ON THE SINGULAR SETS OF A MODULE II

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Throughout this note, A and B will denote a (non-trivial) commutative Noetherian ring with a multiplicative identity element and M will denote a non-zero finitely generated A-module.

For every non-negative integer k, the set $S^*(M) = \{ p \in \text{Spec}(A) \mid \text{depth } M_p + \dim A/p \leq k \}$ is called the singular set of M with respect to k.

It is known that when the ring A is homomorphic image of a biequidimensional regular ring, then the singular sets of M are all closed in the Zariski topology on Spec(A) (see [3, ch. IV, 5]).

A development of this famous theorem has been recently shown in the sense that if A is a homomorphic image of a biequidimensional Gorenstein ring, the singular sets of M are still closed (Sec[2]).

The purpose of this article is to show that if B is a homomorphic image of a Cohen-Macaulay local ring, then $S^*(N)$ is closed, for every finitely generated B-module N.

First we prove some preliminary lemmas which help us to conclude the subsequent main theorem. From now on, A will denote a Cohen-Macaulay local ring with the unique maximal ideal m, and Â (respectively ˆM) will denote the maximal ideal of A (respectively M).

1. Proposition. Let $\phi : A \rightarrow \hat{A}$ be the natural homomorphism. Then for every $q \in \text{Spec}(\hat{A}), S^*(M) \Longleftrightarrow p = q^\circ \in S^*_k(M)$ (for any ideal J we write $J^\circ$ for $\phi^{-1}(J)$).

Proof. By [5,23.3], $\text{depth}_{A_q} (M_p \otimes_{A_p} \hat{A})$
depth_{\hat{\mathcal{A}}_p}(M_p) + \text{depth}(\hat{\mathcal{A}}_q/p\mathcal{A}_p \hat{\mathcal{A}}_q), \quad \text{since},
\phi : \mathcal{A}_p \to \hat{\mathcal{A}}_q
\frac{p}{s} \to \phi(s)

is a flat homomorphism. Also we have
\begin{align*}
M_p \otimes_{\mathcal{A}_p} \hat{\mathcal{A}}_q &\equiv (M \otimes_{\mathcal{A}_p} \mathcal{A}_p) \otimes_{\mathcal{A}_p} \hat{\mathcal{A}}_q \\
&\equiv M \otimes_{\mathcal{A}_p} \mathcal{A}_q = \mathcal{M}_A(\mathcal{A}_q) \otimes_{\mathcal{A}_p} \hat{\mathcal{A}}_q \\
&\equiv \hat{M} \otimes_{\mathcal{A}_q} \hat{\mathcal{A}}_q = \hat{\mathcal{M}}_q.
\end{align*}

Thus we conclude that
\text{depth}_{\hat{\mathcal{A}}_q}(\hat{\mathcal{M}}_q) = \text{depth}_{\mathcal{A}_p}(M_p) + \text{depth}(\hat{\mathcal{A}}_q/p\mathcal{A}_p \hat{\mathcal{A}}_q).

On the other hand, since $\mathcal{A}$ is Cohen-Macaulay, $\hat{\mathcal{A}}$ is a cohen-Macaulay local ring; whence, by corollary of [5;23.3], $\hat{\mathcal{A}}_q/p\mathcal{A}_p \hat{\mathcal{A}}_q$ is a Cohen-Macaulay ring. But
\begin{align*}
\hat{\mathcal{A}}_q/p\mathcal{A}_p \hat{\mathcal{A}}_q &= \hat{\mathcal{A}}_q/p \hat{\mathcal{A}}_q.
\end{align*}

Hence
\text{depth}(\hat{\mathcal{A}}_q/p \hat{\mathcal{A}}_q) = \text{dim}(\hat{\mathcal{A}}_q/p \hat{\mathcal{A}}_q).

Moreover, by [5;15.1],
\begin{align*}
\text{ht } q = \text{ht } p + \text{dim}(\hat{\mathcal{A}}_q/p \hat{\mathcal{A}}_q).
\end{align*}

Hence
\begin{align*}
\text{depth}_{\hat{\mathcal{A}}_q}(\hat{\mathcal{M}}_q) &= \text{depth}_{\mathcal{A}_p}(M_p) + \text{ht } q - \text{ht } p \\
&= \text{depth}_{\mathcal{A}_p}(M_p) + \text{dim } \mathcal{A} - \text{ht } p.
\end{align*}

From which we get, by [5;17.4],
\begin{align*}
\text{depth}_{\hat{\mathcal{A}}_q}(\hat{\mathcal{M}}_q) + \text{dim}(\hat{\mathcal{A}}_q) &= \text{depth}_{\mathcal{A}_p}(M_p) + \text{dim } \mathcal{A} - \text{ht } p \\
&= \text{depth}_{\mathcal{A}_p}(M_p) + \text{dim } \mathcal{A} - \text{ht } p.
\end{align*}

The result now follows.

2. Proposition. With the same assumption as in Proposition 1. Let $p,q \in \text{Spec}(\mathcal{A})$ be prime ideals such that $p \subseteq q$ and $p \in S_k^*(M)$. Then $\hat{p} \in S_k^*(M)$.

Proof. Since $\phi : \mathcal{A}_p \to \hat{\mathcal{A}}_q$ is a faithfully flat homomorphism, there exists $q \in \text{Spec}(\hat{\mathcal{A}})$ for which $(q^\vee)^\oplus = p^\oplus$ (by [5;7.3]). But $\phi$ has the going down property (see [5;9.5]). Hence there is a prime ideal $q \in \text{Spec}(\hat{\mathcal{A}})$ such that $q^\vee = p$ and $q \subseteq q$. By Proposition 1, this implies that $q \subseteq S_k^*(M)$. But $\hat{\mathcal{A}}$ is a homomorphic image of a regular local ring (see [5;29.4(ii)]); thus by [3], $S_k^*(M)$ is a closed subset of $\text{Spec}(\hat{\mathcal{A}})$ (note that, every Cohen-Macaulay local ring is bi-quidimensional ring). This implies that $\hat{q} \subseteq S_k^*(\hat{M})$. Again from Proposition 1, this in turn implies that $q^\vee = \hat{p} \subseteq S_k^*(M)$ as required.

3. Lemma. (See [4,ch.1, §6, Ex. 1]) Let $R \subseteq T$ be rings and $p$ a minimal prime ideal in $R$. Then there exists in $T$ a prime ideal contracting to $p$.

Proof. Let $p$ be a minimal prime ideal of $R$. Set $S = \text{Rej}_{R} p$, and

$$K = \{a \mid a \cap S = \phi & a \text{ is an ideal of } T\}.$$ 

Then $K$ have a maximal element which is prime ideal of $T$. Let $q$ be such prime ideal. Since $(q \cap R) \cap S = \phi$, we have $(q \cap R) \subseteq p$ and consequently $q \cap R = p$.

We now turn to the main theorem of the note.

4. Theorem. For every positive integer $k$, $S_k^*(M)$ is a closed subset of $\text{Spec}(\mathcal{A})$.

Proof: Since $S_k^*(M)$ is closed in $\text{Spec}(\hat{\mathcal{A}})$, there exists an ideal $J$ of $\hat{\mathcal{A}}$ such that $V(J) = S_k^*(\hat{M})$. It is enough to show that

$$V(J^\vee) = S_k^*(M).$$

Let $p \in S_k^*(M)$. Hence there is $q \in S_k^*(\hat{A})$ such that $q^\vee = p$. Hence $q \in S_k^*(\hat{M})$. Thus $J \subseteq q$; this implies that $p^\vee \subseteq q^\vee = p$; i.e., $p \in V(J^\vee)$.

Now let $p \in V(J^\vee)$. $\phi$ induces the one-to-one homomorphism.
\[ \varphi : A/J^c \rightarrow \hat{A}/J \]
\[ a + J^c \rightarrow \varphi(a) + J. \]

There is also a minimal prime ideal of \( J^c \) as \( p \) such that
\[ J^c \subseteq p \subseteq \hat{p}. \]

Now by Lemma 3, there is \( q/J \) in Spec(\( \hat{A}/J \)) such that
\[ \hat{\varphi}^{-1}(q/J) = p/J^c. \]

Hence \( p = q^c \) and \( J \subseteq q \). Hence \( q \in V(J) = \mathcal{S}^*_k(M) \). It follows from Proposition 1 that \( p \in \mathcal{S}^*_k(M) \). By Proposition 2, we conclude that \( p \in \mathcal{S}^*_k(M) \).

Hence \( V(J^c) = \mathcal{S}^*_k(M) \) and \( \mathcal{S}^*_k(M) \) is closed as claimed.

5. **Corollary.** Let \( B \) be a homomorphic image of \( A \). Then for every finitely generated \( B \)-module \( N \) the singular sets \( \mathcal{S}^*_k(N) \) are closed.

**Proof.** Let \( f : A \rightarrow B \) be the relevant ring epimorphism. By \([1,5]\), for every non-negative integer \( k \),
\[ \mathcal{S}^*_k(N) = \{ p \in \text{Spec}(B) : f^{-1}(p) \subseteq \mathcal{S}^*_k(N | \_A) \} \]
in which \( N | \_A \) is the module \( N \) to be considered by restriction of scalars by means of \( f \). Since \( \mathcal{S}^*_k(N | \_A) \) is a closed subset of \( \text{Spec}(A) \), and \( f^* : \text{Spec}(B) \rightarrow \text{Spec}(A) \) is a continuous map, we conclude that \( f^{-1}\mathcal{S}^*_k(N | \_A) = \mathcal{S}^*_k(N) \) is a closed subset of \( \text{Spec}(B) \).

**References**


