The Existence of a Topolinear Isomorphism on an infinite dimensional Hilbert Space $H$ Corresponding to a Homeomorphism on it's Projective Space $P(H)$

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Abstract
In this paper we prove a theorem which states the relationship between the topolinear isomorphisms on an infinite dimensional Hilbert Space $H$ and the Homeomorphisms on projective Space $P(H)$. This theorem is proved by E.Artin in the finite dimensional case.

Key words: Topolinear Isomorphism, Hilbert Space, Homeomorphism, Projective.
Introduction

The following \( H \) is an infinite dimensional separable Hilbert Space and \( P(H) \) is its projective space which is given a smooth structure as in [2]. We mean by \( [x] \in P(H) \) the one dimensional vector subspace of \( H \) generated by \( x \in H \). \( [x] + [y] \) means the two dimensional subspace generated by \( x, y \in H \). in fact \( [z] \subseteq [x] + [y] \) means that there exists \( a, b \in \mathbb{R} \) such that \( z = ax + by \) and if \( [z] \neq [x] \), There exists a unique \( dy \) such that \( [z] = [x + dy] \). We quote some necessary statemtents from [2].

**Theorem 1.1** Let \( S \) be a unit sphere in a normed linear space \( B \) and \( T : B \rightarrow B \) a linear bijection, and \( \tilde{T} \) be the induced bijection on \( S \)

\[
\tilde{T} : S \rightarrow S
\]

defined by \( \tilde{T}(u) = \frac{T(u)}{||T(u)||} \) for \( u \in S \subseteq B \).

If \( T \) is a homeomorphism then \( T \) is also homeomorphism.

We are ready to state the theorem which is the main goal of this paper

**Theorem 1.2** Let \( f : P(H) \rightarrow P(H) \) be a homeomorphism such that

\[
[x] \subseteq [y] + [z] \rightarrow f[x] \subseteq f[y] + f[z].
\]

Then there exists a topolinear isomorphism \( T : H \rightarrow H \) such that the induced transformation \( P(H) \rightarrow P(H) \) agrees with \( f \).

**Proof:** the hypothesis implies that if \( [x] \subseteq [y] + [z] \) then \( f^{-1}[x] \subseteq f^{-1}[y] + f^{-1}[z] \) and by induction on \( k \), we get that if \( [z] \subseteq [z_1] + \cdots + [z_k] \) then \( f[z] \subseteq f[z_1] + \cdots + f[z_k] \), and \( f^{-1}[z] \subseteq f^{-1}[z_1] + \cdots + f^{-1}[z_k] \).

Let \( \{x_i\} \) be a Hamel basis for \( H \) where \( i \) is an arbitrary element of a set \( A \). It is clear that if \( f[x_i] = [y_i] \) then \( \{y_i\} \) is also a Hamel basis for \( H \).

Now we choose an element of \( A \) call it 1, then for any \( i \neq 1 \) the line

\[
L_i = [x_1 + x_i] \subseteq [x_1] + [x_i]
\]

where \( L_i \) is not coincide with \( [x_i] \) or \( [x_1] \), consequently

\[
fL_i \subseteq [y_1] + [y_i]
\]

and \( fL_i \) is not coincide with \( [y_i] \) or \( [y_1] \). Then, for some unique \( d_i \in \mathbb{R} \) we have

\[
fL_i = [y_1 + d_i y_i].
\]

by choosing a suitable \( y_i \) we may assume that \( d_i = 1 \). Then

\[
\text{for } i \in A, \quad f[x_i] = [y_i] \tag{1}
\]

and for \( i \neq 1, f[x_1 + x_i] = [y_1 + y_i] \).

Now we choose another index from \( A \), call it 2. Then for \( a \in \mathbb{R} \)

\[
L = [x_1 + ax_2] \subseteq [x_1] + [x_2] \quad \text{where } L \neq [x_2]
\]

Therefore

\[
fL \subseteq [y_1] + [y_2], \quad \text{where } fL \neq [y_2].
\]

Then for a unique \( a' \in \mathbb{R} \) we have

\[
fL = [y_1 + a'y_1].
\]
Introduction

If the following $H$ is an infinite dimensional separable Hilbert Space and $P(H)$ is its projective space which is given a smooth structure as in [2]. We mean by $[x] \in P(H)$ the one dimensional vector subspace of $H$ generated by $x \in H$. $[x]+[y]$ means the two dimensional subspace generated by $x, y \in \hat{H}$, in fact $[z] \subset [x]+[y]$ means that there exists $a,b \in \mathbb{R}$ such that $z = ax + by$ and if $[z] \neq [x]$, There exists a unique $\frac{z}{z}$ such that $[z] = [x + dy]$. We quote some necessary statments from [2].

Theorem 1.1 Let $S$ be a unit sphere in a normed vector space $B$ and $T : B \rightarrow B$ a linear bijective transformation, and $\hat{T}$ be the induced bijective transformation

$$\hat{T} : S \rightarrow S$$

defined by $\hat{T}(u) = \frac{T(u)}{\|T(u)\|}$ for $u \in S \subset B$. If $T$ is a homeomorphism then $T$ is also a homeomorphism.

We are ready to state the theorem which is the goal of this paper.

Theorem 1.2 Let $f : P(H) \rightarrow P(H)$ be a homeomorphism such that

$$[x] \subset [y] + [z] \rightarrow f[x] \subset f[y] + f[z].$$

Then there exists a topolinear isomorphism $T : P(H) \rightarrow H$ such that the induced transformation $f^* : P(H) \rightarrow P(H)$ agrees with $f$.

Proof. the hypothesis implies that if $[x] \subset [y] + [z]$ then $f^{-1}([x]) \subset f^{-1}([y]) + f^{-1}([z])$ and by induction on $k$, we get that if $[z] \subset [z_1] + \cdots + [z_k]$ then $f[z] \subset f[z_1] + \cdots + f[z_k]$, and $f^{-1}([z]) \subset f^{-1}([z_1]) + \cdots + f^{-1}([z_k])$.

Let $\{x_i\}$ be a Hamel basis for $H$ where $i$ is an arbitrary element of a set $A$. It is clear that if $f[x_i] = [y_i]$ then $\{y_i\}$ is also a Hamel basis for $H$.

Now we choose an element of $A$ call it 1, then for any $i \neq 1$ the line

$$L_i = [x_1 + x_i] \subset [x_1] + [x_i]$$

where $L_i$ is not coincide with $[x_i]$ or $[x_1]$, consequently

$$f[L_i] \subset [y_i] + [y_i]$$

and $f[L_i]$ is not coincide with $[y_i]$ or $[y_1]$. Then, for some unique $d_i \in \mathbb{R}$ we have

$$f[L_i] = [y_1 + d_i y_i].$$

by choosing a suitable $y_i$ we may assume that $d_i = 1$. Then

$$f[x_i] = [y_i] \quad (1)$$

and for $i \neq 1$, $f[x_1 + x_i] = [y_1 + y_i].$

Now we choose another index from $A$, call it 2. Then for $a \in \mathbb{R}$

$$L = [x_1 + ax_2] \subset [x_1] + [x_2]$$

where $L \neq [x_2]$.

Therefore

$$f[L] \subset [y_1] + [y_2]$$

where $f[L] \neq [y_2]$.

Then for a unique $a' \in \mathbb{R}$ we have

$$f[L] = [y_1 + a'y_2].$$
Now we define
\[ \mu : R \rightarrow R \]
by \( \mu(a) = a' \) and we will show that \( \mu \) is the identity function on \( R \). Since
\[ [x_1 + ax_2] \neq [x_1 + bx_2] \text{ if } a \neq b \]
it follows that \( a' \neq b' \), then \( \mu \) is injective. We have also from (1) that
\[ 0' = 0 \text{ and } 1' = 1. \quad (2) \]
Now, we will show that for any \( i \in \mathcal{A} \)
\[ f[x_1 + ax_i] = [y_1 + a'y_i] \]
For any fixed \( i \neq 1, 2 \) in \( \mathcal{A} \) we have
\[ f[x_1 + ax_i] = [y_1 + by_i] . \]
On the other hand \( L = [ax_2 - ax_i] \subset [x_2] + [x_i] \)
with \( L \neq [x_i] \), and so \( fL \subset [y_2] + [y_i] \) with
\( fL \neq [y_i] \). Consequently, \( fL = [y_2 + dy_i] \) for
some unique \( d \). On the other hand,
\[ L \subset [x_1 + ax_2] + [x_1 + ax_i] \text{ with } L \neq [x_1 + ax_i] . \]
Then as before \( fL = (y_1 + a'y_2) + d'(y_1 + by_i) \)
and it follows that \( d' = -b' \). But
\[ L \subset [x_1 + x_2] + [x_1 + x_i] \text{ with } L \neq [x_1 + x_i] \]
and by (1)
\[ fL \subset [y_1 + y_2] + [y_1 + y_i] \text{ with } fL \neq [y_1 + y_i] \]
Then for some unique \( h \) we have \( fL = [y_1 + y_2 +\]
\( h(y_1 + y_i)] \), consequently \( d = -1 \) and \( b = a' \),
then for all \( i \in \mathcal{A} \) and \( a \in R \) we have
\[ f[x_1 + ax_i] = [y_1 + a'y_i] . \quad (3) \]
Now we are going to prove that \( \mu \) is surjective. Choose a finite number of \( n \) vectors of \( \{x_i \} \)
including \( x_1 \) and \( x_2 \) say \( x_1, x_2, \ldots, x_n \). Then
induction we have
\[ f[x_1 + a_2x_2 + \ldots + a_nx_n] = [y_1 + a'_2y_2 + \ldots + a'_nx_n] \]
and it follows that
\[ f[a_2x_2 + \ldots + a_nx_n] = [a'_2y_2 + \ldots + a'_ny_n] . \quad (4) \]
Let \( L = [y_1 + by_2] \) be a point of \( P(H) \), since
is bijective, then there exists some \( v \in \hat{H} \) such that \( L = f[v] \), then \( v \) can be written as a linear combination of \( x_j \) including \( x_1, x_2 \). For the purpose we can use the above set \( x_1, x_2, \ldots, x_n \)
then
\[ v = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n . \]
By (5) we have \( \alpha_1 \neq 0 \) and consequently,
\[ L = f[x_1 + \beta_2x_2 + \ldots + \beta_nx_n] \text{ with } \beta_j = \frac{\alpha_j}{\alpha_1} . \]
Then by (4) \( \beta'_2 = b \) and consequently \( \mu \) is surjective.

To show that \( \mu(a + b) = \mu(a) + \mu(b) \) we consider the line \( L = [x_1 + (a + b)x_2 + x_3] \). Then
by (2) and (3) we have
\[ fL = [y_1 + (a + b)'y_2 + y_3] \]
but
\[ L \subset [x_1 + ax_2] + [bx_2 + x_3] \text{ and } L \neq [bx_2 + x_3] . \]
By (4) and (5)
\[ fL \subset [y_1 + a'y_2] + [y'_2 + y_3] \text{ with } fL \neq [y'_2 + y_3] . \]
References


