The Existence of a Topolinear Isomorphism on an
infinite dimensional Hilbert Space $H$ Corresponding to
a Homeomorphism on it's Projective Space $P(H)$

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Abstract

In this paper we prove a theorem which states the relationship between the topolinear iso-
morphisms on an infinite dimentional Hilbert Space $H$ and the Homeomorphisms on projective
Space $P(H)$. This theorem is proved by E.Artin in the finite dimentional case.

Key words: Topolinear Isomorphism, Hilbert Space, Homeomorphism, Projective.
Introduction

The following $H$ is an infinite dimensional separable Hilbert Space and $P(H)$ is its projective space which is given a smooth structure as in [2]. We mean by $[x] \in P(H)$ the one dimensional vector subspace of $H$ generated by $x$, in fact $H = H - 0$.

$[x] + [y]$ means the two dimensional subspace generated by $x, y \in \hat{H}$. in fact $[z] \subset [x] + [y]$ means that there exists $a, b \in \hat{R}$ such that $z = ax + by$ and if $[z] \neq [x]$, There exists a unique $d \in \hat{R}$ such that $[z] = [x + dy]$. We quote some easy statments from [2].

**Theorem 1.1** Let $S$ be a unit sphere in a normed vector space $B$ and $T : B \to B$ a linear bijective transformation, and $\tilde{T}$ be the induced bijective transformation

$$\tilde{T} : S \to S$$

defined by $\tilde{T}(u) = \frac{T(u)}{||T(u)||}$ for $u \in S \subset B$.

This is homeomorphism then $T$ is also homeomorphism.

Now we are ready to state the theorem which is the goal of this paper.

**Theorem 1.2** Let $f : P(H) \to P(H)$ be a homeomorphism such that

$$[x] \subset [y] + [z] \to f[x] \subset f[y] + f[z].$$

Then there exists a toplinear isomorphism $T : P(H) \to H$ such that the induced transformation $f : P(H) \to P(H)$ agrees with $f$.

**Proof.** the hypothesis implies that if $[x] \subset [y] + [z]$ then $f^{-1}[x] \subset f^{-1}[y] + f^{-1}[z]$ and by induction on $k$, we get that if $[z] \subset [z_1] + \cdots + [z_k]$ then $f[z] \subset f[z_1] + \cdots + f[z_k]$, and $f^{-1}[z] \subset f^{-1}[z_1] + \cdots + f^{-1}[z_k]$.

Let $\{x_i\}$ be a Hamel basis for $H$ where $i$ is an arbitrary element of a set $A$. It is clear that if $f[x_i] = [y_i]$ then $\{y_i\}$ is also a Hamel basis for $H$.

Now we choose an element of $A$ call it $i$, then for any $i \neq 1$ the line

$$L_i = [x_1 + x_i] \subset [x_1] + [x_i]$$

where $L_i$ is not coincide with $[x_i]$ or $[x_1]$, consequently

$$fL_i \subset [y_1] + [y_i]$$

and $fL_i$ is not coincide with $[y_i]$ or $[y_1]$. Then, for some unique $d_i \in \hat{R}$ we have

$$fL_i = [y_1 + d_i y_i].$$

by choosing a suitable $y_i$ we may assume that $d_i = 1$. Then

$$f[x_i] = [y_i] \quad (1)$$

and for $i \neq 1$, $f[x_1 + x_i] = [y_1 + y_i]$.

Now we choose another index from $A$, call it $2$. Then for $a \in \hat{R}$

$$L = [x_1 + ax_2] \subset [x_1] + [x_2] \text{ where } L \neq [x_2]$$

Therefore

$$fL \subset [y_1] + [y_2], \text{ where } fL \neq [y_2].$$

Then for a unique $a' \in \hat{R}$ we have

$$fL = [y_1 + a' y_2].$$
Introduction

The following $H$ is an infinite dimensional separable Hilbert Space and $P(H)$ is its positive cone space which is given a smooth structure as in [2]. We mean by $[x] \in P(H)$ the two dimensional vector subspace of $H$ generated by $x, y \in \hat{H}$. In fact $[z] \subseteq [x] + [y]$ means the two dimensional subspace generated by $x,y \in \hat{H}$. In fact $[z] \subseteq [x] + [y]$ means that there exists $a,b \in \hat{R}$ such that $z = ax + by$ and if $[z] \neq [x]$, there exists a unique scalar $c$ such that $[z] = [x + dy]$. We quote some easily statments from [2].

**Theorem 1.1** Let $S$ be a unit sphere in a normed space $B$ and $T : B \to B$ a linear bijective transformation, and $\hat{T}$ be the induced bijective transformation

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defined by $\hat{T}(u) = \frac{T(u)}{||T(u)||}$ for $u \in S \subseteq B$.

If $T$ is a homeomorphism then $T$ is also homeomorphism.

Now we are ready to state the theorem which is the goal of this paper.

**Theorem 1.2** Let $f : P(H) \to P(H)$ be a homeomorphism such that

$$[x] \subseteq [y] \Rightarrow f[x] \subseteq f[y] + f[z].$$

Then there exists a topological isomorphism $T : H \to H$ such that the induced transformation $P(H) \to P(H)$ agrees with $f$.

**Proof.** the hypothesis implies that if $[x] \subseteq [y] + [z]$ then $f^{-1}[x] \subseteq f^{-1}[y] + f^{-1}[z]$ and by induction on $k$, we get that if $[z] \subseteq [z_1] + \ldots + [z_k]$ then $f[z] \subseteq f[z_1] + \ldots + f[z_k]$, and $f^{-1}[z] \subseteq f^{-1}[z_1] + \ldots + f^{-1}[z_k]$.

Let $\{x_i\}$ be a Hamel basis for $H$ where $i$ is an arbitrary element of a set $A$. It is clear that if $f(x_i) = [y_i]$ then $\{y_i\}$ is also a Hamel basis for $H$.

Now we choose an element of $A$ call it $i$, then for any $i \neq 1$ the line

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$$fL_i = [y_i + d_i y_i]$$

by choosing a suitable $y_i$ we may assume that $d_i = 1$. Then

for $i \in A$,

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and for $i \neq 1$, $f[x_i + x_i] = [y_i + y_i]$.

Now we choose another index from $A$, call it $2$. Then for $a \in \hat{R}$

$$L = [x_1 + ax_2] \subseteq [x_1] + [x_2]$$

where $L \neq [x_2]$

Therefore

$$fL \subseteq [y_1] + [y_2], \quad \text{where } fL \neq [y_2].$$

Then for a unique $a' \in \hat{R}$ we have

$$fL = [y_1 + a'y_2].$$
Now we define
\[ \mu : R \rightarrow R \]
by \( \mu(a) = a' \) and we will show that \( \mu \) is the identity function on \( R \). Since
\[ [x_1 + ax_2] \neq [x_1 + bx_2] \text{ if } a \neq b \]
it follows that \( a' \neq b' \), then \( \mu \) is injective. We have also from (1) that
\[ 0' = 0 \; \text{ and } \; 1' = 1. \tag{2} \]
Now, we will show that for any \( i \in A \)
\[ f[x_1 + ax_i] = [y_1 + a'y_i] \]
For any fixed \( i \neq 1, 2 \) in \( A \) we have
\[ f[x_1 + ax_i] = [y_1 + by_i] . \]
On the other hand \( L = [ax_2 - ax_1] \subset [x_2] + [x_i] \) with \( L \neq [x_i] \), and so \( fL \subset [y_2] + [y_i] \) with \( fL \neq [y_i] \). Consequently, \( fL = [y_2 + dy_i] \) for some unique \( d \). On the other hand,
\[ L \subset [x_1 + ax_2] + [x_1 + ax_i] \text{ with } L \neq [x_1 + ax_i] . \]
Then as before \( fL = ([y_1 + a'y_2] + d'(y_1 + by_i)] \) and it follows that \( d = -\frac{b}{a'} \). But
\[ L \subset [x_1 + x_2] + [x_1 + x_i] \text{ with } L \neq [x_1 + x_i] \]
and by (1)
\[ fL \subset [y_1 + y_2] + [y_1 + y_i] \text{ with } fL \neq [y_1 + y_i] \]
Then for some unique \( h \) we have \( fL = [y_1 + y_2 + h(y_1 + y_i)] \), consequently \( d = -1 \) and \( b = a' \), then for all \( i \in A \) and \( a \in R \) we have
\[ f[x_1 + ax_i] = [y_1 + a'y_i] . \tag{3} \]
Now we are going to prove that \( \mu \) is surjective. Choose a finite number of \( n \) vectors of \( \{z \} \)
including \( x_1 \) and \( x_2 \) say \( x_1, x_2, \ldots, x_n \). Then induction we have
\[ f[x_1 + ax_2 + \cdots + a_n x_n] = [y_1 + a'y_2 + \cdots + a'_n y_n] . \tag{4} \]
and it follows that
\[ f[a_2 x_2 + \cdots + a_n x_n] = [a'_2 y_2 + \cdots + a'_n y_n] . \tag{5} \]
Let \( L = [y_1 + by_2] \) be a point of \( P(H) \), since \( \mu \)
is bijective, then there exists some \( v \in H \) such that \( L = f[v] \), then \( v \) can be written as a linear combination of \( x_j \) including \( x_1, x_2 \). For the purpose we can use the above set \( x_1, x_2, \ldots, x_n \), then
\[ v = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n . \]
By (5) we have \( \alpha_1 \neq 0 \) and consequently,
\[ L = f[x_1 + \beta_2 x_2 + \cdots + \beta_n x_n] \text{ with } \beta_j = \frac{\alpha_j}{\alpha_1} . \]
Then by (4) \( \beta_2' = b \) and consequently \( \mu \) is surjective.
To show that \( \mu(a + b) = \mu(a) + \mu(b) \) we consider the line \( L = [x_1 + (a + b)x_2 + x_3] \). Then by (2) and (3) we have
\[ fL = [y_1 + (a + b)y_2 + y_3] \]
but
\[ L \subset [x_1 + ax_2] + [bx_2 + x_3] \text{ and } L \neq [bx_2 + x_3] . \]
By (4) and (5)
\[ fL \subset [y_1 + a'y_2] + [y_1 + hy_3] \text{ with } h = -1 \]
and
\[ fL = [y_1 + a'y_2 + hy_3] . \]
and so \( fL = [(y_1 + a'y_2) + \lambda(b'y_2 + y_3)] \) for some \( \lambda \). It follows that \( \lambda = 1 \) and so
\[
\mu(a + b) = (a + b)' = a' + b' = \mu(a) + \mu(b). \tag{6}
\]
Similarly by considering a line \([x_1 + (ab)x_2 + x_3]\), we get
\[
\mu(ab) = \mu(a), \mu(b) \tag{7}
\]
thus \( \mu \) is a bijective mapping satisfying (2),(6) and (7) and therefore it is the identity mapping \( \mu = I \).

Consequently
\[
f(x_1 + \cdots + a_kx_k) = [a_1y_1 + \cdots + a_ky_k]. \tag{8}
\]
The equation (8) has been derived by fixing \( a_1, \ldots, a_k \) from the Hamel basis \( \{x_i\} \). Since it still holds for \( a_1, a_2 \) zeros, it follows that (8) is true for any finite combination of vectors in \( \{x_i\} \).

If \( x \in H \), then \( x = \sum a_i x_i \) (a finite sum).

So we define a linear map \( T : H \to H \) by \( T(x) = \sum a_i y_i \).

The \( T \) is also a bijection and it induces a map \( \overline{T} : P(H) \to P(H) \)
\[
\overline{T}[x] = [T(x)] = [\sum a_i y_i] = f[x]
\]
consequently, \( \overline{T} \) agrees with \( f \).

The bijection \( \tilde{T} : S \to S \) defined by \( T \) as in theorem 1.1 is a homeomorphism. This follows from the commutative diagram
\[
\begin{array}{ccc}
P(H) & \xrightarrow{f} & P(H) \\
\uparrow \phi & & \uparrow \phi \\
S & \xrightarrow{\tilde{T}} & S
\end{array}
\tag{9}
\]

because \( f \) is supposed a homeomorphism and \( \phi \) is the local diffeomorphism between \( S \) and \( P(H) \), it follows from Theorem 1.1 that \( \tilde{T} \) is a homeomorphism.