The Existence of a Topolinear Isomorphism on an infinite dimensional Hilbert Space $H$ Corresponding to a Homeomorphism on it's Projective Space $P(H)$

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Abstract

In this paper we prove a theorem which states the relationship between the topolinear isomorphisms on an infinite dimensional Hilbert Space $H$ and the Homeomorphisms on projective Space $P(H)$. This theorem is proved by E.Artin in the finite dimentional case.

Key words: Topolinear Isomorphism, Hilbert Space, Homeomorphism, Projective.
Introduction

The following \( H \) is an infinite dimensional separable Hilbert Space and \( P(H) \) is its positive definite space which is given a smooth structure as in [2]. We mean by \([x] \in P(H)\) the \( n\)-dimensional vector subspace of \( H \) generated by \( x \in H = H - 0 \).

\([x] + [y]\) means the two dimensional subspace generated by \( x, y \in H \). in fact \([z] \subset [x] + [y]\) if and only if there exists \( a, b \in H \) such that \( z = ax + by \). and if \([z] \neq [x]\), There exists a unique \( z \) such that \([z] = [x + d]y\). We quote some of the previous statments from [2].

**Theorem 1.1** Let \( S \) be a unit sphere in a normed vector space \( B \) and \( T : B \to B \) a linear bijective transformation, and \( \hat{T} \) be the induced bijective transformation

\[
\hat{T} : S \to S
\]

defined by \( \hat{T}(u) = \frac{T(u)}{\|T(u)\|} \) for \( u \in S \subset B \).

If \( \hat{T} \) is a homeomorphism then \( T \) is also homeomorphism.

We are ready to state the theorem which is the goal of this paper

**Theorem 1.2** Let \( f : P(H) \to P(H) \) be a homeomorphism such that

\( [x] \subset [y] + [z] \to f[x] \subset f[y] + f[z] \).

Then there exists a topolinear isomorphism \( T : P(H) \to H \) such that the induced transformation \( f : P(H) \to P(H) \) agrees with \( f \).

**Proof.** the hypothesis implies that if \([x] \subset [y] + [z]\) then \( f^{-1}[x] \subset f^{-1}[y] + f^{-1}[z] \) and by induction on \( k \), we get that if \([z] \subset [z_1] + \cdots + [z_k]\) then \( f[z] \subset f[z_1] + \cdots + f[z_k] \), and \( f^{-1}[z] \subset f^{-1}[z_1] + \cdots + f^{-1}[z_k] \).

Let \( \{x_i\} \) be a Hamel basis for \( H \) where \( i \) is an arbitrary element of a set \( A \). It is clear that if \( f[x_i] = [y_i] \) then \( \{y_i\} \) is also a Hamel basis for \( H \).

Now we choose an element of \( A \) call it 1, then for any \( i \neq 1 \) the line

\[
L_i = [x_1 + x_i] \subset [x_1] + [x_i]
\]

where \( L_i \) is not coincide with \([x_i] \) or \([x_1] \), consequently

\[
fL_i \subset [y_1] + [y_i]
\]

and \( fL_i \) is not coincide with \([y_i] \) or \([y_1] \). Then, for some unique \( d_i \in R \) we have

\[
fL_i = [y_1 + d_i y_i].
\]

by choosing a suitable \( y_i \) we may assume that \( d_i = 1 \). Then

\[
\text{for } i \in A, \quad f[x_i] = [y_i] \quad (1)
\]

and for \( i \neq 1 \), \( f[x_1 + x_i] = [y_1 + y_i] \).

Now we choose another index from \( A \), call it 2. Then for \( a \in R \)

\[
L = [x_1 + ax_2] \subset [x_1] + [x_2] \text{ where } L \neq [x_2]
\]

Therefore

\[
fL \subset [y_1] + [y_2], \text{ where } fL \neq [y_2].
\]

Then for a unique \( a' \in R \) we have

\[
fL = [y_1 + a' y_2].
\]
Introduction

The following $H$ is an infinite dimensional separable Hilbert Space and $P(H)$ is its projective space which is given a smooth structure as in [2]. We mean by $[x] \in P(H)$ the closed two dimensional vector subspace of $H$ generated by $x, y \in \hat{H}$, in fact $[z] \subset [x] + [y]$ means the two dimensional subspace generated by $x, y \in \hat{H}$, in fact $[z] \subset [x] + [y]$ means the two dimensional subspace generated by $x, y \in \hat{H}$, in fact $[z] \subset [x] + [y]$. We quote some necessary statements from [2].

**Theorem 1.1** Let $S$ be a unit sphere in a normed linear space $B$ and $T : B \rightarrow B$ a linear bijection, and $\tilde{T}$ be the induced bijection transformation

$$\tilde{T} : S \rightarrow S$$

defined by $\tilde{T}(u) = \frac{T(u)}{||T(u)||}$ for $u \in S \subset B$.

If $T$ is a homeomorphism then $T$ is also homeomorphism.

We are ready to state the theorem which is the goal of this paper.

**Theorem 1.2** Let $f : P(H) \rightarrow P(H)$ be a homeomorphism such that

$$[x] \subset [y] + [z] \rightarrow f([x]) \subset f([y]) + f([z]).$$

Then there exists a topological isomorphism $T : P(H) \rightarrow H$ such that the induced transformation $P(H) \rightarrow P(H)$ agrees with $f$.

**Proof.** the hypothesis implies that if $[x] \subset [y] + [z]$ then $f^{-1}[x] \subset f^{-1}[y] + f^{-1}[z]$ and by induction on $k$, we get if $[z] \subset [z_1] + \cdots + [z_k]$ then $f(z) \subset f(z_1) + \cdots + f(z_k)$, and $f^{-1}[z] \subset f^{-1}[z_1] + \cdots + f^{-1}[z_k]$.

Let $\{x_i\}$ be a Hamel basis for $H$ where $i$ is an arbitrary element of a set $A$. It is clear that if $f(x_i) = [y_i]$ then $\{y_i\}$ is also a Hamel basis for $H$.

Now we choose an element of $A$ call it 1, then for any $i \neq 1$ the line

$$L_i = [x_1 + x_i] \subset [x_1] + [x_i]$$

where $L_i$ is not coincide with $[x_i]$ or $[x_1]$. Consequently

$$fL_i \subset [y_1] + [y_i]$$

and $fL_i$ is not coincide with $[y_i]$ or $[y_1]$. Then, for some unique $d_i \in R$ we have

$$fL_i = [y_1 + d_i y_i].$$

by choosing a suitable $y_i$ we may assume that $d_i = 1$. Then

$$f[x_i] = [y_i] \quad (1)$$

and for $i \neq 1$, $f[x_1 + x_i] = [y_1 + y_i]$.

Now we choose another index from $A$, call it 2. Then for $a \in R$

$$L = [x_1 + ax_2] \subset [x_1] + [x_2] \text{ where } L \neq [x_2].$$

Therefore

$$fL \subset [y_1] + [y_2], \text{ where } fL \neq [y_2].$$

Then for a unique $a' \in R$ we have

$$fL = [y_1 + a'y_2].$$
Now we define
\[ \mu : R \rightarrow R \]
by \( \mu(a) = a' \) and we will show that \( \mu \) is the identity function on \( R \). Since
\[ \lfloor x_1 + ax_2 \rfloor \neq \lfloor x_1 + bx_2 \rfloor \text{ if } a \neq b \]
it follows that \( a' \neq b' \), then \( \mu \) is injective. We have also from (1) that
\[ 0' = 0 \text{ and } 1' = 1. \quad (2) \]
Now, we will show that for any \( i \in A \)
\[ f[x_1 + ax_i] = [y_1 + a'y_i] \]
For any fixed \( i \neq 1, 2 \) in \( A \) we have
\[ f[x_1 + ax_i] = [y_1 + by_i]. \]
On the other hand \( L = [ax_2 - ax_1] \subset [x_2] + [x_1] \)
with \( L \neq [x_1] \), and so \( fL \subset [y_2] + [y_1] \) with \( fL \neq [y_1] \). Consequently, \( fL = [y_2 + dy_i] \) for some unique \( d \). On the other hand,
\[ L \subset [x_1 + x_2] + [x_1 + ax_i] \text{ with } L \neq [x_1 + ax_i]. \]
Then as before \( fL = ([y_1 + a'y_2] + d'(y_1 + by_i)) \)
and it follows that \( d = -\frac{b}{a'} \). But
\[ L \subset [x_1 + x_2] + [x_1 + x_i] \text{ with } L \neq [x_1 + x_i] \]
and by (1)
\[ fL \subset [y_1 + y_2] + [y_1 + y_i] \text{ with } fL \neq [y_1 + y_i] \]
Then for some unique \( h \) we have \( fL = [y_1 + y_2 + \ h(y_1 + y_i)] \), consequently \( d = -1 \) and \( b = a' \),
then for all \( i \in A \) and \( a \in R \) we have
\[ f[x_1 + ax_i] = [y_1 + a'y_i]. \quad (3) \]
Now we are going to prove that \( \mu \) is surjective. Choose a finite number of \( n \) vectors of \( \{x \} \)
including \( x_1 \) and \( x_2 \) say \( x_1, x_2, \ldots, x_n \). Then
induction we have
\[ f[x_1 + a_2x_2 + \ldots + a_nx_n] = [y_1 + a'_2y_2 + \ldots + a'_nx_n] \]
and it follows that
\[ f[a_2x_2 + \ldots + a_nx_n] = [a'_2y_2 + \ldots + a'_ny_n]. \quad (4) \]
Let \( L = [y_1 + by_2] \) be a point of \( P(H) \), since \( \mu \)
is bijective, then there exists some \( v \in \hat{H} \) such that \( L = f[v] \), then \( v \) can be written as a linear combination of \( x_j \) including \( x_1, x_2 \). For this purpose we can use the above set \( x_1, x_2, \ldots, x_n \)
then
\[ v = \alpha_1x_1 + \alpha_2x_2 + \ldots + \alpha_nx_n. \]
By (5) we have \( \alpha_1 \neq 0 \) and consequently,
\[ L = f[x_1 + \beta_2x_2 + \ldots + \beta_nx_n] \text{ with } \beta_j = \frac{\alpha_j}{\alpha_1}. \]
Then by (4) \( \beta_2 = b \) and consequently \( \mu \) is surjective.
To show that \( \mu(a + b) = \mu(a) + \mu(b) \) we consider the line \( L = [x_1 + (a+b)x_2 + x_3] \). Then
by (2) and (3) we have
\[ fL = [y_1 + (a+b)y_2 + y_3] \]
but
\[ L \subset [x_1 + ax_2] + [bx_2 + x_3] \text{ and } L \neq [bx_2 + x_3]. \]
By (4) and (5)
\[ fL \subset [y_1 + a'y_2 + y_3] \text{ and } L \neq [by_2 + x_3]. \]
and so \( fL = [(y_1 + a'y_2) + \lambda(b'y_2 + y_3)] \) for some \( \lambda \). It follows that \( \lambda = 1 \) and so

\[
\mu(a + b) = (a + b)' = a' + b' = \mu(a) + \mu(b). \tag{6}
\]

Similarly by considering a line \( x_1 + (ab)x_2 + x_3 = 0 \) we get

\[
\mu(ab) = \mu(a)\mu(b) \tag{7}
\]

thus \( \mu \) is a bijective mapping satisfying (2),(6) and (7) and therefore it is the identity mapping \( \mathbb{H} \). Consequently

\[
f[x_1 + \cdots + a_kx_k] = [a_1y_1 + \cdots + a_ky_k]. \tag{8}
\]

The equation (8) has been derived by fixing \( a_k \) from the Hamel basis \( \{x_i\} \). Since it still holds for \( a_1, a_2 \) zeros, it follows that (8) is true for any finite combination of vectors in \( \{x_i\} \).

if \( x \in \mathbb{H} \), then \( x = \sum a_i x_i \) (a finite sum)

and so we define a linear map

\[
T : \mathbb{H} \to \mathbb{H} \text{ by } T(x) = \sum a_i y_i.
\]

The map \( T \) is also a bijection and it induces a map

\[
\overline{T} : \mathbb{P} H \to \mathbb{P} H
\]

where

\[
\overline{T}[x] = [T(x)] = [\sum a_i y_i] = f[x]
\]

consequently, \( \overline{T} \) agrees with \( f \).

The bijection \( \overline{T} : S \to S \) defined by \( T \) as in theorem 1.1 is a homeomorphism. This follows from the commutative diagram

\[
\begin{array}{c}
P(H) \xrightarrow{f} P(H) \\
\phi \uparrow \quad \uparrow \phi \\
S \xrightarrow{T} S
\end{array}
\]

because \( f \) is supposed a homeomorphism and \( \phi \) is the local diffeomorphism between \( S \) and \( \mathbb{P}(H) \), it follows from Theorem 1.1 that \( T \) is a

References


