ON THE SINGULAR SETS OF A MODULE II

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Throughout this note, A and B will denote a (non-trivial) commutative Noetherian ring with a multiplicative identity element and M will denote a non-zero finitely generated A-module.

For every non-negative integer k, the set
\[ S^*_k(M) = \{ p \in \text{Spec}(A) \mid \text{depth } M_p + \dim A/p \leq k \} \]
is called the singular set of M with respect to k.

It is known that when the ring A is homomorphic image of a biequidimensional regular ring, then the singular sets of M are all closed in the Zariski topology on \( \text{Spec}(A) \) (see[3; ch. IV, 5]).

A development of this famous theorem has been recently shown in the sense that if A is a homomorphic image of a biequidimensional Gorenstein ring, the singular sets of M are still closed (Sec[2]).

The purpose of this article is to show that if B is a homomorphic image of a Cohen-Macaulay local ring, then \( S^*_k(N) \) is closed, for every finitely generated B-module N.

First we prove some preliminary lemmas and propositions which help us to conclude the subsequent theorem. From now on, A will denote a Cohen-Macaulay local ring with the unique maximal ideal \( \mathfrak{m} \), and \( \hat{A} \) (respectively \( \hat{M} \)) will denote the maximal ideal of \( \hat{A} \) (respectively \( \hat{M} \)).

1. Proposition. Let \( \phi : A \to \hat{A} \) be the homomorphism. Then for every \( q \in \text{Spec}(\hat{A}) \),
\[ S^*_k(M) \iff \exists p \in S^*_k(M) \quad \text{such that} \quad q = p^\circ \in S_k^*(M) \] (for any ideal \( J \), we write \( J^\circ \) for \( \phi^{-1}(J) \)).

Proof. By [5,23.3], \( \text{depth}_{A_q} (M_p \otimes_{A_p} \hat{A}_q) = \text{depth}_{\hat{A}_q} (M_p \otimes_{A_p} \hat{A}_q) \).
depth\(\lambda_p(M_p) + \text{depth}(\lambda_q/p\lambda_p\lambda_q)\), since,
\[
\varphi : \lambda_p \rightarrow \lambda_q
\]

is a flat homomorphism. Also we have
\[
M_p \otimes \lambda_p \lambda_q \equiv (M \otimes \lambda_p) \otimes \lambda_q \equiv \lambda_p \otimes \lambda_q \lambda_q
\]
\[
= \lambda_q \otimes \lambda_q
\]

Thus we conclude that
\[
\text{depth}(\lambda_q/\lambda_p) = \text{depth}(\lambda_q/p\lambda_p) \lambda_q.
\]

On the other hand, since \(\lambda\) is Cohen-Macaulay, \(\lambda\) is a Cohen-Macaulay local ring; whence, by corollary of [5,23.3], \(\lambda_q/p\lambda_p\lambda_q\) is a Cohen-Macaulay ring. But
\[
\lambda_q/p\lambda_p\lambda_q = \lambda_q/p\lambda_q.
\]

Hence
\[
\text{depth}(\lambda_q) = \text{depth}(\lambda_q/p\lambda_q).
\]

Moreover, by [5,15.1],
\[
\text{ht} q = \text{ht} p + \text{dim}(\lambda_q/\lambda_p)
\]

Hence
\[
\text{depth}(\lambda_q) = \text{depth}(\lambda_q/p\lambda_q).
\]

From which we get, by [5,17.4],
\[
\text{depth}(\lambda_q) = \text{depth}(\lambda_q/p\lambda_q).
\]

The result now follows.

2. Proposition. With the same assumption as in Proposition 1. Let \(p, q \in \text{Spec}(\lambda)\) be prime ideals such that \(p \subseteq q\) and \(q \subseteq S^*k(M)\). Then \(p \in S^*k(M)\).

Proof. Since \(\varphi : \lambda \rightarrow \lambda\) is a faithfully flat homomorphism, there exists \(q \in \text{Spec}(\lambda)\) for which \((q^c) = p\) (by [5,7.3]). But \(\varphi\) has the going down property (see [5,9.5]). Hence there is a prime ideal \(q \in \text{Spec}(\lambda)\) such that \(q^c = p\) and \(q \subseteq q\). By Proposition 1, this implies that \(q \subseteq S^*k(M)\). But \(\lambda\) is a homomorphic image of a regular local ring (see [5,29.4(ii))]; thus by [3], \(S^*k(M)\) is a closed subset of \(\text{Spec}(\lambda)\) (note that every Cohen-Macaulay local ring is bi-quidimensional ring). This implies that \(q \in S^*k(M)\). Again from Proposition 1, this in turn implies that \((q^c) = p \in S^*k(M)\) as required.

3. Lemma. (See [4, ch. 1, §6, Ex. 1]) Let \(R \subseteq T\) be rings and \(p\) a minimal prime ideal in \(R\). Then there exists in \(T\) a prime ideal contracting to \(p\).

Proof. Let \(p\) be a minimal prime ideal of \(R\). Set \(S = R/p\) and
\[
K = \{a \mid a \in S = p & a\text{ is an ideal of } T\}.
\]

Then \(K\) have a maximal element which is prime ideal of \(T\). Let \(q\) be such prime ideal. Since \((q \cap R) \cap S = \phi\), we have \(q \cap R \subseteq p\) and consequently \(q \cap R = p\).

We now turn to the main theorem of the note.

4. Theorem. For every positive integer \(k\), \(S^*k(M)\) is a closed subset of \(\text{Spec}(\lambda)\).

Proof: Since \(S^*k(M)\) is closed in \(\text{Spec}(\lambda)\), there exists an ideal \(J\) of \(\lambda\) such that \(V(J) = S^*k(M)\). It is enough to show that
\[
V(J^c) = S^*k(M).
\]

Let \(p \in S^*k(M)\). Hence there is \(q \in S^*k(\lambda)\) such that \(q^c = p\). Hence \(q \subseteq S^*k(M)\). Thus \(J \subseteq q\); this implies that \(J^c \subseteq q^c = p\); i.e., \(p \subseteq V(J^c)\).

Now let \(p \subseteq V(J^c)\). \(\varphi\) induces the one-to-one homomorphism
\[ \phi : A/J^c \rightarrow \hat{A}/I \]
\[ a + J^c \rightarrow \phi(a) + I. \]

There is also a minimal prime ideal of \( J^c \) as \( p \) such that
\[ J^c \subseteq p \subseteq p. \]

Now by Lemma 3, there is \( q/J \) in Spec(\( \hat{A}/J \)) such that
\[ \bar{\phi}^{-1}(q/J) = p/J^c. \]

Hence \( p = q^c \) and \( J \subseteq q \). Hence \( q \in V(J) = S^*_k(\hat{M}) \). It follows from Proposition 1 that \( p \in S^*_k(M) \). By Proposition 2, we conclude that \( p \in S^*_k(M) \).

Hence \( V(J^c) = S^*_k(M) \) and \( S^*_k(\hat{M}) \) is closed as claimed.

5. Corollary. Let \( B \) be a homomorphic image of \( A \). Then for every finitely generated \( B \)-module \( N \) the singular sets \( S^*_k(N) \) are closed.

Proof. Let \( f : A \rightarrow B \) be the relevant ring epimorphism. By [1,5], for every non-negative integer \( k \),
\[ S^*_k(N) = \{ p \in \text{Spec}(B) : f^{-1}(p) \in S^*_k(N \mid \_A) \} \]
in which \( N \mid \_A \) is the module \( N \) to be considered by restriction of scalars by means of \( f \). Since \( S^*_k(N \mid \_A) \) is a closed subset of \( \text{Spec}(A) \), and \( f^* : \text{Spec}(B) \rightarrow \text{Spec}(A) \) is a continuous map, we conclude that \( f^* S^*_k(N \mid \_A) = S^*_k(N) \) is a closed subset of \( \text{Spec}(B) \).

References


