasi system with system function $h$. Then $Y$ is reciprocal if and only if

$$H(w)H(-w) = \frac{a'b'(a^2 + w^2)}{ab(a'^2 + w^2)}$$

for some positive numbers $a, b, a', b'$.

Proof. Assume that the input and output of the system satisfies assumption $A$. From

$$C_X(t) = be^{-at}$$

we get

$$S_X(w) = \frac{2ab}{a^2 + w^2}.$$ 

Similarly, for some $a' > 0, b' > 0$, we have

$$C_Y(t) = b'e^{-a't}.$$ 

Therefore

$$S_Y(w) = \frac{2a'b'}{a'^2 + w'^2}.$$ 

But, from lemma 1, we have

$$S_Y(w) = S_{XY}(w)H(w) = S_{XY}(w)H(-w)H(w).$$

Therefore

$$2a'b' = \frac{2ab}{a^2 + w^2}H(w)H(-w)$$

From here we get

$$H(w)H(-w) = \frac{a'b'(a^2 + w^2)}{ab(a'^2 + w'^2)}.$$ 

Now assume that $H$ satisfies the above relation and the input process satisfies assumption $A ((i)-(v))$, we prove that $Y$ satisfies assumption $A ((i)-(v))$. The only property that we have to prove is the reciprocal property of $Y$. From lemma 1 and the given condition on $H$ we have

$$S_Y(w) = \frac{2ab}{a^2 + w^2}.$$ 

Thus,

$$S_Y(w) = \frac{2a'b'}{a'^2 + w'^2}.$$ 

This is the Fourier transform of a function of the following form

$$C_Y(t) = b'e^{-a't}.$$ 

Now from the Jamison result in [1] we conclude that $Y$ has reciprocal properly.

Example: An example of this kind of filters is

$$h(t) = \frac{b'}{bt^2 + b\pi^2}.$$ 

The Fourier transform of $h$ is

$$H(w) = \sqrt{\frac{b}{b}} e^{-\pi u \sqrt{b}}.$$ 

Therefore, for any $a > 0$ we have

$$H(w)H(-w) = \frac{b'}{b}$$

$$= \frac{b'a(a^2 + w^2)}{ba(a'^2 + w'^2)}.$$ 

This filter will take an input with covariance function

$$C_X(t) = be^{-at}$$

to an output with covariance function

$$C_Y(t) = b'e^{-a't}.$$
This filter gives more weight to the most recent input than to the most far inputs.

References

