ON THE SINGULAR SETS OF A MODULE II

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Throughout this note, A and B will denote a (non-trivial) commutative Noetherian ring with a multiplicative identity element and M will denote a non-zero finitely generated A-module.

For every non-negative integer k, the set
\[ S^*_k(M) = \{ p \in \text{Spec}(A) \mid \text{depth } M_p + \dim A/p \leq k \} \]
is called the **singular set of M** with respect to k.

It is known that when the ring A is homomorphic image of a biequidimensional regular ring, then the singular sets of M are all closed in the Zariski topology on Spec(A) (see [3; ch. IV, 5]).

A development of this famous theorem has been recently shown in the sense that if A is a homomorphic image of a biequidimensional Gorenstein ring, the singular sets of M are still closed (see [2]).

The purpose of this article is to show that if B is homomorphic image of a Cohen-Macaulay local ring and \( S^*_k(N) \) is closed, for every finitely generated B-module N.

First we prove some preliminary lemmas which help us to conclude the subsequent main theorem. From now on, A will denote a Cohen-Macaulay local ring with the unique maximal ideal m, and \( \hat{A} \) (respectively \( \hat{M} \)) will denote the maximal completion of A (respectively M).

1. **Proposition.** Let \( \phi : A \rightarrow \hat{A} \) be the natural homomorphism. Then for every \( q \in \text{Spec}(\hat{A}) \), \( S^*_k(M) \Leftrightarrow p = q^\circ \in S^*_k(M) \) (for any ideal J we write \( J^\circ \) for \( \phi^{-1}(J) \)).

**Proof.** By [5; 3.3], \( \text{depth}_{A_\phi} (M_p \otimes \phi^{-1}_p) \)
depth_{A_p}(M_p) + \text{depth}(\wedge_q p A_p \wedge_q), \text{ since,}
\phi : A_p \rightarrow \wedge_q
\begin{array}{c}
\phi(a) \\
\phi(s)
\end{array}
is a flat homomorphism. Also we have
\begin{align*}
M_p \otimes_A \wedge_q &\equiv (M \otimes_A A_p) \otimes_A \wedge_q \equiv M \otimes_A (A_p \otimes_A \wedge_q) \\
&\equiv M \otimes_A \wedge_q \\
&\equiv \wedge_q
\end{align*}
Thus we conclude that
\text{depth}_{\wedge_q}(M_q) = \text{depth}_{A_p} M_p + \text{depth}(\wedge_q p A_p \wedge_q).
On the other hand, since A is Cohen-Macaulay, \wedge_q is a
cohen-Macaulay local ring, whence, by corollary of
[5,23,3], \wedge_q p A_p \wedge_q is a Cohen-Macaulay ring. But
\wedge_q / p A_p \wedge_q = \wedge_q / p \wedge_q.
Hence
\text{depth}(\wedge_q / p \wedge_q) = \text{dim}(\wedge_q / p \wedge_q).
Moreover, by [5,15.1],
\text{ht} q = \text{ht} p + \text{dim}(\wedge_q / p \wedge_q).
Hence
\text{depth}_{\wedge_q}(M_q) = \text{depth}_{A_p}(M_p) + \text{ht} q - \text{ht} p.
From which we get, by [5,17.4],
\text{depth}_{\wedge_q}(M_q) + \text{dim}(\wedge_q / q) = \text{depth}_{A_p}(M_p) + \text{dim}(A / q - \text{ht} p
\quad = \text{depth}_{A_p}(M_p) + \text{dim}(A / q - \text{ht} p
\quad = \text{depth}_{A_p}(M_p) + \text{dim}(\wedge_q / q)
The result now follows.

2. Proposition. With the same assumption as in
Proposition 1. Let p, p \subseteq \text{Spec}(A) be prime ideals such
that p \subseteq p and p \subseteq S^*(M). Then p \subseteq S^*(M).

Proof. Since \phi : A \rightarrow \wedge_q is a faithfully flat
homomorphism, there exists q \subseteq \text{Spec}(A) for which
(q)^c = p (by [5,7.3]). But \phi has the going down
property (see[5,9.5]). Hence there is a prime ideal
q \subseteq \text{Spec}(\wedge_q) such that q^c = p and q \subseteq q. By Proposition
1, this implies that q \subseteq S^*(M). But \wedge_q is a homomorphism image of a regular local ring (see[5,29.4(ii)]); thus by
[3], S^*(M) is a closed subset of Spec(\wedge_q) (note that,
every Cohen-Macaulay local ring is bequidimensional
ring). This implies that q \subseteq S^*(M). Again from
Proposition 1, this in turn implies that
(q)^c = p \subseteq S^*(M) as required.

3. Lemma. (See[4, ch. 1, §6, Ex. 1]) Let R \subseteq T be rings and
p a minimal prime ideal in R. Then there exists in
T a prime ideal contracting to p.

Proof. Let p be a minimal prime ideal of R. Set
S = R - p, and
K = \{a \mid a \cap S = \phi \& a \text{ is an ideal of } T\}.
Then K have a maximal element which is prime ideal
of T. Let q be such prime ideal. Since (q \cap R) \cap S = \phi,
we have (q \cap R) \subseteq p and consequently q \cap R = p.

We now turn to the main theorem of the note.

4. Theorem. For every positive integer k, S^*(M) is a
closed subset of Spec(A).

Proof: Since S^*(M) is closed in Spec(\wedge_q), there exists
an ideal J of \wedge_q such that V(J) = S^*(M). It is enough
to show that
V(J^c) = S^*(M)
Let p \subseteq S^*(M). Hence there is q \subseteq S^*(A) such
that q^c = p. Hence q \subseteq S^*(M). Thus J \subseteq q; this implies
that J^c \subseteq q^c = p; i.e., p \subseteq V(J^c).
Now let p \subseteq V(J^c). \phi induces the one-to-one
5. Corollary. Let \( B \) be a homomorphic image of \( A \). Then for every finitely generated \( B \)-module \( N \) the singular sets \( S^* _k(N) \) are closed.

Proof. Let \( f : A \rightarrow B \) be the relevant ring epimorphism. By \([1;5]\), for every non-negative integer \( k \),

\[
S^* _k(N) = \{ p \in \text{Spec}(B) : f^{-1}(p) \in S^* _k(N | \lambda) \}
\]

in which \( N | \lambda \) is the module \( N \) to be considered by restriction of scalars by means of \( f \). Since \( S^* _k(N | \lambda) \) is a closed subset of \( \text{Spec}(A) \), and \( f^* : \text{Spec}(B) \rightarrow \text{Spec}(A) \) is a continuous map, we conclude that \( f^{-1}S^* _k(N | \lambda) = S^* _k(N) \) is a closed subset of \( \text{Spec}(B) \).

References


