Local And Central Limit Theorems For Intensity Measures In Multitype Branching Random Walk

A. Rahimzadeh Sani: Tarbiat Moallem University of Tehran

Abstract

A multitype branching random walk on the real line R is considered. The positions of n-th generation individuals form a point process with related intensity measure. The purpose of this paper is to study the asymptotic behavior of these intensity measures. The central and local limit theorems are proved.

Introduction

A discrete time multitype (p-type) branching random walk on the real line R is considered. The process starts with a single i-type ancestor located at the origin. This particle splits into a random number of new particles of different types, with probability law depending on i, to make the first generation individuals. Then the particles choose some place in R, to inhabit, according to probability laws which depend on the type and position of their parent particle. Each of these particles behave in the same way, independent from each other and the history of the process. For each fixed i = 1, … , p, let $Z_i^n = (Z_{i1}^n, \ldots, Z_{ip}^n)$ be the vector of point processes that give the positions of different types of individuals in generation n descended from an i-type one in generation zero and located at the origin. So for each fixed i, j =1, …, p, Z_{ij}^n is the point process of the positions of j-type individuals in generation n. The multitype branching random walk $\{Z_i^n\}$ is considered quite frequently, see [3], [4],and [5]. Let μ_{ij}^{n*}, be the intensity measure of the point process Z_{ij}^n in the sense that, for all Borel measurable sets $A \subset \mathcal{R}, \mu_{ij}^{n*} = E[Z_{ij}^n(A)]$,(see Notations). Establishing a local limit theorem for the point processes in branching random walk, when the generation size tends to infinity, is considered frequently see, for a single type, [1] and [2]. In the multitype case, [4] has a

MS2000 subject classification. Primary 60J80, 60F05.

Keywords and phrases: Local limit theorem, central limit theorem, multitype branching random walk, intensity measures, kernels of operators.

similar result when the space of residence is a lattice. One of the techniques that is common to all of these papers is the use of similar results for related intensity measures. In setting a local limit theorem for the multitype branching random walk $\{Z_{ij}^{n}(.)\}$ in non-lattice space of residence, that we have in mind , it seems necessary to set a local limit theorem for the intensity measures, that is our Theorem 2.2.

The discussion that we follow in this paper is based on the convolution of matrices of intensity measures that are almost surely finite. Since our method is closely related to the convolution of distribution functions in Markov additive process in [6], we will give the definition and some normalizing, such as tilting and centering of measures in Section 3, to be able to apply the results of [6]. The proofs of the main results are also given in Section 3. Section1 is the introduction and in Section 2 we give the notations and some preliminary results, which is followed by setting the main results.

Notations and some preliminary results

We consider a multitype branching random walk $\{Z_i^n\} = \{(Z_{i1}^n, ..., Z_{ip}^n)\}$ on the real line R with a single i-type ancestor located at the origin. For each fixed i, j = 1, …, p, let the point process $\{Z_{ij}^{n}(r);\ r = 1, 2, ...\}$ give the positions of the j-type individuals in generation n. With a slight abuse of notation, let also $Z_{ij}^{n}(.)$ be the counting measure associated with this point process in the sense that, for any Borel measurable set $A \subset \mathcal{R}$, $Z_{ij}^n(A) = \#\{r : Z_{ij}^n(r) \in A\}$, which is almost surely finite. The point process Z_{ij}^{n} has the intensity measure μ_{ij}^{n*}, defined inductively: if $\mu^{n*} = \{\mu_{ij}^{n*}\}_{p\times p}$ then $\mu_{ij}^{(n+1)*} = \sum_{k=1}^{p} \mu_{ik} \otimes \mu_{Kj}^{n*}$ where "$\otimes$" is ordinary convolution of measures and $\mu_{ij}(\cdot) := \mu_{ij}^{1*}(\cdot)$ is the intensity measure of Z_{ij}^1. Define the Laplace transforms $m_{ij}(\lambda)$ with complex arguments λ by:

$$m_{ij}(\lambda) = \int_{\mathcal{R}} e^{-\lambda x} \mu_{ij}(dx), \qquad \lambda \in C.$$

Let $L = \cap_{i,j} \operatorname{int}\{\lambda = \theta + i\eta \in C \ : m_{ij}(\theta) < \infty\}$. Then L is an open convex subset of C and $L_0 = L \cap \mathcal{R}$ is an open interval (see[3]). Define $M(\lambda) = \{m_{ij}(\lambda)\}_{p\times p}$ and let $M^n(\lambda) = \{m_{ij}^n(\lambda)\}_{p\times p}$ be its n-th power. Then

$$m_{ij}^n(\lambda) = \int_{\mathcal{R}} e^{-\lambda x} \mu_{ij}^{n*}(dx) = E\left[\sum_r e^{-\lambda Z_{ij}^n(r)}\right].$$

The matrix A is called positive regular if all the entries are non-negative and for some n, all the entries of A^n are positive numbers (see[5]). The eigenvalue $\rho(\lambda)$ is called *maximum-modulus eigenvalue* of $M(\lambda)$, if it is a simple eigenvalue of $M(\lambda)$ and for any other eigenvalue $\rho_1(\lambda)$ of $M(\lambda)$, $|\rho_1(\lambda)|<|\rho(\lambda)|$. Assume:

$A(1)$: *The process* $\{Z_i^n\}$ *is positive regular in the sense that the matrix* $M=\{m_{ij}(0)\}=\{E[Z_{ij}(\mathcal{R})]\}$ *is positive regular.*

The entries of the matrix $M(\lambda)$ are complex-valued analytic functions in λ; and for those values of $\lambda=\theta\in\Re$, the matrix $M(\lambda)$ is positive regular. So, the conditions of Theorem 1 in [3] hold and we single out this fact here in the next lemma.

Lemma 2.1. *Let* $L\subset\mathcal{C}$ *be open and for all i, j,* $m_{ij}(\lambda)$ *be analytic in* $\lambda\in L$. *Also let for all* $\theta\in L_0=L\cap\mathcal{R}$, $M(\theta)$ *be positive regular. Then there is an open set* $\Omega\subseteq L$ *contatining* $L_\circ$ *such that for any* $\lambda\in\Omega$, $M(\lambda)$ *has a simple maximum – modulus eigenvalue* $\rho(\lambda)$, *with related left and right eigenvectors* $u(\lambda)$ *and* $\mathrm{v}(\lambda)$ *with the properties that:*

(a) $\rho(\lambda), u(\lambda)$ *and* $\mathrm{v}(\lambda)$ *are nonzero analytic functions in* $\lambda\in\Omega$ *; and for the real argument , they become positive;*

(b) $u(\lambda)$ *and* $\mathrm{v}(\lambda)$ *have nonzero components and are normalized so that* $u(\lambda)^T\,\mathrm{v}(\lambda)=1$ *and* $\sum_{i=1}^{p}u_i(\lambda)=1$.

Let $\theta\in\Omega_0=\Omega\cap\mathcal{R}$. The multitype branching random walk is strongly nonlattice when it is positive regular and, for some (k,l) and some $\theta\in\Omega_\circ$,

$$\left|\frac{m_{kl}(\theta+i\eta)}{m_{kl}(\theta)}\right|=1$$

only whenever $\eta=0$. When the process is strongly nonlattice, then for all $\theta\in\Omega_\circ$, $\rho(\theta)$ is strictly log-convex (see[3]). We set the next assumption:

A(2): The process $\{Z_i^n\}$ *is strongly non-lattice.*

For any $\theta\in\Omega_\circ$, define $\Lambda(\theta)=\log\rho(\theta)$ then by *A(2)*, $\Lambda''(\theta)>0$, and we denote it by $\sigma^2=\Lambda''(\theta)$. For $a=-\rho'(\theta)/\rho(\theta)$ define $\Lambda^*(a)=-\theta\rho'(\theta)/\rho(\theta)+\log\rho(\theta)$. The next assumption states that the process $\{Z_{ij}^n\}$ is supercritical. This implies that, with a positive probability, there are individuals alive in all generations. Assume:

A(3): *For each* $\theta\in\Omega_\circ$, $\Lambda^*(-\rho'(\theta)/\rho(\theta))>0.$

Let g_{ij}^{n} be the density function of $\tilde{\mu}_{\theta,ij}^{(n)}(dx)$, defined in (3.2) with related characteristic functions $\phi_{ij}^{(n)}$. Our last two assumptions are:

A(4): $0 \in \Omega_{\circ}$.

A(5): For any $\theta \in \Omega_{\circ}$, the characteristic function $\phi_{ij}^{(1)}(x)$ of $\tilde{\mu}_{\theta,ij}^{(1)}$ is absolutely integrable.

Now we set the main results, the first is a local limit theorem for tilted measure $\tilde{\mu}_{\theta,ij}^{(n)}(.)$.

Theorem 2.2. *Suppose A(1)-A(5) hold, $\theta \in \Omega_{\circ}$ and $a = -\rho(\theta)/\rho(\theta)$. Then for any fixed i, j,*

$$\lim_{n}\left|\sqrt{n}\,g_{ij}^{n}(x\sqrt{n}) - \frac{1}{\sqrt{2\pi\sigma}}e^{\frac{-x^2}{2\sigma^2}}\right| = 0$$

uniformly in $x \in \mathcal{R}$.

By estimating the integral of the normal density $N(0,\sigma^2)$ over A we get

Theorem2.3. *Let the hypothesis of Theorem 2.2 hold. Then for any bounded measurable set A, and fixed i, j,*

$$\lim_{n\to\infty}[\sqrt{2n\pi}\;\sigma\mu_{\theta,ij}^{n*}(na + A)] = u_j(\theta)\mathrm{v}_j(\theta)\,|\,A\,|$$

where, $|A|$ is the Lebesgue measure of A.

We conclude this section with a centeral limit theorem for the intensity measures $\{\mu_{ij}^{n*}\}$ by applying the Riemman integral estimate in Theorem 2.3. It is the Theorem 8 in [3] with a new proof based on our local limit Theorem 2.2.

Theorem2.4. *Suppose A(1)-A(5) hold, $\theta \in \Omega_{\circ}$, b>0 and $a = -\rho(\theta)/\rho(\theta)$. Then for any fixed i, j, and any bounded measureable set A,*

$$\lim_{n\to\infty}[\sqrt{2n\pi}\;\sigma\, e^{-n\Lambda^*(a)}\mu_{\theta,ij}^{n*}(na + A)] = u_j(\theta)\mathrm{v}_j(\theta)\int_A e^{\theta x}dx$$

uniformly in $|A| \le b$, where, $\Lambda^*(a) = \log\rho(\theta) - \theta\rho'(\theta)/\rho(\theta)$ and $|A|$ is the Lebesgue measure of A.

The tilted versions of $\{\mu_{ij}^{n*}\}$ and the proof of the main results

In this section we introduce different tilted versions of the intensity measures $\{\mu_{ij}^{n*}\}$ to find an equivalent version with the measures that are introduced in [6]. The exact study of these intensity measures are via the kernels of operators. In fact all the measures that are introduced in this section, are the kernels of operators on the set E={1, 2, …, p}$\times\mathcal{R}$ (see [1], [4], and [5]). For any $\theta\in\Omega_\circ$ and i, j =1, …, p, define the tilted measure $\mu_{\theta,ij}$ by

$$\mu_{\theta,ij}(dx)=\frac{\mathrm{v}_j(\theta)}{\mathrm{v}_i(\theta)}\frac{e^{-\theta x}}{\rho(\theta)}\mu_{ij}(dx).$$

Then the n-fold convolution of $\mu_\theta=\{\mu_{\theta,ij}\}$ is defined inductively:

$$\begin{aligned}\mu_{\theta,ij}^{(n+1)*}(dx)&=\sum_k\int_{\mathcal{R}}\mu_{\theta,ik}(dy)\,\mu_{\theta,kj}^{n*}(dx-y)\\ &=\frac{\mathrm{v}_\mathrm{j}(\theta)}{\mathrm{v}_\mathrm{i}(\theta)}\frac{e^{-\theta x}}{\rho(\theta)^{(n+1)}}\mu_{ij}^{(n+1)*}(dx).\end{aligned}\tag{3.1}$$

The mean drift of measure μ_θ is $a=-\rho'(\theta)/\rho(\theta)$, (see[4]). Define the centered measure $\bar{\mu}_{\theta,ij}$ by $\bar{\mu}_{\theta,ij}(dx)=\mu_{\theta,ij}(a+dx)$. Then $\bar{\mu}_\theta=\{\bar{\mu}_{\theta,ij}\}$, is a centered stochastic kernel on E with zero drift and invariant measure $\pi(\theta)$. Its n-fold convolution is $\bar{\mu}_{\theta,ij}^{n*}(dx)=\bar{\mu}_{\theta,ij}^{n*}(na+dx)$. Let *A(1)* hold and $\theta\in\Omega_\circ$ then $P=(p_{ij})=\{\bar{\mu}_{\theta,ij}(\Re)\}$ is a positive regular stochastic matrix with stationary measure (see[4])

$$\pi(\theta)=(\pi_1(\theta),...,\pi_p(\theta)),\ \pi_k(\theta)=u_k(\theta)\mathrm{v}_k(\theta),\ (k=1,\ ...,\ p).$$

Since P is positive regular, so for large values of n, $P^n=(p_{ij}^n)$ has all positive entries. Define $\tilde{\mu}_{\theta,ij}^{(n)}(dx)=\tilde{\mu}_{\theta,ij}^{n*}(dx)/p_{ij}^n$ whenever $p_{ij}^n>0$, and let $\tilde{\mu}_{\theta,ij}^{(n)}(dx)\equiv 0$ for $p_{ij}^n=0$. Thus we have

$$\bar{\mu}_{\theta,ij}^{n*}(dx)=p_{ij}^n\tilde{\mu}_{\theta,ij}^{(n)}(dx).\tag{3.2}$$

Then $\tilde{\mu}_{\theta,ij}^{(n)}(dx)$ is a probability measure and its characteristic function is

$$\phi_{ij}^{(n)}(\eta)=\int_{\mathcal{R}}e^{-i\eta x}\tilde{\mu}_{\theta,ij}^{(n)}(dx).$$

By the next proposition we will show that all $\phi_{ij}^{(n)}$ are absolutely integrable and their related density functions exist.

Proposition 3.1. *Suppose A(1)-A(5) hold and* $\theta\in\Omega_\circ$ *with* $a=-\rho'(\theta)/\rho(\theta)$. *Then for any i,j and n,* $\phi_{ij}^{(n)}(\eta)$ *is absolutely integrable and the density function* g_{ij}^n *exist.*

Proof. Define $\varphi(\eta)=\max_{i,j}|\phi_{ij}^{(1)}(\eta)|$ for $\eta\in\mathrm{R}$, then by *A(5),* φ is absolutely integrable and $\circ\le\varphi(\eta)<1$ for all $\eta\ne 0$ with $\lim_{|\eta|\to\infty}\varphi(\eta)=0$. Let i, j be fixed. For each n, using the convolution of measures and then a change of variable $x\to x+y$ give

$$\left|p_{ij}^{n+1}\phi_{ij}^{(n+1)}(\eta)\right| = \left|\int_{\Re} e^{-i\eta x}\,\overline{\mu}_{\theta,ij}^{(n+1)*}(dx)\right|$$
$$= \left|\sum_k \int_{\Re,x} \mathrm{e}^{-i\eta x}\int_{\Re,x}\overline{\mu}_{\theta,ik}(dy)\,\overline{\mu}_{\theta,kj}^{n*}(dx-y)\right|$$
$$= \left|\sum_{\mathrm{k}} p_{ik}\,p_{kj}^{n}\,\phi_{ik}^{(1)}(\eta)\phi_{kj}^{(n)}(\eta)\right|$$
$$\leq p_{ij}^{n+1}\varphi^{(n+1)}(\eta). \tag{3.3}$$

Then induction and $|\phi_{ij}^{(1)}(\eta)| \leq \varphi(\eta)$ imply $|\phi_{ij}^{(n)}(\eta)| \leq (\varphi(\eta))^n$ for all $\eta \in \Re$ and i, j. Since φ is integrable and $\varphi^n \leq \varphi$, we get the absolute integrability of all $\varphi_{ij}^{(n)}$ and hence the existence of their density functions g_{ij}^{n}.

Now by comparing the result of [6], our measure $\overline{\mu}_\theta = (\overline{\mu}_{\theta,ij})$ is the same as the measure $\bar{v}$ in [6] with $\tilde{\mu}_\theta$ in place of $\tilde{v}$. Thus the results of [6] hold and we can use it to prove our results:

Proof of theorem 2.3. Let g_{ij}^{n} be the density function of the measure $\tilde{\mu}_{\theta,ij}^{(n)}$. All the conditions of Therorem 2.3 in [6] hold, with g_{ij}^{n} as the density function of the measure $\tilde{v}_{ij}^{(n)} / p_{ij}^{n}$. Moreover it should be noted that, for any $\theta \in \Omega_\circ$ we have a tilted measure $\tilde{\mu}_\theta$ which is equivalent to $\tilde{v}$ in [6]. This completes the proof.

Proof of Theorem 2.3. Let $\mathrm{A} \subset [-b,b]$ be a bounded measurable set and i, j be fixed. For any n we can write

$$\left|\sqrt{2\pi n}\;\sigma\tilde{\mu}_{\theta,ij}^{(n)}(na+A) - |A|\right| \leq \sqrt{2\pi}\,\sigma\left|\sqrt{n}\tilde{\mu}_{\theta,ij}^{(n)}(A) - \int_A \frac{1}{\sqrt{2\pi}\sigma}e^{\frac{-x^2}{2n\sigma^2}}dx\right|$$
$$+\sqrt{2\pi}\sigma\left|\int_A\left[\frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2n\sigma^2}} - \frac{1}{\sqrt{2\pi}\sigma}\right]dx\right|$$
$$= \mathrm{A_n} + \mathrm{B_n} \quad \text{say.} \tag{3.4}$$

By applying the integral to the uniformly convergent sequences in Theorem 2.2 over the set A, we get

$$\lim_{n\to\infty}\left|\sqrt{n}\tilde{\mu}_{\theta,ij}^{n*}(c+A) - \int_A \frac{1}{\sqrt{2\pi}\sigma}e^{\frac{-x^2}{2n\sigma^2}}dx\right| = 0,$$

Uniformly in $A \subset [-b,b]$. This implies that as $n \to \infty$, $A_n \to 0$. It is enough to show that $B_n \to 0$ as $\mathrm{n} \to \infty$. By the properties of the normal distribution, for some constant $C > 0$ and all x, $\left|f_\sigma(x) - f_\sigma(0)\right| \leq Cx^2$. Thus we have

$$|B_n| = \sqrt{2\pi}\sigma \frac{C}{n}\int_A x^2 dx \le \frac{C_1}{n}\int_{-b}^{b} x^2 dx := \frac{C_2}{n},$$

where, C_1 and C_2 are constants which complete the proof of $B_n \to 0$, as $n \to \infty$. Since $\mu_{\theta,ij}^{n*}(na+A)/p_{ij}^n = \tilde{\mu}_{\theta,ij}^{(n)}(A)$ and $u_j(\theta)\mathrm{v}_j(\theta)$ is the stationary distribution of p^n, then by applying the definition of $\tilde{\mu}_{ij}^{(n)}$ in (3.4) we get

$$\lim_{n\to\infty}\left[\sqrt{2\pi n}\sigma\mu_{\theta,ij}^{n*}(na+A)\right] = u_j(\theta)\mathrm{v}_j(\theta)\,|\,A\,|$$

and completing the proof.

Proof of Theorem 2.4. Let $A \subset [-b,b] = I$ be a measurable set. For any fixed i, j, and n, from the equation (3.1) we can write

$$\mu_{ij}^{n*}(na+A) = \left[\frac{\mathrm{v}_i(\theta)}{\mathrm{v}_j(\theta)}\rho(\theta)^n\right]\int_{na+A} e^{\theta x}\mu_{\theta,ij}^{n*}(dx).$$

By changing the variable *x* to *x*+*na* and recalling that $a = -\rho'(\theta)/\rho(\theta)$, we get

$$\mu_{ij}^{n*}(na+A) = \frac{\mathrm{v}_i(\theta)}{\mathrm{v}_j(\theta)} e^{n\Lambda^*(a)}\int_A e^{\theta x}\overline{\mu}_{\theta,ij}^{n*}(dx)$$

where, $\Lambda^*(a) = \log\rho(\theta) - \theta\rho'(\theta)/\rho(\theta)$. Define $f(x) = 1_A(x)e^{\theta x}$ for all $x \in I$. Let $\{A_r : r\}$ be a finite partition of I, then for each r, define $M_r = \sup\{f(x) : x \in A_r\}$ and $m_r = \inf\{f(x) : x \in A_r\}$. For each $x \in I$, let $\bar{f}(x) = \sum_r M_r 1_{Ar}(x)$ and $\bar{f}(\mathrm{x}) = \sum_r m_r 1_{Ar}(x)$. When $x \in A$ we have $\underline{f}(x) \le f(x) \le \overline{f}(x)$, so for this fixed partition $\{A_r : r\}$,

$$\mu_{ij}^{n*}(na+A) \le \frac{\mathrm{v}_i(\theta)}{\mathrm{v}_j(\theta)} e^{n\Lambda^*(a)}\sum_r M_r\overline{\mu}_{\theta,ij}^{n*}(A_r). \tag{3.5}$$

Let $M = \sum_r M_r$ and $\varepsilon > 0$ be fixed. Form Theorem 2.3, for any fixed *r*, there is an N_1 such that, for all $n \ge N_1$ we have

$$\left|\sqrt{2\pi n}\sigma\,\overline{\mu}_{\theta,ij}^{n*}(A_r) - \overline{u}_j(\theta)v_j(\theta)|A_r|\right| \le \frac{\varepsilon}{M}. \tag{3.6}$$

We choose N_1 large enough such that (3.6) holds for all *r*. Then applying (3.6) in (3.5) for large $n \ge N_1$ implies

$$\sqrt{2\pi n}\,\sigma\,\mu_{ij}^{n*}(na+A) \le \left[\frac{\mathrm{v}_i(\theta)}{\mathrm{v}_j(\theta)} e^{n\Lambda^*(a)}\right]\sum_r M_r\left[u_j(\theta)\mathrm{v}_j(\theta)|A_r| + \frac{\varepsilon}{M}\right]$$

$$= e^{n\Lambda^*(a)}\left[\mathrm{L}\int_A \overline{f}(\theta)dx + \frac{\mathrm{v}_i(\theta)}{\mathrm{v}_j(\theta)}\varepsilon\right]$$

where $L = u_j(\theta)\mathrm{v}_i(\theta)$. Thus , for any $\varepsilon > 0$, we have

$$J_n =: \sqrt{2\pi n}\,\sigma\, e^{-n\Lambda^*(a)} \mu_{ij}^{n*}(na + A) \le \mathrm{L}\int_{\mathrm{A}} \bar{f}(x)dx + \frac{\mathrm{v}_i(\theta)}{\mathrm{v}_j(\theta)}\varepsilon.$$

By taking limsup of both sides of the above inequality, as $n \to \infty$, we get $\limsup_{n\to\infty} J_n \le L\int_A f(x)dx + (\mathrm{v}_i(\theta)/\mathrm{v}_j(\theta))\varepsilon$. Since ε is arbitrary, we will have $\limsup_{n\to\infty}$ $\le L\int_A \bar{f}(x)dx$. This inequality holds for all partitions of I, thus by taking infimum on all partitions of I and taking into account that J_n is independent of these partitions, we get

$$\limsup_{n\to\infty} J_n \le L\int_A f(x)dx. \tag{3.7}$$

With a similar argument we can prove that $\liminf_{n\to\infty} J_n \ge L\int_A f(x)dx$, which in conjunction with (3.7) complete the proof .

Acknowledgement

The author wold like to thank professor J. D. Biggins for many valuable discussions.

References

1. J. Barral, Generalized vector multiplicative cascades. *Adv. Appl. Prob.* 33(2001) 874-895.
2. J.D. Biggins, Uniform convergence of martingales in the branching random walk. *Ann. Probab*. 20(1992) 137-151.
3. J.D. Biggins, And A. Rahimzade Sani, Convergence results on multitype, multivariate branching random walks. *Adv. Appl. Probab*. 37(2005) 681-705.
4. M. Bramson, P. Ney, And J. Tao, The population composition of a multitype branching random walk. *Ann. Appl. Probab*. 2(1992) 575-596.
5. C. J. Mode, *Multitype Branching Processes*. New York, American Elsevier(1971).
6. A. Rahimzade Sani, Central and local limit theorems in markov dependent random variables. Bulletin of the Iranian Mathematical Socity Vol. 34 No. 1(2008) 23-35.