The Existence of a Topolinear Isomorphism on an infinite dimensional Hilbert Space $H$ Corresponding to a Homeomorphism on it’s Projective Space $P(H)$

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Abstract

In this paper we prove a theorem which states the relationship between the topolinear isomorphisms on an infinite dimensional Hilbert Space $H$ and the Homeomorphisms on projective Space $P(H)$. This theorem is proved by E.Artin in the finite dimensional case.

Key words: Topolinear Isomorphism, Hilbert Space, Homeomorphism, Projective.
Introduction

The following \( H \) is an infinite dimensional separable Hilbert Space and \( P(H) \) is its projection space which is given a smooth structure as in [2]. We mean by \( [x] \in P(H) \) the \( k \)-dimensional vector subspace of \( H \) generated by \( x \in H = H - 0 \).

\([x] + [y]\) means the two dimensional subspace generated by \( x, y \in H \). in fact \( [z] \subseteq [x] + [y] \) implies that there exists \( a, b \in \mathbb{R} \) such that \( z = ax + by \) and if \( [z] \neq [x] \), There exists a unique \( \frac{a}{b} \) such that \( [z] = [x + dy] \). We quote some necessary statements from [2].

**Theorem 1.1** Let \( S \) be a unit sphere in a normed linear space \( B \) and \( T : B \rightarrow B \) a linear bijective transformation, and \( \tilde{T} \) be the induced bijective transformation

\[ \tilde{T} : S \rightarrow S \]

defined by \( \tilde{T}(u) = \frac{T(u)}{||T(u)||} \) for \( u \in S \subseteq B \).

If \( \tilde{T} \) is homeomorphism then \( T \) is also homeomorphism.

Now we are ready to state the theorem which is the goal of this paper.

**Theorem 1.2** Let \( f : P(H) \rightarrow P(H) \) be a homeomorphism such that

\[ [x] \subseteq [y] + [z] \rightarrow f[x] \subseteq f[y] + f[z]. \]

Then there exists a topological isomorphism \( T : P(H) \rightarrow H \) such that the induced transformation \( T \cdot P(H) \rightarrow P(H) \) agrees with \( f \).

**Proof.** the hypothesis implies that if \( [x] \subseteq [y] + [z] \) then \( f^{-1}[x] \subseteq f^{-1}[y] + f^{-1}[z] \) and by induction on \( k \), we get that if \( [z] \subseteq [z_1] + \ldots + [z_k] \) then \( f[z] \subseteq f[z_1] + \ldots + f[z_k] \) and \( f^{-1}[z] \subseteq f^{-1}[z_1] + \ldots + f^{-1}[z_k] \).

Let \( \{x_i\} \) be a Hamel basis for \( H \) where \( i \) is an arbitrary element of a set \( A \). It is clear that if \( f[x_i] = [y_i] \) then \( \{y_i\} \) is also a Hamel basis for \( H \).

Now we choose an element of \( A \) call it 1, then for any \( i \neq 1 \) the line

\[ L_i = [x_1 + x_i] \subseteq [x_1] + [x_i] \]

where \( L_i \) is not coincide with \( [x_i] \) or \( [x_1] \), consequently

\[ fL_i \subseteq [y_1] + [y_i] \]

and \( fL_i \) is not coincide with \([y_i] \) or \([y_1] \). Then, for some unique \( d_i \in \mathbb{R} \) we have

\[ fL_i = [y_1 + d_i y_i]. \]

by choosing a suitable \( y_i \) we may assume that \( d_i = 1 \). Then

\[ f[x_i] = [y_i] \quad (1) \]

and for \( i \neq 1 \), \( f[x_1 + x_i] = [y_1 + y_i] \). Then for another index from \( A \), call it 2. Then for \( a \in \mathbb{R} \)

\[ L = [x_1 + ax_2] \subseteq [x_1] + [x_2] \quad \text{where} \quad L \neq [x_2] \]

Therefore

\[ fL \subseteq [y_1] + [y_2], \quad \text{where} \quad fL \neq [y_2]. \]

Then for a unique \( a' \in \mathbb{R} \) we have

\[ fL = [y_1 + a'y_2]. \]
Introduction

The following $H$ is an infinite dimensional separable Hilbert Space and $P(H)$ is its positive cone space which is given a smooth structure as in [2]. We mean by $[x] \in P(H)$ the one dimensional vector subspace of $H$ generated by $x \in H = H - 0$.

$[x]+[y]$ means the two dimensional subspace generated by $x, y \in H$. In fact $[z] \subseteq [x]+[y]$ means that there exists $a, b \in \mathbb{R}$ such that $z = ax + by$, and if $[z] \neq [x]$, there exists a unique $\beta \in \mathbb{R}$ such that $[z] = [x + \beta y]$. We quote some necessary statements from [2].

**Theorem 1.1** Let $S$ be a unit sphere in a normed linear space $B$ and $T : B \rightarrow B$ a linear bijective transformation, and $\hat{T}$ be the induced bijective transformation

$$\hat{T} : S \rightarrow S$$

defined by $\hat{T}(u) = \frac{T(u)}{|T(u)|}$ for $u \in S \subseteq B$. If $\hat{T}$ is homeomorphism then $T$ is also homeomorphism.

Now we are ready to state the theorem which is the goal of this paper.

**Theorem 1.2** Let $f : P(H) \rightarrow P(H)$ be a homeomorphism such that

$$[x] \subset [y]+[z] \rightarrow f([x]) \subset f([y])+f([z]).$$

Then there exists a topological isomorphism $T : P(H) \rightarrow H$ such that the induced transformation $T : P(H) \rightarrow P(H)$ agrees with $f$.

**Proof.** The hypothesis implies that if $[x] \subset [y]+[z]$ then $f^{-1}[x] \subset f^{-1}[y]+f^{-1}[z]$ and by induction on $k$, we get that if $[z] \subset [z_1]+\ldots+[z_k]$ then $f([z]) \subset f([z_1]+\ldots+[z_k])$, and $f^{-1}[z] \subset f^{-1}[z_1]+\ldots+f^{-1}[z_k]$.

Let $\{x_i\}$ be a Hamel basis for $H$ where $i$ is an arbitrary element of a set $A$. It is clear that if $f(x_i) = [y_i]$ then $\{y_i\}$ is also a Hamel basis for $H$.

Now we choose an element of $A$ call it 1, then for any $i \neq 1$ the line

$$L_i = [x_1 + x_i] \subset [x_1] + [x_i]$$

where $L_i$ is not coincide with $[x_i]$ or $[x_1]$, consequently

$$fL_i \subset [y_1] + [y_i]$$

and $fL_i$ is not coincide with $[y_i]$ or $[y_1]$. Then, for some unique $d_i \in \mathbb{R}$ we have

$$fL_i = [y_1 + d_i y_i].$$

by choosing a suitable $y_i$ we may assume that $d_i = 1$. Then

$$f[x_i] = [y_i] \quad (1)$$

and for $i \neq 1$, $f[x_1 + x_i] = [y_1 + y_i]$.

Now we choose another index from $A$, call it 2. Then for $a \in \mathbb{R}$

$$L = [x_1 + ax_2] \subset [x_1] + [x_2] \text{ where } L \neq [x_2]$$

Therefore

$$fL \subset [y_1] + [y_2], \text{ where } fL \neq [y_2].$$

Then for a unique $a' \in \mathbb{R}$ we have

$$fL = [y_1 + a' y_2].$$
Now we define 
\[\mu : R \to R\]
by \(\mu(a) = a'\) and we will show that \(\mu\) is the identity function on \(R\). Since 
\[\begin{align*}
[x_1 + ax_2] &\neq [x_1 + bx_2] \text{ if } a \neq b
\end{align*}\]
it follows that \(a' \neq b'\), then \(\mu\) is injective. We have also from (1) that 
\[0' = 0 \text{ and } 1' = 1. \tag{2}\]
Now, we will show that for any \(i \in \mathcal{A}\)
\[f[x_1 + ax_i] = [y_1 + a'y_i]\]
For any fixed \(i \neq 1, 2\) in \(\mathcal{A}\) we have
\[f[x_1 + ax_i] = [y_1 + by_i].\]
On the other hand \(L = [ax_2 - ax_1] \subset [x_2] + [x_i]\)
with \(L \neq [x_i]\), and so \(fL \subset [y_2] + [y_i]\) with 
\[fL \neq [y_i].\]
Consequently, \(fL = [y_2 + dy_i]\) for some unique \(d\). On the other hand,
\[L \subset [x_1 + ax_2] + [x_1 + ax_i]\text{ with } L \neq [x_1 + ax_i].\]
Then as before \(fL = ([y_1 + a'y_2] + d'(y_1 + by_i)]\)
and it follows that \(d = -\frac{b'}{a'}\). But
\[L \subset [x_1 + x_2] + [x_1 + x_i]\text{ with } L \neq [x_1 + x_i]\]
and by (1)
\[fL \subset [y_1 + y_2] + [y_1 + y_i]\text{ with } fL \neq [y_1 + y_i].\]
Then for some unique \(h\) we have \(fL = [y_1 + y_2 +
\[h(y_1 + y_i)]\), consequently \(d = -1\) and \(b = a'\),
then for all \(i \in \mathcal{A}\) and \(a \in R\) we have
\[f[x_1 + ax_i] = [y_1 + a'y_i]. \tag{3}\]
Now we are going to prove that \(\mu\) is surjective. Choose a finite number of \(n\) vectors of \(\mathcal{A}\) including \(x_1\) and \(x_2\) say \(x_1, x_2, \ldots, x_n\). Then
\[f[x_1 + ax_2 + \cdots + a_n x_n] = [y_1 + a'_2 y_2 + \cdots + a'_n y_n]\]
and it follows that
\[f[a_2 x_2 + \cdots + a_n x_n] = [a'_2 y_2 + \cdots + a'_n y_n]. \tag{4}\]
Let \(L = [y_1 + by_2]\) be a point of \(P(H)\), since \(\mu\) is bijective, then there exists some \(v \in H\) such that \(L = f[v]\), then \(v\) can be written as a linear combination of \(x_j\) including \(x_1, x_2\). For this purpose we can use the above set \(x_1, x_2, \ldots, x_n\), then
\[v = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n.\]
By (5) we have \(\alpha_1 \neq 0\) and consequently,
\[L = f[x_1 + \beta_2 x_2 + \cdots + \beta_n x_n]\text{ with } \beta_j = \frac{\alpha_j}{\alpha_1}.\]
Then by (4) \(\beta_2' = b\) and consequently \(\mu\) is surjective.

To show that \(\mu(a + b) = \mu(a) + \mu(b)\) we consider the line \(L = [x_1 + (a + b)x_2 + x_3]\). Then
by (2) and (3) we have
\[fL = [y_1 + (a + b)' y_2 + y_3]\]
but
\[L \subset [x_1 + ax_2] + [bx_2 + x_3]\text{ and } L \neq [bx_2 + x_3].\]
By (4) and (5)
\[fL \subset [y_1 + a'y_2 + y_3] + [y_2 + y_3]\text{ with } fL \neq [y_2 + y_3].\]
and so \( fL = [(y_1 + a'y_2) + \lambda(b'y_2 + y_3)] \) for some \( \lambda \). It follows that \( \lambda = 1 \) and so

\[
\mu(a + b) = (a + b)' = a' + b' = \mu(a) + \mu(b).
\]

Similarly by considering a line \( [x_1 + (ab)x_2 + x_3] \), we get

\[
\mu(ab) = \mu(a) \mu(b)
\]

Thus \( \mu \) is a bijective mapping satisfying (2), (6) and (7) and therefore it is the identity mapping \( I \). Consequently

\[
f[x_1 + \cdots + a_k x_k] = [a_1 y_1 + \cdots + a_k y_k].
\]

The equation (8) has been derived by fixing \( x \) from the Hamel basis \( \{x_i\} \). Since it still holds for \( a_1, a_2 \) zeros, it follows that (8) is true for any finite combination of vectors in \( \{x_i\} \).

If \( x \in H \), then \( x = \sum a_i x_i \) (a finite sum).

Hence we define a linear map

\[
T : H \rightarrow H \quad \text{by} \quad T(x) = \sum a_i y_i
\]

The map \( T \) is also a bijection and it induces a map

\[
\overline{T} : P(H) \rightarrow P(H)
\]

\[
\overline{T}[x] = [T(x)] = [\sum a_i y_i] = f[x]
\]

Consequently, \( \overline{T} \) agrees with \( f \).

The bijection \( \overline{T} : S \rightarrow S \) defined by \( T \) as in theorem 1.1 is a homeomorphism. This follows from the commutative diagram

\[
P(H) \xrightarrow{f} P(H)
\]

\[
\phi \uparrow \quad \uparrow \phi
\]

\[
S \xrightarrow{T} S
\]

because \( f \) is supposed a homeomorphism and \( \phi \) is the local diffeomorphism between \( S \) and \( (P(H), \phi) \). It follows from Theorem 1.1 that \( T \) is a homeomorphism.