The Existence of a Topolinear Isomorphism on an infinite dimensional Hilbert Space $H$ Corresponding to a Homeomorphism on it’s Projective Space $P(H)$

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Abstract

In this paper we prove a theorem which states the relationship between the topolinear isomorphisms on an infinite dimensional Hilbert Space $H$ and the Homeomorphisms on projective Space $P(H)$. This theorem is proved by E. Artin in the finite dimensional case.

Key words: Topolinear Isomorphism, Hilbert Space, Homeomorphism, Projective.
Introduction

The following $H$ is an infinite dimensional separable Hilbert Space and $P(H)$ is its projective space which is given a smooth structure as in [2]. We mean by $[x] \in P(H)$ the one dimensional vector subspace of $H$ generated by $x$, i.e. $[x] = Hx = \{yx : y \in H\}$. The two dimensional subspace generated by $x, y \in H$. in fact $[z] \subset [x] + [y]$ means that there exists a $t \in \mathbb{R}$ such that $z = x + ty$ and if $[z] \neq [x]$, there exists a unique $b \in \mathbb{R}$ such that $[z] = [x + dy]$. We quote some necessary statements from [2].

**Theorem 1.1** Let $S$ be a unit sphere in a normed vector space $B$ and $T : B \rightarrow B$ a linear bijective transformation, and $\hat{T}$ be the induced bijective transformation

$$\hat{T} : S \rightarrow S$$

such that $\hat{T}(u) = \frac{T(u)}{||T(u)||}$ for $u \in S \subset B$. If $T$ is homeomorphism then $\hat{T}$ is also homeomorphism.

We are ready to state the theorem which is the main result of this paper.

**Theorem 1.2** Let $f : P(H) \rightarrow P(H)$ be a homeomorphism such that

$$[x] \subset [y] + [z] \rightarrow f[x] \subset f[y] + f[z].$$

Then there exists a topolinear isomorphism $T : P(H) \rightarrow H$ such that the induced transformation $f : P(H) \rightarrow P(H)$ agrees with $f$.

**Proof.** the hypothesis implies that if $[x] \subset [y] + [z]$ then $f^{-1}[x] \subset f^{-1}[y] + f^{-1}[z]$ and by induction on $k$, we get that if $[z] \subset [z_1] + \cdots + [z_k]$ then $f[z] \subset f[z_1] + \cdots + f[z_k]$, and $f^{-1}[z] \subset f^{-1}[z_1] + \cdots + f^{-1}[z_k]$.

Let $\{x_i\}$ be a Hamel basis for $H$ where $i$ is an arbitrary element of a set $A$. It is clear that if $f[x_i] = [y_i]$ then $\{y_i\}$ is also a Hamel basis for $H$.

Now we choose an element of $A$ call it 1, then for any $i \neq 1$ the line

$$L_i = [x_1 + x_i] \subset [x_1] + [x_i]$$

where $L_i$ is not coincide with $[x_i]$ or $[x_1]$, consequently

$$fL_i \subset [y_1] + [y_i]$$

and $fL_i$ is not coincide with $[y_i]$ or $[y_1]$. Then, for some unique $d_i \in \mathbb{R}$ we have

$$fL_i = [y_1 + d_iy_i].$$

by choosing a suitable $y_i$ we may assume that $d_i = 1$. Then

$$f[x_i] = [y_i] \quad (1)$$

and for $i \neq 1$, $f[x_1 + x_i] = [y_1 + y_i]$.

Now we choose another index from $A$, call it 2. Then for $a \in \mathbb{R}$

$$L = [x_1 + ax_2] \subset [x_1] + [x_2] \quad \text{where} \quad L \neq [x_2]$$

Therefore

$$fL \subset [y_1] + [y_2], \quad \text{where} \quad fL \neq [y_2].$$

Then for a unique $a' \in \mathbb{R}$ we have

$$fL = [y_1 + a'y_2].$$
Introduction

The following $H$ is an infinite dimensional separable Hilbert Space and $P(H)$ is its positive cone space which is given a smooth structure as in [2]. We mean by $[x] \in P(H)$ the one dimensional vector subspace of $H$ generated by $x, y \in \hat{H}$. in fact $[z] \subseteq [x] + [y]$ means the two dimensional subspace spanned by $x, y \in \hat{H}$. in fact $[z] \subseteq [x] + [y]$ means that there exists $a, b \in \mathbb{R}$ such that $z = ax + by$ and if $[z] \neq [x]$, there exists a unique $\hat{b} \in \hat{H}$ such that $[z] = [x + dy]$. We quote some necessary statements from [2].

**Theorem 1.1** Let $S$ be a unit sphere in a normed space $B$ and $T : B \to B$ a linear bijective transformation, and $\hat{T}$ be the induced bijective transformation

\[ \hat{T} : S \to S \]

defined by $\hat{T}(u) = \frac{T(u)}{||T(u)||}$ for $u \in S \subset B$. If $T$ is a homeomorphism then $T$ is also a homeomorphism.

We are ready to state the theorem which is the goal of this paper.

**Theorem 1.2** Let $f : P(H) \to P(H)$ be a bijection such that

\[ [x] \subseteq [y] + [z] \rightarrow f([x]) \subseteq f([y]) + f([z]). \]

Then there exists a topological isomorphism $T : H \to H$ such that the induced transformation $T : P(H) \to P(H)$ agrees with $f$.

**Proof.** The hypothesis implies that if $[x] \subset [y] + [z]$ then $f^{-1}([x]) \subset f^{-1}([y]) + f^{-1}([z])$ and by induction on $k$, we get that if $[z] \subseteq [z_1] + \cdots + [z_k]$ then $f([z]) \subseteq f([z_1]) + \cdots + f([z_k])$, and $f^{-1}([z]) \subseteq f^{-1}([z_1]) + \cdots + f^{-1}([z_k])$.

Let $\{x_i\}$ be a Hamel basis for $H$ where $i$ is an arbitrary element of a set $\mathcal{A}$. It is clear that if $f(x_i) = [y_i]$ then $\{y_i\}$ is also a Hamel basis for $H$.

Now we choose an element of $\mathcal{A}$ call it 1, then for any $i \neq 1$ the line

\[ L_i = [x_1 + x_i] \subset [x_1] + [x_i] \]

where $L_i$ is not coincide with $[x_i]$ or $[x_1]$, consequently

\[ fL_i \subset [y_1] + [y_i] \]

and $fL_i$ is not coincide with $[y_i]$ or $[y_1]$. Then, for some unique $d_i \in \mathbb{R}$ we have

\[ fL_i = [y_1 + d_i y_i] \]

by choosing a suitable $y_i$ we may assume that

\[ d_i = 1. \]

Then

\[ f[x_i] = [y_i] \quad (1) \]

and for $i \neq 1, f[x_1 + x_i] = [y_1 + y_i].$

Now we choose another index from $\mathcal{A}$, call it 2. Then for $a \in \mathbb{R}$

\[ L = [x_1 + ax_2] \subset [x_1] + [x_2] \quad \text{where} \quad L \neq [x_2] \]

Therefore

\[ fL \subset [y_1] + [y_2], \quad \text{where} \quad fL \neq [y_2]. \]

Then for a unique $a' \in \mathbb{R}$ we have

\[ fL = [y_1 + a' y_2]. \]
Now we define 
\[ \mu : R \rightarrow R \]
by \( \mu(a) = a' \) and we will show that \( \mu \) is the identity function on \( R \). Since
\[ [x_1 + ax_2] \neq [x_1 + bx_2] \text{ if } a \neq b \]
it follows that \( a' \neq b' \), then \( \mu \) is injective. We have also from (1) that
\[ 0' = 0 \text{ and } 1' = 1. \] (2)

Now, we will show that for any \( i \in A \)
\[ f[x_1 + ax_i] = [y_1 + a'y_i] \]
For any fixed \( i \neq 1,2 \) in \( A \) we have
\[ f[x_1 + ax_i] = [y_1 + by_i]. \]

On the other hand \( L = [ax_2 - ax_i] \subset [x_2] + [x_i] \) with \( L \neq [x_i] \), and so \( fL \subset [y_2] + [y_i] \) with \( fL \neq [y_i] \). Consequently, \( fL = [y_2 + dy_i] \) for some unique \( d \). On the other hand,
\[ L \subset [x_1 + ax_2] + [x_1 + ax_i] \text{ with } L \neq [x_1 + ax_i]. \]

Then as before \( fL = ([y_1 + a'y_2] + d'(y_1 + by_i)] \) and it follows that \( d = \frac{b}{a'} \). But
\[ L \subset [x_1 + x_2] + [x_1 + x_i] \text{ with } L \neq [x_1 + x_i] \]
and by (1)
\[ fL \subset [y_1 + y_2] + [y_1 + y_i] \text{ with } fL \neq [y_1 + y_i] \]
Then for some unique \( h \) we have \( fL = [y_1 + y_2 + h(y_1 + y_i)] \), consequently \( d = -1 \) and \( b = a' \),
then for all \( i \in A \) and \( a \in R \) we have
\[ f[x_1 + ax_i] = [y_1 + a'y_i]. \] (3)

Now we are going to prove that \( \mu \) is surjective. Choose a finite number of \( n \) vectors of \( \mathcal{A} \) including \( x_1 \) and \( x_2 \) say \( x_1, x_2, \ldots, x_n \). Then by induction we have
\[ f[x_1 + a_2 x_2 + \cdots + a_n x_n] = [y_1 + a'_2 y_2 + \cdots + a'_n y_n] \]
and it follows that
\[ f[a_2 x_2 + \cdots + a_n x_n] = [a'_2 y_2 + \cdots + a'_n y_n]. \] (1)


Let \( L = [y_1 + by_2] \) be a point of \( P(H) \), since \( \mu \) is bijective, then there exists some \( v \in \tilde{H} \) such that \( L = f[v] \), then \( v \) can be written as a linear combination of \( x_j \) including \( x_1, x_2 \). For this purpose we can use the above set \( x_1, x_2, \ldots, x_n \) then
\[ v = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n. \]

By (5) we have \( \alpha_1 \neq 0 \) and consequently,
\[ L = f[x_1 + \beta_2 x_2 + \cdots + \beta_n x_n] \text{ with } \beta_j = \frac{\alpha_j}{\alpha_1}. \]

Then by (4) \( \beta_2 = b \) and consequently \( \mu \) is surjective.

To show that \( \mu(a + b) = \mu(a) + \mu(b) \) we consider the line \( L = [x_1 + (a + b)x_2 + x_3] \). Then by (2) and (3) we have
\[ fL = [y_1 + (a + b')y_2 + y_3] \]
but
\[ L \subset [x_1 + ax_2] + [bx_2 + x_3] \text{ and } L \neq [bx_2 + x_3]. \]
By (4) and (5)
\[ fL \subset [y_1 + a'y_2 + h'y_3] \text{ with } h' \neq 0 \].
and so \( fL = [(y_1 + a'y_2) + \lambda(b'y_2 + y_2)] \) for some \( \lambda \).

It follows that \( \lambda = 1 \) and so

\[
\mu(a+b) = (a+b') = a' + b' = \mu(a) + \mu(b). \quad (6)
\]

Similarly by considering a line \( x_1 + (ab)x_2 + x_3 \), we get

\[
\mu(ab) = \mu(a) \cdot \mu(b) \quad (7)
\]

thus \( \mu \) is a bijective mapping satisfying (2),(6) and (7) and therefore it is the identity mapping \( f \).

Consequently

\[
f[x_1 \cdots + a_k x_k] = [a_1 y_1 \cdots + a_k y_k]. \quad (8)
\]

The equation (8) has been derived by fixing \( a_1, \ldots, a_k \) from the Hamel basis \( \{x_i\} \). Since it still holds for \( a_1, a_2 \) zeros, it follows that (8) is true for any finite combination of vectors in \( \{x_i\} \).

If \( x \in H \), then \( x = \sum a_i x_i \) ( a finite sum ).

So we define a linear map

\[
T : H \rightarrow H \text{ by } T(x) = \sum a_i y_i
\]

The \( T \) is also a bijection and it induces a map

\[
\overline{T} : P(H) \rightarrow P(H)
\]

\[
\overline{T}[x] = [T(x)] = [\sum a_i y_i] = f[x]
\]

consequently, \( \overline{T} \) agrees with \( f \).

The bijection \( \overline{T} : S \rightarrow S \) defined by \( T \) as in Theorem 1.1 is a homeomorphism. This follows from the commutative diagram

\[
\begin{array}{ccc}
P(H) & \xrightarrow{f} & P(H) \\
\phi \uparrow & & \uparrow \phi \\
S & \xrightarrow{T} & S
\end{array}
\]

because \( f \) is supposed a homeomorphism and \( \phi \) is the local diffeomorphism between \( S \) and \( P(H) \), it follows from Theorem 1.1 that \( T \) is a

References


