ON THE SINGULAR SETS OF A MODULE II

Dr M. H. BIJAN-ZADEH
Department of Mattematics, U. T. E

A. TEHRANIAN
Research Unit, Islamic Azad University

Dr. M. Toosi
Department of Matematus, Shahid Beheshti University

Throughout this note, A and B will denote a (non-trivial) commutative Noetherian ring with a multiplicative identity element and M will denote a non-zero finitely generated A-module.

For every non-negative integer k, the set
\[ S^*_k(M) = \{ p \in \text{Spec}(A) \mid \text{depth } M_p + \dim A/p \leq k \} \]
is called the singular set of M with respect to k.

It is known that when the ring A is homomorphic image of a biequidimensional regular ring, then the singular sets of M are all closed in the Zariski topology on Spec(A) (see[3; ch. IV, 5]).

A development of this famous theorem has been recently shown in the sense that if A is a homomorphic image of a biequidimensional Gorenstein ring, the singular sets of M are still closed (Sec[2]).

The purpose of this article is to show that if B is a homomorphic image of a Cohen-Macaulay local ring and B is a finitely generated B-module N.

First we prove some preliminary lemmas and propositions which help us to conclude the subsequent main theorem. From now on, A will denote a Cohen-Macaulay local ring with the unique maximal ideal m, and \( \hat{A} \) (respectively \( \hat{M} \)) will denote the maximal completion of A (respectively M).

1. Proposition. Let \( \phi : A \rightarrow \hat{A} \) be the natural homomorphism. Then for every \( q \in \text{Spec}(\hat{A}) \),
\[ S^*_q(M) \iff p = q^\circ \in S^*_p(M) \text{ (for any ideal } J, \text{ we write } J^\circ \text{ for } \phi^{-1}(J)) \]

Proof. By [5, 23.3], \( \text{depth}_{A_q} (M_q \otimes_{A_q} \hat{A}_q) \).
depth\(\mathcal{A}_p(M_p) + \text{depth}(\mathcal{A}_q/p\mathcal{A}_q)\), since,
\[
\bar{q} : \mathcal{A}_p \to \mathcal{A}_q
\]
is a flat homomorphism. Also we have
\[
\begin{align*}
\mathcal{M}_p \otimes_{\mathcal{A}_p} \mathcal{A}_q & \equiv (\mathcal{M} \otimes_{\mathcal{A}_p} \mathcal{A}_p) \otimes_{\mathcal{A}_p} \mathcal{A}_q \\
& \equiv \mathcal{M} \otimes_{\mathcal{A}_p} \mathcal{A}_q \\
& \equiv \mathcal{M} \otimes_{\mathcal{A}_q} \mathcal{A}_q
\end{align*}
\]
Thus we conclude that
\[
\text{depth} \mathcal{A}_q(M_q) = \text{depth} \mathcal{A}_p(M_p) + \text{depth} \mathcal{A}_q(p\mathcal{A}_q)
\]
On the other hand, since \(\mathcal{A}\) is Cohen-Macaulay, \(\mathcal{A}\) is a Cohen-Macaulay local ring, whence, by corollary of
\[5;23.3\], \(\mathcal{A}_q/p\mathcal{A}_q \mathcal{A}_q\) is a Cohen-Macaulay ring. But
\[
\mathcal{A}/p\mathcal{A}_q \mathcal{A}_q = \mathcal{A}/p\mathcal{A}_q
\]
Hence
\[
\text{depth}(\mathcal{A}/p\mathcal{A}_q) = \dim(\mathcal{A}/p\mathcal{A}_q).
\]
Moreover, by [5;15.1],
\[
 \text{ht} \ q = \text{ht} \ p + \dim(\mathcal{A}/p\mathcal{A}_q).
\]
Hence
\[
\text{depth} \mathcal{A}_q(M_q) = \text{depth} \mathcal{A}_p(M_p) + \text{ht} \ q - \text{ht} \ p.
\]
From which we get, by [5;17.4],
\[
\text{dept} \mathcal{A}_q(M_q) + \dim(\mathcal{A}/q) = \text{depth} \mathcal{A}_p(M_p) + \dim \mathcal{A} - \text{ht} \ p
\]
\[
= \text{depth} \mathcal{A}_p(M_p) + \dim \mathcal{A} - \text{ht} \ p
\]
\[
= \text{depth} \mathcal{A}_p(M_p) + \dim(\mathcal{A}/q)
\]
The result now follows.

2. **Proposition.** With the same assumption as in Proposition 1. Let \(p,q \subseteq \text{Spec}(\mathcal{A})\) be prime ideals such that \(p \subseteq q\) and \(p \subseteq S^*_k(M).\) Then \(p \subseteq S^*_k(M).\)

**Proof.** Since \(\varphi : \mathcal{A} \to \mathcal{A}\) is a faithfully flat homomorphism, there exists \(q' \subseteq \text{Spec}(\mathcal{A})\) for which \((q')^\varphi = p\) (by [5;7.3]). But \(\varphi\) has the going down property (see [5;9.5]). Hence there is a prime ideal \(q \subseteq \text{Spec}(\mathcal{A})\) such that \((q')^\varphi = p\) and \(q \subseteq q'.\) By Proposition 1, this implies that \(q \subseteq S^*_k(M).\) But \(\mathcal{A}\) is a homomorphism image of a regular local ring (see [5;29.4(ii)]); thus by [3], \(S^*_k(M)\) is a closed subset of \(\text{Spec}(\mathcal{A})\) (note that, every Cohen-Macaulay local ring is bi-equidimensional ring). This implies that \(q \subseteq S^*_k(M).\) Again from Proposition 1, this in turn implies that \((q')^\varphi = p \subseteq S^*_k(M)\) as required.

3. **Lemma.** (See [5, ch. 1, §6, Ex. 1]) Let \(R \subseteq T\) be rings and \(p\) a minimal prime ideal in \(R.\) Then there exists in \(T\) a prime ideal contracting to \(p.\)

**Proof.** Let \(p\) be a minimal prime ideal of \(R.\) Set \(S = R - p,\) and
\[
K = \{a \mid a \cap S = \varnothing \text{ and } a \text{ is an ideal of } T\}.
\]
Then \(K\) have a maximal element which is prime ideal of \(T.\) Let \(q\) be such prime ideal. Since \((q \cap R) \cap S = \varnothing,\) we have \((q \cap R) \subseteq p\) and consequently \(q \cap R = p.
\]
We now turn to the main theorem of the note.

4. **Theorem.** For every positive integer \(k, S^*_k(M)\) is a closed subset of \(\text{Spec}(\mathcal{A}).\)

**Proof.** Since \(S^*_k(M)\) is closed in \(\text{Spec}(\mathcal{A}),\) there exists an ideal \(J\) of \(\mathcal{A}\) such that \(V(J) = S^*_k(M).\) It is enough to show that
\[
V(J^\varphi) = S^*_k(M)
\]
Let \(p \subseteq S^*_k(M).\) Hence there is \(q \subseteq S^*_k(\mathcal{A})\) such that \(q^\varphi = p.\) Hence \(q \subseteq S^*_k(M).\) Thus \(J \subseteq q;\) this implies that \(q^\varphi \subseteq q^\varphi q = p;\) i.e., \(p \subseteq V(J^\varphi).\)

Now let \(p \subseteq V(J^\varphi).\) \(\varphi\) induces the one-to-one homomorphism
\[ \phi : A / J^c \to \hat{A} / J \]

\[ a + J^c \to \phi(a) + J. \]

There is also a minimal prime ideal of \( J^c \) as \( p \) such that

\[ J^c \subseteq p \subseteq \hat{p}. \]

Now by Lemma 3, there is \( q/J \) in Spec(\( \hat{A}/J \)) such that

\[ \phi^{-1}(q/J) = p/J^c. \]

Hence \( p = q^c \) and \( J \subseteq q \). Hence \( q \in V(J) = S^*_k(M) \). It follows from Proposition 1 that \( p \in S^*_k(M) \). By Proposition 2, we conclude that \( p \in S^*_k(M) \).

Hence \( V(J^c) = S^*_k(M) \) and \( S^*_k(M) \) is closed as claimed.

5. Corollary. Let \( B \) be a homomorphic image of \( A \). Then for every finitely generated \( B \)-module \( N \) the singular sets \( S^*_k(N) \) are closed.

Proof. Let \( f : A \to B \) be the relevant ring epimorphism. By \([1;5]\), for every non-negative integer \( k \),

\[ S^*_k(N) = \{ p \in \text{Spec}(B) : f^{-1}(p) \in S^*_k(N | A) \} \]

in which \( N | A \) is the module \( N \) to be considered by restriction of scalars by means of \( f \). Since \( S^*_k(N | A) \) is a closed subset of \( \text{Spec}(A) \), and \( f^* : \text{Spec}(B) \to \text{Spec}(A) \) is a continuous map, we conclude that \( f^{*\ast}S^*_k(N | A) = S^*_k(N) \) is a closed subset of \( \text{Spec}(B) \).

References


