Characterization of Filters Preserving Reciprocality

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ABSTRACT

In this paper we characterize the system function of a linear filter that its output will be a reciprocal process whenever its input is a reciprocal one.

Introduction

Let \( X = \{ X(t), -\infty < t < \infty \} \) be a process defined on some complete probability space \((\Omega, \mathcal{F}, P)\). The notion of reciprocity was first defined by Jamison [1], and studied in some extend by pasha [2] and [3]. The process \( X \) has reciprocal property on \((-\infty, \infty)\) if for each \( n \in \mathbb{N} \), and for each reals \( u < v \), and for each reals \( t_1, \ldots, t_n \) in the complement of interval \((u, v)\), and finally for each \( t \in (u, v) \), the conditional distribution of \( X_t \) given \( X_u, X_v, X_{t_1}, \ldots, X_{t_n} \) is the same as the conditional distribution of \( X_t \) given \( X_u \) and \( X_v \).

In [2] a martingale representation of Gaussian stationary reciprocal processes is given. In [3] the notion of reciprocity is generalized. Jamison [1] proved that the covariance function of Gaussian stationary reciprocal processes with zero mean is of the following form

\[ C_X(t) = E(X(s)X(t+s)) = be^{-\alpha |t|} \quad t \in \mathbb{R}, \]

Key words: Gaussian, Stationary, Reciprocal, Filters.
for some positive numbers a and b. It is clear that $\sigma^2(X(t)) = b$.

In this paper we make the following assumptions:

Assumption A: we assume that the process $X$ satisfies the following conditions:

(i) $X$ is Gaussian,

(ii) $X$ is stationary,

(iii) The mean of $X_t$ is zero

(iv) The covariance function of $X_t$ is continuous,

(v) $X$ has reciprocal property on $(-\infty, \infty)$.

**Linear filters**

Let $X$ be the input of a linear filter with quasi system function $h$, i.e.

$$h(t) = 0, \quad t \leq 0.$$  

Let $Y = \{y(t), -\infty < t < \infty\}$ be the output of the system, i.e.

$$Y(t) = \int_{0}^{\infty} h(t)X(t-s)ds.$$  

It is well known that if the process $X$ is Gaussian and stationary then the output process $Y$ also is Gaussian and stationary.

In the following we want to determine the function $h$ so that if $X$ satisfies assumption $A$, then $Y$ satisfies the assumption $A$, specifically it has reciprocal property.

If $X$ is stationary then the covariance function of $Y$ is given by

$$C_Y(t) = E(y(t+s)y(s)) = \int_{0}^{\infty} h(s)C_X(s+t)ds = C_X(t) * h(-s)$$

where $\ast$ stands for the convolution of the function $C_X(t)$ and $h_1(t) = h(-t)$.

We will use the following notions in the sequel:

$$C_X(t) = E(X(t+s)X(s)), \quad C_Y(t) = E(Y(t+s)Y(s))$$

$$C_{XY}(t) = E(X(t+s)Y(s))$$

$$S_X(w) = \int_{-\infty}^{\infty} e^{-iwt}C_X(t)dt,$$

$$H(w) = \int_{0}^{\infty} e^{-iwt}h(t)dt.$$  

$S_Y(w), S_{XY}(w)$ will be defined similarly.

**Lemma.** Let $X$ satisfies assumption $A$ ((i)-(iv)), then

$$S_{XY}(w) = S_X(w)H(-w)$$

$$S_Y(w) = S_{XY}(w)H(w).$$

**Proof.** We have

$$C_{XY} = \int_{0}^{\infty} h(s)C_X(s+t)ds$$

$$= \int_{-\infty}^{\infty} h(s)C_X(s+t)ds$$

$$= \int_{-\infty}^{\infty} h(-s)C_X(t-s)ds$$

$$= (C_X * h_1)(t)$$

where $h_1(t) = h(-t)$. Therefore by taking the Fourier transform we will have

$$S_{XY}(w) = S_X(w)H_1(w)$$

$$= S_X(w)H(-w).$$

Where $H_1(w)$ is the fourier transform of $h_1$, which is equal to $H(-w)$. A similar computation will prove the second equality.

Now we have the following theorem.
Theorem 1. Let $X$ satisfy assumption $A$ ((i)-(v)) and $C_X(t) = be^{-\alpha|t|}$. Let $Y$ be the output of the linear quasi system with system function $h$. Then $Y$ is reciprocal if and only if

$$H(w)H(-w) = \frac{a'b'(a^2 + w^2)}{ab(a^2 + w^2)}$$

for some positive numbers $a, b, a', b'$.

**Proof.** Assume that the input and output of the system satisfies assumption $A$. From

$$C_X(t) = be^{-\alpha|t|}$$

we get

$$S_X(w) = \frac{2ab}{a^2 + w^2}.$$  

Similarly, for some $a' > 0$, $b' > 0$, we have

$$C_Y(t) = b'e^{-\alpha'|t|},$$

Therefore

$$S_Y(w) = \frac{2a'b'}{a'^2 + w^2}.$$  

But, from lemma 1, we have

$$S_Y(w) = S_{XY}(w)H(w) = S_{XY}(w)H(-w)H(w)$$

Therefore

$$\frac{2a'b'}{a'^2 + w^2} = \frac{2ab}{a^2 + w^2}H(w)H(-w)$$

From here we get:

$$H(w)H(-w) = \frac{a'b'(a^2 + w^2)}{ab(a^2 + w^2)}.$$  

Now assume that $H$ satisfies the above relation and the input process satisfies assumption $A$ ((i)-(v)), we prove that $Y$ satisfies assumption $A$ ((i)-(v)). The only property that we have to prove is the reciprocal property of $Y$. From lemma 1 and the given condition on $H$ we have

$$S_Y(w) = S_X(w)H(w)H(-w) = \frac{2ab}{a^2 + w^2} \frac{a'b'(a^2 + w^2)}{ab(a^2 + w^2)}.$$  

Thus,

$$S_Y(w) = \frac{2a'b'}{a^2 + w^2}.$$  

This is the Fourier transform of a function of the following form

$$C_Y(t) = b'e^{-\alpha'|t|}.$$  

Now from the Jamison result in [1] we conclude that $Y$ has reciprocal property.

**Example:** An example of this kind of filters is

$$h(t) = \frac{b'}{b't^2 + b'n^2},$$

The Fourier transform of $h$ is

$$H(w) = \sqrt{\frac{b'}{b}}e^{-\pi w} \sqrt{\frac{b}{b'}}$$

Therefore, for any $a > 0$ we have

$$H(w)H(-w) = \frac{b'}{b} = \frac{b'a(a^2 + w^2)}{ba(a^2 + w^2)}.$$  

This filter will take an input with covariance function

$$C_X(t) = be^{-\alpha|t|}$$  

to an output with covariance function

$$C_Y(t) = b'e^{-\alpha'|t|}.$$
This filter gives more weight to the most recent input than to the most far inputs.

References

