Characterization of Filters Preserving Reciprocal

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ABSTRACT

In this paper we characterize the system function of a linear filter that its output will be a reciprocal process whenever its input is a reciprocal one.

Introduction

Let \( X = \{X(t), -\infty < t < \infty\} \) be a process defined on some complete probability space \((\Omega, \mathcal{F}, P)\). The notion of reciprocal-ality was first defined by Jamison [1], and studied in some extent by Pasha [2] and [3]. The process \( X \) has reciprocal property on \((-\infty, \infty)\) if for each \( n \in \mathbb{N} \), and for each reals \( u < v \), and for each reals \( t_1, \ldots, t_n \) in the complement of interval \((u, v)\), and finally for each \( t \in (u, v)\), the conditional distribution of \( X_t \) given \( X_u, X_{u_1}, \ldots, X_{u_n} \) is the same as the conditional distribution of \( X_t \) given \( X_u \) and \( X_v \).

In [2] a martingale representation of Gaussian stationary reciprocal processes is given. In [3] the notion of reciprocality is generalized. Jamison [1] proved that the covariance function of Gaussian stationary reciprocal processes with zero mean is of the following form

\[
C_X(t) = E(X(s)X(t+s)) = be^{-at} \quad t \in \mathbb{R},
\]

Key words: Gaussian, Stationary, Reciprocal, Filters.
for some positive numbers $a$ and $b$. It is clear that $\sigma^2(X(t)) = b$.

In this paper we make the following assumptions:
Assumption A: we assume that the process $X$ satisfies the following conditions:

(i) $X$ is Gaussian,
(ii) $X$ is stationary,
(iii) The mean of $X_t$ is zero
(iv) The covariance function of $X_t$ is continuous,
(v) $X$ has reciprocal property on $(-\infty, \infty)$.

Linear filters

Let $X$ be the input of a linear filter with quasi system function $h$, i.e.

\[ h(t) = 0, \quad t \leq 0. \]

Let $Y = \{y(t), -\infty < t < \infty\}$ be the output of the system, i.e.

\[ Y(t) = \int_{0}^{\infty} h(t)X(t-s)ds. \]

It is well known that if the process $X$ is Gaussian and stationary then the output process $Y$ also is Gaussian and stationary. In the following we want to determine the function $h$ so that if $X$ satisfies assumption A, then $Y$ satisfies the assumption A, specifically it has reciprocal property.

If $X$ is stationary then the covariance function of $Y$ is given by

\[ C_Y(t) = E(y(t+s)y(s)) = \int_{0}^{\infty} h(s)C_X(s+t)ds = C_X(t) \ast h(-s) \]

where $\ast$ stands for the convolution of the function $C_X(t)$ and $h_1(t) = h(-t)$.

We will use the following notions in the sequel:

\[ C_X(t) = E(X(t+s)X(s)), \]
\[ C_Y(t) = E(Y(t+s)Y(s)) \]
\[ C_{XY}(t) = E(X(t+s)Y(s)) \]
\[ S_X(w) = \int_{-\infty}^{\infty} e^{-i\omega t}C_X(t)dt, \]
\[ H(w) = \int_{0}^{\infty} e^{-i\omega t}h(t)dt \]
\[ S_{XY}(w) \text{ and } S_{XY}(w) \text{ will be defined similarly.} \]

Lemma. Let $X$ satisfies assumption A ((i)-(iv)), then

\[ S_{XY}(w) = S_X(w)H(-w) \]
\[ S_Y(w) = S_{XY}(w)H(w). \]

Proof. We have

\[ C_{XY} = \int_{0}^{\infty} h(s)C_X(s+t)ds \]
\[ = \int_{-\infty}^{\infty} h(s)C_X(s+t)ds \]
\[ = \int_{-\infty}^{\infty} h(-s)C_X(t-s)ds \]
\[ = (C_X \ast h_1)(t) \]

where $h_1(t) = h(-t)$. Therefore by taking the Fourier transform we will have

\[ S_{XY}(w) = S_X(w)H_1(w) \]
\[ = S_X(w)H(-w). \]

Where $H_1(w)$ is the Fourier transform of $h_1$, which is equal to $H(-w)$. A similar computation will prove the second equality.

Now we have the following theorem.
Theorem 1. Let $X$ satisfies assumption $A (\text{(i)-(v)})$ and $C_{X}(t) = be^{-a|t|}$. Let $Y$ be the output of the linear quasi system with system function $h$. Then $Y$ is reciprocal if and only if

$$H(w)H(-w) = \frac{a'b'(a^2 + w^2)}{ab(a'^2 + w^2)}$$

for some positive numbers $a, b, a', b'$.

Proof. Assume that the input and output of the system satisfies assumption $A$. From

$$C_{X}(t) = be^{-a|t|}$$

we get

$$S_{X}(w) = \frac{2ab}{a^2 + w^2}.$$  

Similarly, for some $a' > 0, b' > 0$, we have

$$C_{Y}(t) = b'e^{-a'|t|},$$

Therefore

$$S_{Y}(w) = \frac{2a'b'}{a'^2 + w^2}.$$  

But, from lemma 1, we have

$$S_{Y}(w) = S_{X}(w)H(w) = S_{XY}(w)H(w).$$

Therefore

$$\frac{2a'b'}{a'^2 + w^2} = \frac{2ab}{a^2 + w^2}H(w)H(-w).$$

From here we get

$$H(w)H(-w) = \frac{a'b'(a^2 + w^2)}{ab(a'^2 + w^2)}.$$  

Now assume that $H$ satisfies the above relation and the input process satisfies assumption $A (\text{(i)-(v)})$, we prove that $Y$ satisfies assumption $A (\text{(i)-(v)})$. The only property that we have to prove is the reciprocal property of $Y$. From lemma 1 and the given condition on $H$ we have

$$S_{Y}(w) = S_{X}(w)H(w)H(-w) = \frac{2ab}{a^2 + w^2} \frac{a'b'(a^2 + w^2)}{ab(a'^2 + w^2)}.$$  

Thus,

$$S_{Y}(w) = \frac{2a'b'}{a'^2 + w^2}.$$  

This is the Fourier transform of a function of the following form

$$C_{Y}(t) = b'e^{-a'|t|}.$$  

Now from the Jamison result in [1] we conclude that $Y$ has reciprocal properly.

Example: An example of this kind of filters is

$$h(t) = \frac{b'}{bt^2 + b\pi^2}.$$  

The Fourier transform of $h$ is

$$H(w) = \sqrt{\frac{b'}{b}} e^{-\pi w\sqrt{b'}}.$$  

Therefore, for any $a > 0$ we have

$$H(w)H(-w) = \frac{b'}{b} = \frac{b'a(a^2 + w^2)}{ba(a'^2 + w^2)}.$$  

This filter will take an input with covariance function

$$C_{X}(t) = be^{-a|t|}$$

to an output with covariance function

$$C_{Y}(t) = b'e^{-a'|t|}.$$
This filter gives more weight to the most recent input than to the most far inputs.

References

