ON THE SINGULAR SETS OF A MODULE II

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Throughout this note, A and B will denote a (non-trivial) commutative Noetherian ring with a multiplicative identity element and M will denote a non-zero finitely generated A-module.

For every non-negative integer k, the set
\[ S_k(M) = \{ p \in \text{Spec}(A) \mid \text{depth } M_p + \dim A/p \leq k \} \]
is called the singular set of M with respect to k.

It is known that when the ring A is homomorphic image of a biequidimensional regular ring, then the singular sets of M are all closed in the Zariski topology on Spec(A) (see [3; ch. IV, 5]).

A development of this famous theorem has been recently shown in the sense that if A is a homomorphic image of a biequidimensional Gorenstein ring, the singular sets of M are still closed (Sec[2]).

The purpose of this article is to show that if B is a homomorphic image of a Cohen-Macaulay local ring, then \( S^*_k(N) \) is closed, for every finitely generated B-module N.

First we prove some preliminary lemmas and propositions which help us to conclude the subsequent main theorem. From now on, A will denote a Cohen-Macaulay local ring with the unique maximal ideal \( m \), and \( \hat{A} \) (respectively \( \hat{M} \)) will denote the maximal A and M respectively.

1. Proposition. Let \( \phi : A \to \hat{A} \) be the natural homomorphism. Then for every \( q \in \text{Spec}(\hat{A}) \),
\[ S^*_k(M) \iff q^c \in S^*_k(M) \]
(for any ideal \( J \) we write \( J^c \) for \( \phi^{-1}(J) \)).

Proof. By [5,23.3], depth_{A_q} (M_p \otimes A_p)
\[ \text{depth}_{\hat{A}_p}(M_p) + \text{depth}(\hat{A}_q/pA_p\hat{A}_q) \], since.
\[
\varphi : A_p \rightarrow \hat{A}_q
\]
is a flat homomorphism. Also we have
\[
M_p \otimes_{A_p} \hat{A}_q \equiv (M \otimes AA_p) \otimes_{A_p} \hat{A}_q \equiv M \otimes A(A_p \otimes \hat{A}_q)
\equiv M \otimes \hat{A}_q \otimes \hat{A}_q
\equiv M \otimes \hat{A}_q \equiv M_q.
\]
Thus we conclude that
\[
\text{depth}(\hat{A}_q(M_q)) = \text{depth}_{A_p}(M_p) + \text{depth}(\hat{A}_q/pA_p\hat{A}_q).
\]
On the other hand, since \( \hat{A} \) is a Cohen-Macaulay, \( \hat{A}_q \) is a Cohen-Macaulay local ring; whence, by corollary of [5;23.3], \( \hat{A}_q/pA_p\hat{A}_q \) is a Cohen-Macaulay ring. But
\[
\hat{A}_q/pA_p\hat{A}_q = \hat{A}_q/p\hat{A}_q.
\]
Hence
\[
\text{depth}(\hat{A}_q(M_q)) = \text{depth}_{A_p}(M_p) + \text{ht}_{\hat{A}_q} - \text{ht}_p.
\]
Moreover, by [5;15.1],
\[
\text{ht}_{\hat{A}_q} = \text{ht}_p + \dim(\hat{A}_q/p\hat{A}_q).
\]
Hence
\[
\text{depth}(\hat{A}_q(M_q)) = \text{depth}_{A_p}(M_p) + \text{ht}_{\hat{A}_q} - \text{ht}_p.
\]
From which we get, by [5;17.4],
\[
\text{depth}(\hat{A}_q(M_q)) + \dim(\hat{A}_q) = \text{depth}_{A_p}(M_p) + \dim \hat{A}_q - \text{ht}_p
\]
\[
= \text{depth}_{A_p}(M_p) + \dim \hat{A}_q - \text{ht}_p
\]
\[
= \text{depth}_{A_p}(M_p) + \dim(\hat{A}_q).
\]

The result now follows.

2. Proposition. With the same assumption as in Proposition 1. Let \( p, q \subseteq \text{Spec}(\hat{A}) \) be prime ideals such that \( p \subseteq \hat{p} \) and \( p \subseteq \hat{s}_k(M) \). Then \( \hat{p} \subseteq \hat{s}_k(M) \).

Proof. Since \( \varphi : A \rightarrow \hat{A} \) is a faithfully flat homomorphism, there exists \( \hat{q} \subseteq \text{Spec}(\hat{A}) \) for which \( (\hat{q})^c = \hat{p} \) (by [5;7.3]). But \( \varphi \) has the going down property (see[5;9.5]). Hence there is a prime ideal \( q \subseteq \text{Spec}(\hat{A}) \) such that \( q^c = p \) and \( q \subseteq q \). By Proposition 1, this implies that \( q \subseteq s^c_k(M) \). But \( \hat{A} \) is a homomorphism image of a regular local ring (see[5;29.4(ii)]); thus by [3], \( s^c_k(M) \) is a closed subset of \( \text{Spec}(\hat{A}) \) (note that, every Cohen-Macaulay local ring is biinjective ring). This implies that \( \hat{q} \subseteq s^c_k(M) \). Again from Proposition 1, this in turn implies that \( \hat{q}^c = \hat{p} \subseteq \hat{s}_k(M) \) as required.

3. Lemma. (See[4, ch. 1, §6, Ex. 1]) Let \( R \subseteq T \) be rings and \( p \) a minimal prime ideal in \( R \). Then there exists in \( T \) a prime ideal contracting to \( p \).

Proof. Let \( p \) be a minimal prime ideal of \( R \). Set \( S = R - p \) and
\[
K = \{a \mid a \cap S = \varnothing \ & a \text{is an ideal of } T\}.
\]
Then \( K \) have a maximal element which is prime ideal of \( T \). Let \( q \) be such prime ideal. Since \( (q \cap R) \cap S = \varnothing \), we have \( q \cap R \subseteq p \) and consequently \( q \cap R = p \).

We now turn to the main theorem of the note.

4. Theorem. For every positive integer \( k \), \( s^c_k(M) \) is a closed subset of \( \text{Spec}(A) \).

Proof. Since \( s^c_k(M) \) is closed in \( \text{Spec}(\hat{A}) \), there exists an ideal \( J \) of \( \hat{A} \) such that \( V(J) = s^c_k(M) \). It is enough to show that
\[
V(J^c) = s^c_k(M)
\]
Let \( p \subseteq s^c_k(M) \). Hence there is \( q \subseteq s^c_k(\hat{A}) \) such that \( q^c = p \). Hence \( q \subseteq s^c_k(\hat{M}) \). Thus \( J \subseteq q \); this implies that \( J^c \subseteq q^c = p \); i.e., \( p \subseteq V(J^c) \).

Now let \( p \subseteq V(J^c) \). \( \varphi \) induces the one-to-one homomorphism
\[ \phi : A / J^c \rightarrow \tilde{\Lambda} / \tilde{J} \]
\[ a + J^c \rightarrow \phi(a) + J. \]

There is also a minimal prime ideal of \( J^c \) as \( p \) such that
\[ J^c \subseteq p \subseteq \tilde{p}. \]

Now by Lemma 3, there is \( q/J \) in \( \text{Spec}(\tilde{\Lambda}/J) \) such that
\[ \tilde{\phi}^{-1}(q/J) = p/J^c. \]

Hence \( p = q^\circ \) and \( J \subseteq q. \) Hence \( q \in V(J) = S^*_{\tilde{k}}(\tilde{M}). \) It
follows from Proposition 1 that \( p \in S^*_{\tilde{k}}(M). \) By
Proposition 2, we conclude that \( p \in S^*_{\tilde{k}}(M). \)

Hence \( V(J^c) = S^*_{\tilde{k}}(M) \) and \( S^*_{\tilde{k}}(M) \) is closed as
claimed.

5. Corollary. Let \( B \) be a homomorphic image of \( A. \)
Then for every finitely generated \( B \)-module \( N \) the
singular sets \( S^*_k(N) \) are closed.

Proof. Let \( f : A \rightarrow B \) be the relevant ring epimorphism.
By [1;5], for every non-negative integer \( k, \)
\[ S^*_k(N) = \{ p \in \text{Spec}(B) : f^{-1}(p) \in S^*_k(N | A) \} \]
in which \( N | A \) is the module \( N \) to be considered by
restriction of scalars by means of \( f. \) Since \( S^*_k(N | A) \) is
a closed subset of \( \text{Spec}(A), \) and \( f^*: \text{Spec}(B) \rightarrow \text{Spec}(A) \)
is a continuous map, we conclude that
\[ f^{-1}S^*_k(N | A) = S_k(N) \] is a closed subset of \( \text{Spec}(B). \)

References

[1.] Kh. Ahmadi-Amoli, M. Tousi, On the singular sets of modules, 10

[4.] I. Kaplanski. Commutative Rings, The University of Chicago