ON THE SINGULAR SETS OF A MODULE II

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Throughout this note, A and B will denote a (non-trivial) commutative Noetherian ring with a multiplicative identity element and M will denote a non-zero finitely generated A-module.

For every non-negative integer k, the set

\[ S_k^*(M) = \{ p \in \text{Spec}(A) \mid \text{depth} M_p + \dim A/p \leq k \} \]

is called the singular set of M with respect to k.

It is known that when the ring A is homomorphic image of a biequidimensional regular ring, then the singular sets of M are all closed in the Zariski topology on Spec(A) (see[3; ch. IV, 5]).

A development of this famous theorem has been recently shown in the sense that if A is a homomorphic image of a biequidimensional Gorenstein ring, the singular sets of M are still closed (Sec[2]).

The purpose of this article is to show that if B is a homomorphic image of a Cohen-Macaulay local ring, then \( S_k^*(N) \) is closed, for every finitely generated B-module N.

First we prove some preliminary lemmas which help us to conclude the subsequent main theorem. From now on, A will denote a Cohen-Macaulay local ring with the unique maximal ideal \( \mathfrak{m} \), and \( \hat{A} \) (respectively \( \hat{M} \)) will denote the maximal completion of A (respectively M).

1. Proposition. Let \( \phi : A \to \hat{A} \) be the natural homomorphism. Then for every \( q \in \text{Spec}(\hat{A}) \),

\[ S^*_q(M) \iff p = q^\circ \in S^*_q(M) \text{ (for any ideal } J \text{ we write } J^\circ \text{ for } \phi^{-1}(J)) \]

Proof. By [5;23.3], \( \text{depth}_{A_q} (M_p \otimes_{A_p} \hat{A}) \).
depth_{\hat{A} p}(M_p) + \text{ depth}(\hat{A}_q/p\hat{A}_p \hat{A}_q), \text{ since,}
\bar{\varphi} : \hat{A}_p \rightarrow \hat{A}_q
\bar{\varphi} \bar{\varphi}(s) = \varphi(s)
\text{ is a flat homomorphism. Also we have}
M_p \otimes_{\hat{A}_p} \hat{A}_q \cong (M \otimes_{\hat{A}_p} \hat{A}_p) \otimes_{\hat{A}_p} \hat{A}_q = M \otimes_{\hat{A}_p} (\hat{A}_p \otimes_{\hat{A}_p} \hat{A}_q)
\cong M \otimes_{\hat{A}_p} \hat{A}_q = \hat{M}_q \otimes_{\hat{A}_q} \hat{A}_q
\text{ Thus we conclude that}
\text{ depth}_{\hat{A}_p}(\hat{M}_q) = \text{ depth}_{\hat{A}_p}(M_p) + \text{ depth}(\hat{A}_p/p\hat{A}_p \hat{A}_q).
\text{ On the other hand, since } \hat{A} \text{ is Cohen-Macaulay, } \hat{A}_q \text{ is a Cohen-Macaulay local ring; whence, by corollary of}
[5;23.3], \hat{A}_q/p\hat{A}_p \hat{A}_q \text{ is a Cohen-Macaulay ring. But}
\hat{A}_p/p\hat{A}_p \hat{A}_q = \hat{A}_p/p\hat{A}_q.
\text{ Hence}
\text{ depth}(\hat{A}_p/p\hat{A}_q) = \text{ dim}(\hat{A}_p/p\hat{A}_q).
\text{ Moreover, by [5;15.1],}
\text{ ht } q = \text{ ht } p + \text{ dim}(\hat{A}_p/p\hat{A}_q).
\text{ Hence}
\text{ depth}_{\hat{A}_p}(\hat{M}_q) = \text{ depth}_{\hat{A}_p}(M_p) + \text{ ht } q - \text{ ht } p.
\text{ From which we get, by [5;17.4],}
depth_{\hat{A}_p}(\hat{M}_q) + \text{ dim}(\hat{A}_p/q) = \text{ depth}_{\hat{A}_p}(M_p) + \text{ dim}(\hat{A}_p) - \text{ ht } p
\quad = \text{ depth}_{\hat{A}_p}(M_p) + \text{ dim}(\hat{A}_p) - \text{ ht } p
\quad = \text{ depth}_{\hat{A}_p}(M_p) + \text{ dim}(\hat{A}_p/q).
\text{ The result now follows.}

2. Proposition. With the same assumption as in Proposition 1. Let \( p \in \text{Spec}(\hat{A}) \) be prime ideals such that \( p \subseteq \hat{p} \) and \( p \in S^*_k(M) \). Then \( \hat{p} \in S^*_k(M) \).
\text{ Proof. Since } \varphi : \hat{A} \rightarrow \hat{A} \text{ is a faithfully flat homomorphism, there exists } q \in \text{Spec}(\hat{A}) \text{ for which } (q^c)^c = \hat{p} \text{ (by [5;7.3]). But } \varphi \text{ has the going down property (see[5;9.5]). Hence there is a prime ideal } q \in \text{Spec}(\hat{A}) \text{ such that } q^c = p \text{ and } q \subseteq q. \text{ By Proposition 1, this implies that } q \in S^*_k(M). \text{ But } \hat{A}_q \text{ is a homomoronic image of a regular local ring (see[5;29.4(ii)]); thus by}
[3], S^*_k(M) \text{ is a closed subset of } \text{Spec}(\hat{A}) \text{ (note that, every Cohen-Macaulay local ring is biquidimensional ring). This implies that } q \in S^*_k(M). \text{ Again from Proposition 1, this in turn implies that}
(\hat{q})^c = \hat{p} \in S^*_k(M) \text{ as required.}

3. Lemma. (See[4,ch.1, §6, Ex. 1]) Let \( R \subseteq T \) be rings and \( p \) a minimal prime ideal in \( R \). Then there exists in \( T \) a prime ideal contracting to \( p \).
\text{ Proof. Let } p \text{ be a minimal prime ideal of } R. \text{ Set } S = R - p, \text{ and }
K = \{ a \mid a \cap S = \varphi & a \text{ is an ideal of } T \}.
\text{ Then } K \text{ have a maximal element which is prime ideal of } T. \text{ Let } q \text{ be such prime ideal. Since } (q \cap R) \cap S = \varnothing, \text{ we have } (q \cap R) \subseteq p \text{ and consequently } q \cap R = p.
\text{ We now turn to the main theorem of the note.}

4. Theorem. For every positive integer } k, S^*_k(M) \text{ is a closed subset of } \text{Spec}(\hat{A}).
\text{ Proof: Since } S^*_k(M) \text{ is closed in } \text{Spec}(\hat{A}), \text{ there exists an ideal } J \text{ of } \hat{A} \text{ such that } V(J) = S^*_k(M). \text{ It is enough to show that}
V(J^c) = S^*_k(M).
\text{ Let } p \in S^*_k(M). \text{ Hence there is } q \in S^*_k(\hat{A}) \text{ such that } q^c = p. \text{ Hence } q \in S^*_k(M). \text{ Thus } J \subseteq q; \text{ this implies that } J^c \subseteq q^c = p; \text{ i.e., } p \in V(J^c).
\text{ Now let } p \in V(J^c). \varphi \text{ induces the one-to-one homomorphism}

\( \tilde{\varphi} : A/ J^c \to \tilde{\Lambda}/J \)

\( a + J^c \to \varphi(a) + J. \)

There is also a minimal prime ideal of \( J^c \) as \( p \) such that

\[ J^c \subseteq p \subseteq \tilde{p}. \]

Now by Lemma 3, there is \( q/J \) in \( \text{Spec}(\tilde{\Lambda}/J) \) such that

\[ \tilde{\varphi}^{-1}(q/J) = p/J^c. \]

Hence \( p = q^\circ \) and \( J \subseteq q \). Hence \( q \in V(J) = S^*_k(\tilde{M}) \). It follows from Proposition 1 that \( p \in S^*_k(M) \). By Proposition 2, we conclude that \( p \in S^*_k(M) \).

Hence \( V(J^c) = S^*_k(M) \) and \( S^*_k(M) \) is closed as claimed.

5. **Corollary.** Let \( B \) be a homomorphic image of \( A \).

Then for every finitely generated \( B \)-module \( N \) the singular sets \( S^*_k(N) \) are closed.

**Proof.** Let \( f : A \to B \) be the relevant ring epimorphism.

By \([1;5] \), for every non-negative integer \( k \),

\[ S^*_k(N) = \{ p \in \text{Spec}(B) : f^{-1}(p) \in S^*_k(N | \Lambda) \} \]

in which \( N | \Lambda \) is the module \( N \) to be considered by restriction of scalars by means of \( f \).

Since \( S^*_k(N | \Lambda) \) is a closed subset of \( \text{Spec}(A) \), and \( f^* : \text{Spec}(B) \to \text{Spec}(A) \) is a continuous map, we conclude that \( f^{-1}S^*_k(N | \Lambda) = S^*_k(N) \) is a closed subset of \( \text{Spec}(B) \).

**References**


