ON THE SINGULAR SETS OF A MODULE II

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Throughout this note, A and B will denote a (non-trivial) commutative Noetherian ring with a multiplicative identity element and M will denote a non-zero finitely generated A-module.

For every non-negative integer k, the set
\[ S^*_k(M) = \{ p \in \text{Spec}(A) \mid \text{depth } M_p + \dim A/p \leq k \} \]
is called the singular set of M with respect to k.

It is known that when the ring A is homomorphic image of a biequidimensional regular ring, then the singular sets of M are all closed in the Zariski topology on Spec(A) (see [3, ch. IV, 5]).

A development of this famous theorem has been recently shown in the sense that if A is a homomorphic image of a biequidimensional Gorenstein ring, the singular sets of M are still closed (Sec[2]).

The purpose of this article is to show that if B is a homomorphic image of a Cohen-Macaulay local ring, then \( S^*_k(N) \) is closed, for every finitely generated B-module N.

First we prove some preliminary lemmas and propositions which help us to conclude the subsequent main theorem. From now on, A will denote a Cohen-Macaulay local ring with the unique maximal ideal \( m \), and \( \hat{A} \) (respectively \( \hat{M} \)) will denote the maximal ideal of A (respectively M).

1. Proposition. Let \( \phi : A \to \hat{A} \) be a homomorphism. Then for every \( q \in \text{Spec}(\hat{A}) \),
\[ S^*_k(M) \iff p = q^0 \in \hat{S}^*_k(M) \text{ (for any } \hat{J} \text{ we write } J^0 \text{ for } \phi^{-1}(\hat{J})). \]

Proof. By [5,23.3], \[ \text{depth}_{\hat{A}^0} (M_p \otimes \hat{A}^0) = \text{depth}_A (M_p) - 1 \]
depth_{\mathfrak{A}_p}(M_p) + \text{depth}(\mathfrak{A}_q/p\mathfrak{A}_p\mathfrak{A}_q), \text{ since,}
\begin{align*}
\varphi : \mathfrak{A}_p &\rightarrow \mathfrak{A}_q \\
\begin{array}{c}
\mathfrak{a} \\
\mathfrak{s}
\end{array} &\mapsto \begin{array}{c}
\varphi(\mathfrak{a}) \\
\varphi(S)
\end{array}
\end{align*}
is a flat homomorphism. Also we have
\begin{align*}
M_p \otimes_{\mathfrak{A}_p} \mathfrak{A}_q &\equiv (M \otimes_{\mathfrak{A}_p} \mathfrak{A}_p) \otimes_{\mathfrak{A}_p} \mathfrak{A}_q \\
&\equiv M \otimes_{\mathfrak{A}_q} \mathfrak{A}_q \\
&\equiv \mathfrak{M} \otimes_{\mathfrak{A}_q} \mathfrak{A}_q \\
&\equiv \mathfrak{M}
\end{align*}
Thus we conclude that
\begin{align*}
\text{depth}_{\mathfrak{A}_q}(M_q) = \text{depth}_{\mathfrak{A}_p}(M_p) + \text{depth}(\mathfrak{A}_q/p\mathfrak{A}_p\mathfrak{A}_q).
\end{align*}
On the other hand, since \(\mathfrak{A}\) is Cohen-Macaulay, \(\mathfrak{A}\) is a Cohen-Macaulay local ring; whence, by corollary of [5,23.3], \(\mathfrak{A}_q/p\mathfrak{A}_p\mathfrak{A}_q\) is a Cohen-Macaulay ring. But
\[
\mathfrak{A}_q/p\mathfrak{A}_p\mathfrak{A}_q = \mathfrak{A}_q/p\mathfrak{A}_q.
\]
Hence
\[
\text{depth}(\mathfrak{A}_q) = \text{dim}(\mathfrak{A}_q).
\]
Moreover, by [5,15.1],
\[
\text{height } q = \text{height } p + \text{dim}(\mathfrak{A}_q).
\]
Hence
\[
\text{depth}_{\mathfrak{A}_q}(M_q) = \text{depth}_{\mathfrak{A}_p}(M_p) + \text{height } q - \text{height } p.
\]
From which we get, by [5,17.4],
\[
\text{depth}_{\mathfrak{A}_q}(M_q) + \text{dim}(\mathfrak{A}_q) = \text{depth}_{\mathfrak{A}_p}(M_p) + \text{dim}(\mathfrak{A}_q) - \text{height } p
\]
\[
= \text{depth}_{\mathfrak{A}_p}(M_p) + \text{dim}(\mathfrak{A}_q) - \text{height } p
\]
\[
= \text{depth}_{\mathfrak{A}_q}(M_q) + \text{dim}(\mathfrak{A}_q)
\]
The result now follows.

2. Proposition. With the same assumption as in Proposition 1, let \(\mathfrak{p} , \mathfrak{q} \subset \text{Spec}(\mathfrak{A})\) be prime ideals such that \(\mathfrak{p} \subset \mathfrak{q}\) and \(p \subset \text{Spec}(\mathfrak{A})\). Then \(\mathfrak{p} \subset \text{Spec}(\mathfrak{A})\).

Proof. Since \(\varphi : \mathfrak{A} \rightarrow \mathfrak{A}\) is a faithfully flat homomorphism, there exists \(\mathfrak{q} \subset \text{Spec}(\mathfrak{A})\) for which \((\mathfrak{q}^\varphi)^c = p\) (by [5,7.3]). But \(\varphi\) has the going down property (see [5,9.5]). Hence there is a prime ideal \(q \subset \text{Spec}(\mathfrak{A})\) such that \(q^c = p\) and \(q \subset q\). By Proposition 1, this implies that \(q \subset S^*_k(\mathfrak{M})\). But \(\mathfrak{A}\) is a homomorphic image of a regular local ring (see [5,29.4(ii)]); thus by [3], \(S^*_k(\mathfrak{M})\) is a closed subset of \(\text{Spec}(\mathfrak{A})\) (note that every Cohen-Macaulay local ring is bimequidimensional ring). This implies that \(q \subset S^*_k(\mathfrak{M})\). Again from Proposition 1, this in turn implies that \((q^c)^c = p \subset S^*_k(\mathfrak{M})\) as required.

3. Lemma. (See [4, ch.1, §6, Ex. 1]) Let \(R \subset T\) be rings and \(p \subset \text{Spec}(\mathfrak{A})\). Then there exists in \(T\) a prime ideal contracting to \(p\).

Proof. Let \(p\) be a minimal prime ideal of \(R\). Set \(S = R - p\), and
\[
K = \{a \mid a \subset S = \mathfrak{p} \& a \text{ is an ideal of } T\}.
\]
Then \(K\) have a maximal element which is prime ideal of \(T\). Let \(q\) be such prime ideal. Since \((q \cap R) \subset S = \mathfrak{p}\), we have \((q \cap R) \subset p\) and consequently \(q \cap R = p\).

We now turn to the main theorem of the note.

4. Theorem. For every positive integer \(k\), \(S^*_k(\mathfrak{M})\) is a closed subset of \(\text{Spec}(\mathfrak{A})\).

Proof: Since \(S^*_k(\mathfrak{M})\) is closed in \(\text{Spec}(\mathfrak{A})\), there exists an ideal \(J\) of \(\mathfrak{A}\) such that \(V(J) = S^*_k(\mathfrak{M})\). It is enough to show that
\[
V(J^c) = S^*_k(\mathfrak{M})
\]
Let \(p \subset S^*_k(\mathfrak{M})\). Hence there is \(q \subset S^*_k(\mathfrak{A})\) such that \(q^c = p\). Hence \(q \subset S^*_k(\mathfrak{M})\). Thus \(J \subset q\); this implies that \(J^c \subset q^c = p\); i.e., \(p \subset V(J^c)\).

Now let \(p \subset V(J^c)\). \(\varphi\) induces the one-to-one homomorphism
\( \phi : A / J^c \rightarrow \hat{A} / J \)
\( a + J^c \rightarrow \phi(a) + J. \)

There is also a minimal prime ideal of \( J^c \) as \( p \) such that
\( J^c \subseteq p \subseteq \hat{p}. \)

Now by Lemma 3, there is \( q/J \) in Spec(\( \hat{A}/J \)) such that
\( \phi^{-1}(q/J) = p/J^c \)

Hence \( p = q^c \) and \( J \subseteq q. \) Hence \( q \in V(J) = S^*_{k}(\hat{M}). \) It follows from Proposition 1 that \( p \in S^*_{k}(M). \) By Proposition 2, we conclude that \( p \in S^*_{k}(M). \)

Hence \( V(J^c) = S^*_{k}(M) \) and \( S^*_{k}(M) \) is closed as claimed.

5. Corollary. Let \( B \) be a homomorphic image of \( A. \)
Then for every finitely generated \( B \)-module \( N \) the singular sets \( S^*_{k}(N) \) are closed.

Proof. Let \( f : A \rightarrow B \) be the relevant ring epimorphism.
By \([1,5]\), for every non-negative integer \( k, \)
\( S^*_{k}(N) = \{ p \in \text{Spec}(B) : f^{-1}(p) \in S^*_{k}(N \mid _{A}) \} \)
in which \( N \mid _{A} \) is the module \( N \) to be considered by restriction of scalars by means of \( f. \) Since \( S^*_{k}(N \mid _{A}) \) is a closed subset of \( \text{Spec}(A) \), and \( f^{\ast} : \text{Spec}(B) \rightarrow \text{Spec}(A) \) is a continuous map, we conclude that \( f^{\ast -1}S^*_{k}(N \mid _{A}) = S^*_{k}(N) \) is a closed subset of \( \text{Spec}(B). \)

References


