Characterization of Filters Preserving Reciprocity

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ABSTRACT

In this paper we characterize the system function of a linear filter that its output will be a reciprocal process whenever its input is a reciprocal one.

Introduction

Let $X = \{X(t), -\infty < t < \infty\}$ be a process defined on some complete probability space $(\Omega, \mathcal{F}, P)$. The notion of reciprocity was first defined by Jamison [1], and studied in some extend by pasha [2] and [3]. The process $X$ has reciprocal property on $(-\infty, \infty)$ if for each $n \in \mathbb{N}$, and for each reals $u < v$, and for each reals $t_1, \ldots, t_n$ in the complement of interval $(u, v)$, and finally for each $t \in (u, v)$, the conditional distribution of $X_t$ given $X_u, X_{u+1}, \ldots, X_t$, is the same as the conditional distribution of $X_t$ given $X_u$ and $X_t$.

In [2] a martingale representation of Gaussian stationary reciprocal processes is given. In [3] the notion of reciprocity is generalized. Jamison [1] proved that the covariance function of Gaussian stationary reciprocal processes with zero mean is of the following form

$$C_X(t) = E(X(s)X(t+s)) = be^{-\alpha|t|}, \quad t \in \mathbb{R},$$

Key words: Gaussian, Stationary, Reciprocal, Filters.
for some positive numbers a and b. It is clear that $\sigma^2(X(t)) = b$.

In this paper we make the following assumptions:

**Assumption A:** we assume that the process X satisfies the following conditions:

(i) X is Gaussian, 
(ii) X is stationary, 
(iii) The mean of $X_t$ is zero 
(iv) The covariance function of $X_t$ is continuous, 
(v) X has reciprocal property on $(-\infty, \infty)$.

**Linear filters**

Let X be the input of a linear filter with quasi system function h, i.e.

$$h(t) = 0, \quad t \leq 0.$$ 

Let $Y = \{y(t), -\infty < t < \infty\}$ be the output of the system, i.e.

$$Y(t) = \int_{0}^{\infty} h(t)X(t-s)ds.$$ 

It is well known that if the process X is Gaussian and stationary then the output process Y also is Gaussian and stationary. In the following we want to determine the function h so that if X satisfies assumption A, then Y satisfies the assumption A, specifically it has reciprocal property.

If X is stationary then the covariance function of Y is given by

$$C_Y(t) = E(y(t+s)y(s))$$

$$= \int_{0}^{\infty} h(s)C_X(s+t)ds$$

$$= C_X(t) * h(-s)$$

where $*$ stands for the convolution of the function $C_X(t)$ and $h_1(t) = h(-t)$.

We will use the following notions in the sequel:

$$C_X(t) = E(X(t+s)X(s))$$

$$C_Y(t) = E(Y(t+s)Y(s))$$

$$C_{XY}(t) = E(X(t+s)Y(s))$$

$$S_X(w) = \int_{-\infty}^{\infty} e^{-iwt}C_X(t)dt,$$

$$H(w) = \int_{0}^{\infty} e^{-iwt}h(t)dt.$$ 

$S_Y(w), S_{XY}(w)$ will be defined similarly.

**Lemma.** Let X satisfies assumption A ((i)-(iv)), then

$$S_{XY}(w) = S_X(w)H(-w)$$

$$S_Y(w) = S_{XY}(w)H(w).$$

**Proof.** We have

$$C_{XY} = \int_{0}^{\infty} h(s)C_X(s+t)ds$$

$$= \int_{-\infty}^{\infty} h(s)C_X(s+t)ds$$

$$= \int_{-\infty}^{\infty} h(-s)C_X(t-s)ds$$

$$= (C_X * h_1)(t)$$

where $h_1(t) = h(-t)$. Therefore by taking the Fourier transform we will have

$$S_{XY}(w) = S_X(w)H_1(w)$$

$$= S_X(w)H(-w).$$

Where $H_1(w)$ is the fourier transform of $h_1$, which is equal to $H(-w)$. A similar computation will prove the second equality.

Now we have the following theorem.
Theorem 1. Let $X$ satisfies assumption $A$ ((i)-(v)) and $C_X(t) = be^{-a|t|}$. Let $Y$ be the output of the linear quasi system with system function $h$. Then $Y$ is reciprocal if and only if

$$H(w)H(-w) = \frac{a'b'(a^2 + w^2)}{ab(a'^2 + w^2)}$$

for some positive numbers $a, b, a', b'$.

Proof. Assume that the input and output of the system satisfies assumption $A$. From

$$C_X(t) = be^{-a|t|}$$

we get

$$S_X(w) = \frac{2ab}{a^2 + w^2}.$$  

Similarly, for some $a' > 0$, $b' > 0$, we have

$$C_Y(t) = b'e^{-a'|t|},$$

Therefore

$$S_Y(w) = \frac{2ab}{a'^2 + w^2}.$$  

But, from lemma 1, we have

$$S_Y(w) = S_{XY}(w)H(w) = S_{XY}(w)H(-w)H(w).$$

Therefore

$$\frac{2ab}{a'^2 + w^2} = \frac{2ab}{a^2 + w^2}H(w)H(-w)$$

From here we get

$$H(w)H(-w) = \frac{a'b'(a^2 + w^2)}{ab(a'^2 + w^2)}.$$  

Now assume that $H$ satisfies the above relation and the input process satisfies assumption $A$ ((i)-(v)), we prove that $Y$ satisfies assumption $A$ ((i)-(v)). The only property that we have to prove is the reciprocal property of $Y$. From lemma 1 and the given condition on $H$ we have

$$S_Y(w) = S_X(w)H(w).H(-w) = \frac{2ab}{a^2 + w^2} = \frac{a'b'(a^2 + w^2)}{ab(a'^2 + w^2)}.$$  

Thus,

$$S_Y(w) = \frac{2ab}{a'^2 + w^2}.$$  

This is the Fourier transform of a function of the following form

$$C_Y(t) = b'e^{-a'|t|}.$$  

Now from the Jamison result in [1] we conclude that $Y$ has reciprocal property.

Example: An example of this kind of filters is

$$h(t) = \frac{b'}{bt^2 + b\pi^2},$$

The Fourier transform of $h$ is

$$H(w) = \sqrt{\frac{b}{b'}} e^{-\pi u} \sqrt{\frac{b}{b'}}.$$  

Therefore, for any $a > 0$ we have

$$H(w)H(-w) = \frac{b'}{b'} = \frac{b'a(a^2 + w^2)}{ba(a^2 + w^2)}.$$  

This filter will take an input with covariance function

$$C_X(t) = be^{-a|t|}$$

to an output with covariance function

$$C_Y(t) = b'e^{-a'|t|}.$$  

This filter gives more weight to the most recent input than to the most far inputs.

References

