The Existence of a Topolinear Isomorphism on an infinite dimensional Hilbert Space $H$ Corresponding to a Homeomorphism on it’s Projective Space $P(H)$

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Abstract

In this paper we prove a theorem which states the relationship between the topolinear isomorphisms on an infinite dimensional Hilbert Space $H$ and the Homeomorphisms on projective Space $P(H)$. This theorem is proved by E.Artin in the finite dimensional case.

Key words: Topolinear Isomorphism, Hilbert Space, Homeomorphism, Projective.
Introduction

The following $H$ is an infinite dimensional separable Hilbert Space and $P(H)$ is its projection space which is given a smooth structure as in [2]. We mean by $[x] \in P(H)$ the $x$-dimensional vector subspace of $H$ generated by $x \in H = H - 0$.

$[x] + [y]$ means the two dimensional subspace generated by $x, y \in H$. In fact $[z] \subseteq [x] + [y]$ means that there exists $a, b \in \mathbb{R}$ such that $z = ax + by$, and if $[z] \neq [x]$, there exists a unique $\tilde{z} \in [z]$ such that $[z] = [x + dy]$. We quote some auxiliary statements from [2].

**Theorem 1.1** Let $S$ be a unit sphere in a normed vector space $B$ and $T : B \rightarrow B$ a linear bijective transformation, and $\tilde{T}$ be the induced bijective transformation

$$\tilde{T} : S \rightarrow S$$

defined by $\tilde{T}(u) = \frac{T(u)}{||T(u)||}$ for $u \in S \subseteq B$.

This is a homeomorphism then $T$ is also a homeomorphism.

Now we are ready to state the theorem which is the main goal of this paper.

**Theorem 1.2** Let $f : P(H) \rightarrow P(H)$ be a homeomorphism such that

$$[x] \subseteq [y] + [z] \rightarrow f[x] \subseteq f[y] + f[z].$$

Then there exists a topological isomorphism $T : P(H) \rightarrow H$ such that the induced transformation $\tilde{T} : P(H) \rightarrow P(H)$ agrees with $f$.

Proof: the hypothesis implies that if $[x] \subseteq [y] + [z]$ then $f^{-1}[x] \subseteq f^{-1}[y] + f^{-1}[z]$ and by induction on $k$, we get that if $[z] \subseteq [z_1] + \cdots + [z_k]$ then $f[z] \subseteq f[z_1] + \cdots + f[z_k]$, and $f^{-1}[z] \subseteq f^{-1}[z_1] + \cdots + f^{-1}[z_k]$.

Let $\{x_i\}$ be a Hamel basis for $H$ where $i$ is an arbitrary element of a set $A$. It is clear that if $f[x_i] = [y_i]$ then $\{y_i\}$ is also a Hamel basis for $H$.

Now we choose an element of $A$ call it 1, then for any $i \neq 1$ the line

$$L_i = [x_1 + x_i] \subseteq [x_1] + [x_i]$$

where $L_i$ is not coincide with $[x_i]$ or $[x_1]$, consequently

$$fL_i \subseteq [y_1] + [y_i]$$

and $fL_i$ is not coincide with $[y_i]$ or $[y_1]$. Then, for some unique $d_i \in \mathbb{R}$ we have

$$fL_i = [y_1 + d_i y_1].$$

by choosing a suitable $y_i$ we may assume that $d_i = 1$. Then

$$i \in A, \quad f[x_i] = [y_i] \quad (1)$$

and for $i \neq 1$, $f[x_1 + x_i] = [y_1 + y_i]$.

Now we choose another index from $A$, call it 2. Then for $a \in \mathbb{R}$

$$L = [x_1 + ax_2] \subseteq [x_1] + [x_2] \text{ where } L \neq [x_2]$$

Therefore

$$fL \subseteq [y_1] + [y_2], \text{ where } fL \neq [y_2].$$

Then for a unique $a' \in \mathbb{R}$ we have

$$fL = [y_1 + a'y_1].$$
Introduction

The following \( H \) is an infinite dimensional separable Hilbert Space and \( P(H) \) is its positive space which is given a smooth structure as in [2]. We mean by \( [x] \in P(H) \) the one dimensional vector subspace of \( H \) generated by \( x \), that is \( [x] = \{ax | a \in \mathbb{R}\} \).

\( [x] + [y] \) means the two dimensional subspace generated by \( x, y \in H \). In fact \( [z] \subset [x] + [y] \) implies that there exists \( a, b \in \mathbb{R} \) such that \( z = ax + by \) and if \( [z] \neq [x] \), there exists a unique \( \frac{a}{b} \) such that \( [z] = [x + dy] \). We quote some necessary statments from [2].

Theorem 1.1 Let \( S \) be a unit sphere in a normed space \( B \) and \( T : B \rightarrow B \) a linear bijection, and \( \hat{T} \) be the induced bijection on \( \hat{B} \) in \( \hat{B} \),

\[ \hat{T} : S \rightarrow S \]

defined by \( \hat{T}(u) = \frac{T(u)}{||T(u)||} \) for \( u \in S \subset B \).

If \( \hat{T} \) is a homeomorphism then \( T \) is also homeomorphism.

Now we are ready to state the theorem which is the goal of this paper.

Theorem 1.2 Let \( f : P(H) \rightarrow P(H) \) be a homeomorphism such that

\[ [x] \subset [y] + [z] \rightarrow f[x] \subset f[y] + f[z]. \]

Then there exists a topolinear isomorphism \( T : P(H) \rightarrow H \) such that the induced transformation \( f : P(H) \rightarrow P(H) \) agrees with \( f \).

Proof. The hypothesis implies that if \( [x] \subset [y] + [z] \) then \( f^{-1}[x] \subset f^{-1}[y] + f^{-1}[z] \) and by induction on \( k \), we get that if \( [z] \subset [z_1] + \cdots + [z_k] \) then \( f[z] \subset f[z_1] + \cdots + f[z_k] \), and \( f^{-1}[z] \subset f^{-1}[z_1] + \cdots + f^{-1}[z_k] \).

Let \( \{x_i\} \) be a Hamel basis for \( H \) where \( i \) is an arbitrary element of a set \( A \). It is clear that if \( f[x_i] = [y_i] \) then \( \{y_i\} \) is also a Hamel basis for \( H \).

Now we choose an element of \( A \) call it \( 1 \), then for any \( i \neq 1 \) the line

\[ L_i = [x_1 + x_i] \subset [x_1] + [x_i] \]

where \( L_i \) is not coinide with \( [x_i] \) or \( [x_1] \), consequently

\[ fL_i \subset [y_1] + [y_i] \]

and \( fL_i \) is not coinide with \( [y_i] \) or \( [y_1] \). Then, for some unique \( d_i \in \mathbb{R} \) we have

\[ fL_i = [y_1 + d_i y_i]. \]

by choosing a suitable \( y_i \) we may assume that \( d_i = 1 \). Then

for \( i \in A, \quad f[x_i] = [y_i] \quad (1) \]

and for \( i \neq 1, \quad f[x_1 + x_i] = [y_1 + y_i]. \)

Now we choose another index from \( A \), call it 2. Then for \( a \in \mathbb{R} \)

\[ L = [x_1 + ax_2] \subset [x_1] + [x_2] \]

where \( L \neq [x_2] \)

Therefore

\[ fL \subset [y_1] + [y_2], \quad \text{where} \quad fL \neq [y_2]. \]

Then for a unique \( d' \in \mathbb{R} \) we have

\[ fL = [y_1 + d'y_2]. \]
Now we define
\[ \mu : \mathbb{R} \rightarrow \mathbb{R} \]
by \( \mu(a) = a' \) and we will show that \( \mu \) is the identity function on \( \mathbb{R} \). Since
\[ [x_1 + a x_2] \neq [x_1 + b x_2] \] if \( a \neq b \)
it follows that \( a' \neq b' \), then \( \mu \) is injective. We have also from (1) that
\[ 0' = 0 \quad \text{and} \quad 1' = 1. \] (2)
Now, we will show that for any \( i \in \mathcal{A} \)
\[ f[x_1 + a x_i] = [y_1 + a' y_i] \]
For any fixed \( i \neq 1, 2 \) in \( \mathcal{A} \) we have
\[ f[x_1 + a x_i] = [y_1 + b y_i] \]
On the other hand \( L = [a x_2 - a x_i] \subset [x_2] + [x_i] \) with \( L \neq [x_i] \), and so \( fL \subset [y_2] + [y_i] \) with \( fL \neq [y_i] \). Consequently, \( fL = [y_2 + d y_i] \) for some unique \( d \). On the other hand,
\[ L \subset [x_1 + a x_2] + [x_1 + a x_i] \] with \( L \neq [x_1 + a x_i] \).
Then as before \( fL = ([y_1 + a' y_2] + d'(y_1 + b y_i)] \) and it follows that \( d = -\frac{b}{a'} \). But
\[ L \subset [x_1 + x_2] + [x_1 + x_i] \] with \( L \neq [x_1 + x_i] \)
and by (1)
\[ fL \subset [y_1 + y_2] + [y_1 + y_i] \] with \( fL \neq [y_1 + y_i] \)
Then for some unique \( h \) we have \( fL = [y_1 + y_2 + h(y_1 + y_i)] \), consequently \( d = -1 \) and \( b = a' \), then for all \( i \in \mathcal{A} \) and \( a \in \mathbb{R} \) we have
\[ f[x_1 + a x_i] = [y_1 + a' y_i] \] (3)
Now we are going to prove that \( \mu \) is surjective. Choose a finite number of \( n \) vectors of \( \mathcal{A} \) including \( x_1 \) and \( x_2 \) say \( x_1, x_2, \ldots, x_n \). Then
\[ f[x_1 + a_2 x_2 + \cdots + a_n x_n] = [y_1 + a'_2 y_2 + \cdots + a'_n y_n] \] (4)
and it follows that
\[ f[a_2 x_2 + \cdots + a_n x_n] = [a'_2 y_2 + \cdots + a'_n y_n]. \] (5)
Let \( L = [y_1 + b y_2] \) be a point of \( P(H) \), since \( \mu \) is bijective, then there exists some \( v \in \hat{H} \) such that \( L = f[v] \), then \( v \) can be written as a linear combination of \( x_j \) including \( x_1, x_2 \). For the purpose we can use the above set \( x_1, x_2, \ldots, x_n \) then
\[ v = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n. \]
By (5) we have \( \alpha_1 \neq 0 \) and consequently,
\[ L = f[x_1 + \beta_2 x_2 + \cdots + \beta_n x_n] \] with \( \beta_j = \frac{\alpha_j}{\alpha_1} \).
Then by (4) \( \beta'_2 = b \) and consequently \( \mu \) is surjective.
To show that \( \mu(a + b) = \mu(a) + \mu(b) \) we consider the line \( L = [x_1 + (a + b)x_2 + x_3] \). Then by (2) and (3) we have
\[ fL = [y_1 + (a + b') y_2 + y_3] \]
but
\[ L \subset [x_1 + a x_2] + [b x_2 + x_3] \] and \( L \neq [b x_2 + x_3] \).
By (4) and (5)
\[ fL \subset [y_1 + a' y_2] + [y_1' + y_3] \] with \( y_1' = y_1 + \beta_2 y_2 + \cdots + \beta_n y_n \).
and so \( fL = [(y_1 + a'y_2) + \lambda(b'y_2 + y_3)] \) for some \( \lambda \). It follows that \( \lambda = 1 \) and so

\[
\mu(a + b) = (a + b)' = a' + b' = \mu(a) + \mu(b).
\]

Similarly by considering a line \([x_1 + (ab)x_2 + x_3, \ldots + a_kx_k] \) we get

\[
\mu(ab) = \mu(a), \mu(b)
\]

(7)

thus \( \mu \) is a bijective mapping satisfying (2),(6) and (7) and therefore it is the identity mapping \( H \). Consequently

\[
f[x_1 + \cdots + a_kx_k] = [a_1y_1 + \cdots + a_ky_k].
\]

(8)

The equation (8) has been derived by fixing \( a, \ldots, a_k \) from the Hamel basis \( \{x_i\} \). Since it still holds for \( a_1, a_2 \) zeros, it follows that (8) is true for any finite combination of vectors in \( \{x_i\} \).

If \( x \in H \), then \( x = \sum a_ix_i \) (a finite sum)

and so we define a linear map

\[
T : H \rightarrow H \text{ by } T(x) = \sum a_iy_i
\]

and \( T \) is also a bijection and it induces a map

\[
\overline{T} : P(H) \rightarrow P(H)
\]

\[
\overline{T}[x] = [T(x)] = [\sum a_iy_i] = f[x]
\]

consequently, \( \overline{T} \) agrees with \( f \).

The bijection \( \tilde{T} : S \rightarrow S \) defined by \( T \) as in Theorem 1.1 is a homeomorphism. This follows from the commutative diagram

\[
P(H) \xrightarrow{f} P(H) \\
\phi \uparrow \quad \uparrow \phi \\
S \xrightarrow{\tilde{T}} S
\]

because \( f \) is supposed a homeomorphism and \( \phi \) is the local diffeomorphism between \( S \) and \( \phi(H) \). It follows from Theorem 1.1 that \( \tilde{T} \) is a homeomorphism.

References


