ON THE SINGULAR SETS OF A MODULE II

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Throughout this note, A and B will denote a (non-trivial) commutative Noetherian ring with a multiplicative identity element and M will denote a non-zero finitely generated B-module.

For every non-negative integer \( k \), the set
\[
S^*_k(M) = \{ p \in \text{Spec}(A) \mid \text{depth } M_p + \text{dim } A/p \leq k \}
\]
is called the singular set of \( M \) with respect to \( k \).

It is known that when the ring \( A \) is homomorphic image of a biequidimensional regular ring, then the singular sets of \( M \) are all closed in the Zariski topology on \( \text{Spec}(A) \) (see[3; ch. IV, 5]).

A development of this famous theorem has been recently shown in the sense that if \( A \) is a homomorphic image of a biequidimensional Gorenstein ring, the singular sets of \( M \) are still closed (Sec[2]).

The purpose of this article is to show that if \( B \) is homomorphic image of a Cohen-Macaulay local ring \( \hat{\mathfrak{m}} \), then \( S^*_k(N) \) is closed, for every finitely generated \( B \)-module \( N \).

First we prove some preliminary lemmas which help us to conclude the subsequent main theorem. From now on, \( A \) will denote a Cohen-Macaulay local ring with the unique maximal ideal \( \mathfrak{m} \), and \( \hat{A} \) (respectively \( \hat{M} \)) will denote the maximal ideal of \( A \) (respectively \( M \)).

1. Proposition. Let \( \phi : A \rightarrow \hat{A} \) be the natural homomorphism. Then for every \( q \in \text{Spec}(\hat{A}) \),
\[
S^*_k(M) \iff P = q^\circ \in S^*_k(M) \quad \text{(for any ideal } J, \text{ we write } J^\circ \text{ for } \phi^{-1}(J)).
\]

Proof. By [5,23.3],

\[
\text{depth}_{A_q} (M_p \otimes_{A_p} \hat{A}^\circ)
\]
depth_{A_p}(M_p) + \text{depth}((\hat{A}_q/pA_p\hat{A}_q))\), since.

\[ \hat{\varphi}: A_p \rightarrow \hat{A}_q \]

\[ \frac{a}{s} \mapsto \frac{\varphi(a)}{\varphi(s)} \]

is a flat homomorphism. Also we have

\[ M_p \otimes_{A_p} \hat{A}_q \equiv (M \otimes_{A_p} A_p) \otimes_{A_p} \hat{A}_q = M \otimes_{A_p} (A_p \otimes_{A_p} \hat{A}_q) \]

\[ = M \otimes_{A_p} \hat{A}_q = M \otimes_{\hat{A}_q} \hat{A}_q \equiv (M \otimes_{\hat{A}} \hat{A}) \otimes_{\hat{A}} \hat{A}_q \equiv \hat{M} \otimes_{\hat{A}_q} \hat{A}_q = \hat{M}_q. \]

Thus we conclude that

\[ \text{depth}_{\hat{A}_q}(\hat{M}_q) = \text{depth}_{A_p}(M_p) + \text{depth}(\hat{A}_q/pA_p\hat{A}_q). \]

On the other hand, since \( A \) is Cohen-Macaulay, \( \hat{A} \) is a Cohen-Macaulay local ring; whence, by corollary of [5;23.3], \( \hat{A}_q/pA_p\hat{A}_q \) is a Cohen-Macaulay ring. But

\[ \hat{A}_q/pA_p\hat{A}_q = \hat{A}_q/p\hat{A}_q. \]

Hence

\[ \text{depth}_{\hat{A}_q}(\hat{M}_q) = \text{depth}_{A_p}(M_p) + \text{ht}_q \cdot \text{ht}_p. \]

Moreover, by [5;15.1],

\[ \text{ht}_q = \text{ht}_p + \text{dim}(\hat{A}_q/p\hat{A}_q). \]

Hence

\[ \text{depth}_{\hat{A}_q}(\hat{M}_q) = \text{depth}_{A_p}(M_p) + \text{ht}_q \cdot \text{ht}_p. \]

From which we get, by [5;17.4],

\[ \text{depth}_{\hat{A}_q}(\hat{M}_q) + \text{dim}(\hat{A}_q) = \text{depth}_{A_p}(M_p) + \text{dim}(\hat{A}_q) \cdot \text{ht}_p. \]

\[ = \text{depth}_{A_p}(M_p) + \text{dim}(\hat{A}_q) \cdot \text{ht}_p. \]

The result now follows.

2. Proposition. With the same assumption as in Proposition 1. Let \( p, q \in \text{Spec}(\hat{A}) \) be prime ideals such that \( p \subseteq q \) and \( p \in S^*_k(M) \). Then \( \hat{p} \in S^*_k(M) \).

Proof. Since \( \varphi: A \rightarrow \hat{A} \) is a faithfully flat homomorphism, there exists \( \hat{q} \in \text{Spec}(\hat{A}) \) for which \( (\hat{q})^c = \hat{p} \) (by [5;7.3]). But \( \varphi \) has the going down property (see [5;9.5]). Hence there is a prime ideal \( q \in \text{Spec}(\hat{A}) \) such that \( q^c = p \) and \( q \subseteq q \). By Proposition 1, this implies that \( q \in S^*_k(M) \). But \( \hat{A} \) is a homomorphic image of a regular local ring (see [5;29.4(ii)]); thus by [3], \( S^*_k(M) \) is a closed subset of \( \text{Spec}(\hat{A}) \) (note that, every Cohen-Macaulay local ring is biequidimensional ring). This implies that \( \hat{q} \in S^*_k(M) \). Again from Proposition 1, this in turn implies that \( (\hat{q})^c = \hat{p} \in S^*_k(M) \) as required.

3. Lemma. (See [4,ch.1, §6, Ex. 1]) Let \( R \subseteq T \) be rings and \( p \) a minimal prime ideal in \( R \). Then there exists in \( T \) a prime ideal contracting to \( p \).

Proof. Let \( p \) be a minimal prime ideal of \( R \). Set \( S = R - p \) and

\[ K = \{ a \mid a \cap S = \varnothing \text{ \& \ } a \text{ \ is \ an \ ideal \ of } T \}. \]

Then \( K \) have a maximal element which is prime ideal of \( T \). Let \( q \) be such prime ideal. Since \( (q \cap R) \cap S = \varnothing \), we have \( (q \cap R) \subseteq p \) and consequently \( q \cap R = p \).

We now turn to the main theorem of the note.

4. Theorem. For every positive integer \( k \), \( S^*_k(M) \) is a closed subset of \( \text{Spec}(\hat{A}) \).

Proof: Since \( S^*_k(M) \) is closed in \( \text{Spec}(\hat{A}) \), there exists an ideal \( \hat{J} \) of \( \hat{A} \) such that \( V(\hat{J}) = S^*_k(M) \). It is enough to show that

\[ V(J^c) = S^*_k(M). \]

Let \( p \in S^*_k(M) \). Hence there is \( q \in S^*_k(\hat{A}) \) such that \( q^c = p \). Hence \( q \in S^*_k(M) \). Thus \( J \subseteq q \); this implies that \( J^c \subseteq q^c = p \); i.e., \( p \in V(J^c) \).

Now let \( p \in V(J^c) \). \( \varphi \) induces the one-to-one homomorphism
\[ \varphi : \mathfrak{A}/\mathfrak{J}^c \to \mathfrak{A}/\mathfrak{J}^c \]
\[ a + \mathfrak{J}^c \to \varphi(a) + \mathfrak{J}^c. \]

There is also a minimal prime ideal of \( \mathfrak{J}^c \) as \( \mathfrak{p} \) such that
\[ \mathfrak{J}^c \subseteq \mathfrak{p} \subseteq \mathfrak{J}^c. \]

Now by Lemma 3, there is \( \mathfrak{q}/\mathfrak{J} \) in Spec(\( \mathfrak{A}/\mathfrak{J} \)) such that
\[ \varphi^{-1}(\mathfrak{q}/\mathfrak{J}) = \mathfrak{p}/\mathfrak{J}^c. \]

Hence \( \mathfrak{p} = \mathfrak{q}^c \) and \( \mathfrak{J} \subseteq \mathfrak{q} \). Hence, \( \mathfrak{q} \in V(\mathfrak{J}) = \mathfrak{S}^*_k(\mathfrak{M}) \). It
follows from Proposition 1 that \( \mathfrak{p} \in \mathfrak{S}^*_k(\mathfrak{M}) \). By
Proposition 2, we conclude that \( \mathfrak{p} \in \mathfrak{S}^*_k(\mathfrak{M}) \).

Hence \( V(\mathfrak{J}^c) = \mathfrak{S}^*_k(\mathfrak{M}) \) and \( \mathfrak{S}^*_k(\mathfrak{M}) \) is closed as
claimed.

5. **Corollary.** Let \( \mathfrak{B} \) be a homomorphic image of \( \mathfrak{A} \).
Then for every finitely generated \( \mathfrak{B} \)-module \( \mathfrak{N} \) the
singular sets \( \mathfrak{S}^*_k(\mathfrak{N}) \) are closed.

**Proof.** Let \( f : \mathfrak{A} \to \mathfrak{B} \) be the relevant ring epimorphism.
By \([1,5]\), for every non-negative integer \( k \),
\[ \mathfrak{S}^*_k(\mathfrak{N}) = \{ \mathfrak{p} \in \text{Spec}(\mathfrak{B}) : f^{-1}(\mathfrak{p}) \in \mathfrak{S}^*_k(\mathfrak{N} | \mathfrak{A}) \} \]
in which \( \mathfrak{N} | \mathfrak{A} \) is the module \( \mathfrak{N} \) to be considered by
restriction of scalars by means of \( f \). Since \( \mathfrak{S}^*_k(\mathfrak{N} | \mathfrak{A}) \) is
a closed subset of \( \text{Spec}(\mathfrak{A}) \), and \( f^* : \text{Spec}(\mathfrak{B}) \to \text{Spec}(\mathfrak{A}) \) is a continuous map, we conclude that
\[ f^{*-1}\mathfrak{S}^*_k(\mathfrak{N} | \mathfrak{A}) = \mathfrak{S}^*_k(\mathfrak{N}) \] is a closed subset of \( \text{Spec}(\mathfrak{B}) \).

References

[1.] Kh. Ahmad-Ahmadi, M. Tousi, *On the singular sets of modules, to
.arb.\mathfrak{A} \mathfrak{M} \mathfrak{N} \mathfrak{O} \mathfrak{C} \mathfrak{N} \mathfrak{I} \mathfrak{P} \mathfrak{A} \mathfrak{P} \mathfrak{P} \mathfrak{A} *


[4.] I. Kaplanski. *Commutative Rings*, The University of Chicago