Numerical solution of nonlinear Fredholm and Volterra integral equations of the second kind using Haar wavelets and collocation method

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Abstract

In this paper, we present a numerical method for solving nonlinear Fredholm and Volterra integral equations of the second kind which is based on the use of Haar wavelets and collocation method. We use properties of Block Pulse Functions (BPF) for solving Volterra integral equation. Numerical examples show efficiency of the method.

1. Introduction

Integral equations of the Hammerstein type have been one of the most important domains of applications of the ideas and methods of nonlinear functional analysis and in particular of the theory of nonlinear operators of monotone type. Various applied problems arrising in mathematical physics, mechanics and control theory leads to multivalued analogs of the Hammerstein integral equations[15]. In recent years, many different basis functions have been used to solve and reduce integral equations to a system of algebraic equations [1-3], [7-10] and [12-13]. For numerical solution of integral equations quadrature formula methods and spline approximations are used. In the case of this method, systems of algebraic equations must be solved. For large matrices, this requires a huge number of arithmetic operators and a larg storage capacity. A lot of computing time is saved if we succeed in replacing the fully populated transform matrix with a sparse matrix. One possibility for this gives the wavelet method; the wavelet bases

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lead to a sparse matrix representation since (i) the basis functions are usually orthogonal; (ii) most of the functions have a small interval of support. The aim of this paper is to present a numerical method for solving nonlinear Fredholm and Volterra integral equations of Hammerstein type using Haar wavelets defined as following:

Definition: The Haar wavelet is the function defined on the real line R as:

$$H(t)=\begin{cases} 1, & 0 \le t < \frac{1}{2} \\ -1, & \frac{1}{2} \le t < 1 \\ 0, & \text{elsewhere} \end{cases}$$

now for n = 1, 2, . . . , write $n = 2^j + k$ with j = 0, 1, . . . , and k = 0, 1, . . . , 2^j-1 and define $h_n(t) = 2^{\frac{j}{2}} H(2^j t - k)|_{[0,1]}$. Also, define $h_0(t) = 1$ for all t. Here the integer 2^j , j = 0, 1, . . . , indicates the level of the wavelet and k = 0, 1, . . . , $2^j - 1$ is the translation parameter. It can be shown that the sequence $\{h_n\}_{n=0}^{\infty}$ is a complete orthonormal system in $L^2[0, 1]$ and for $f \in C[0, 1]$, the series $\sum_n < f, h_n > h_n$ converges uniformly to f [14], where $< f, h_n >= \int_0^1 f(x) h_n(x) dx$.

2. Function Approximation

A function u(t) defined over the interval [0, 1) may be expanded as:

$$u(t) = \sum_{n=0}^{\infty} u_n h_n(t), \tag{1}$$

with $u_n =< u(t), h_n(t) >$.

In practice, only the first k-term of (1) are considered, where k is a power of 2, that is,

$$u(t) \simeq u_k(t) = \sum_{n=0}^{k-1} u_n h_n(t), \tag{2}$$

with matrix form:

$$u(t) \simeq u_k(t) = \mathbf{u}^t \mathbf{h}(t), \tag{3}$$

where, $\mathbf{u} = [u_0, u_1, \dots, u_{k-1}]^t$ and $\mathbf{h}(t) = [h_0(t), h_1(t), \dots, h_{k-1}(t)]^t$. Similarly, $K(x, t) \in L^2[0, 1)^2$ may be approximated as:

$$K(x,t) \simeq \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} K_{ij} h_i(x) h_j(t)$$

or in matrix form

$$K(x,t) \simeq \mathbf{h}^t(x) \mathbf{K} \mathbf{h}(t) \tag{4}$$

where $\mathbf{K} = [K_{ij}]_{0 \le i,j \le k-1}$ and $K_{ij} =< h_i(x), < K(x, t), h_j(t) >>$.

3. Nonlinear integral equations

Consider the following nonlinear integral equation of the second kind:

$$u(t) = \int_0^1 K(t,x)\varnothing[x, u(x)]\,dx + g(t), \tag{5}$$

where, $K \in L^2\,[0, 1)^2$ and g,h $\in L^2\,[0, 1)$ are known functions and u(t) is the unknown function to be determined. We consider the problem when using a collocation method. By substituting (2) into (5) and evaluating the new equation at the collocation points $t_j \in [0, 1)$ we obtain:

$$\sum_{n=0}^{k-1} u_n h_n(t_j) = \int_0^1 K(t_j, x)\varnothing[x, \textstyle\sum_{n=0}^{k-1} u_n h_n(x)]\,dx + g(t_j), \tag{6}$$

for $j = 1, 2, \ldots, k$. In the iterative solution of this system, many integrals will need to be computed, which usually becomes quite expensive. In particular, the integral on the right side will need to re-evaluated with each new iterate. But if we define

$$W(t) = \varnothing[t, u(t)] \tag{7}$$

from (5) we obtain

$$W(t) = \varnothing[t, \int_0^1 K(t,x)W(x)\,dx + g(t)]. \tag{8}$$

If we approximate $W_k(t)$ as:

$$W(t) \simeq W_k(t) = \sum_{n=0}^{k-1} w_n h_n(t) = \mathbf{w}^t \mathbf{h}(t), \tag{9}$$

where $\mathbf{w} = [w_0, w_1, \ldots, w_{k-1}]^t$, the collocation method for (8) is

$$\sum_{n=0}^{k-1} w_n h_n(t_j) = \varnothing\left[t_j, \quad \sum_{n=0}^{k-1} w_n \int_0^1 K(t_j, x) h_n(x)\,dx + g(t_j)\right], \tag{10}$$

the integral of the right side of (10) need be evaluated only once, since they are dependent only on the basis, not on the unknowns $\{u_n\}$. Many fewer integrals need be calculated to solve this system. In this paper, we consider nonlinear integral equations with degenerate kernel [2].

3.1. Nonlinear Fredholm integral equations of Hammerstein type

Now consider the following nonlinear Fredholm integral equation of the second kind of Hammerstein type:

$$u(x) = \int_0^1 K(x,t)\varnothing[t, u(t)]\,dt + g(x), \tag{11}$$

where, $K \in L^2 [0, 1)^2$ and g, $\phi \in L^2 [0, 1)$ are known functions and u(t) is the unknown function to be determined. Now from equations (7) and (11) we have:

$$u(x) = \int_0^1 K(x, t)W(t)\,dt + g(x), \tag{12}$$

If we approximate equation (12) by

$$u_k(x) = \int_0^1 K(x, t)W_k(t)\,dt + g(x), \tag{13}$$

we have to approximate W_k (t) as (9). By approximating functions K(x, t) and W (t), as before, in the

matrix form we have:

$$K(x, t) \simeq \mathbf{h}^t(x)\mathbf{K}\mathbf{h}(t), \tag{14}$$

$$W(t) \simeq \mathbf{w}^t\mathbf{h}(t), \tag{15}$$

by substituting the approximations (14) and (15) into (8) we obtain:

$$\mathbf{w}^t\mathbf{h}(t) = \phi\left[t, \int_0^1 \mathbf{h}^t(t)\mathbf{K}\mathbf{h}(x)\mathbf{h}^t(x)\mathbf{w}\,dx + g(t)\right]$$
$$= \phi[t, \mathbf{h}^t(t)\mathbf{K}\mathbf{w} + g(t)], \tag{16}$$

where we used the orthonormality of the sequence $\{h_n\}$ on [0, 1) that implies $\int_0^1 \mathbf{h}(x)\mathbf{h}^t(x)\,dx = \mathbf{I}_{k\times k}$, where $\mathbf{I}_{k\times k}$ is the identity matrix of order k.
Evaluating (16) at the collocation points $t_j = \frac{j-0.5}{k}, j = 1, 2, \ldots, k,$ leads to

$$\mathbf{w}^t\mathbf{h}(t_j) = \phi[t_j, \mathbf{h}^t(t_j)\mathbf{K}\mathbf{w} + g(t_j)], \tag{17}$$

which is a nonlinear system of algebraic equations. Solving (17) gives column vector **w**. Therefore, from (9) we can approximate W (t) by W_k (t) and from (13) we get desired approximation u_k (t) for u(t).

3.2. Nonlinear Volterra integral equations of Hammerstein type

Now consider the nonlinear Volterra integral equation of the second kind of Hammerstein type:

$$u(x) = \int_0^x K(x, t)\phi[t, u(t)]\,dt + g(x), \tag{18}$$

as before, we let

$$W(t) = \phi[t, u(t)], \tag{19}$$

by substituting (19) into (18) we obtain:

$$u(x) = \int_0^x K(x,t)W(t)\,dt + g(x), \tag{20}$$

substituting (20) into (19) leads to

$$W(t) = \emptyset\left[t, \int_0^t K(t,x)W(x)\,dx + g(t)\right]. \tag{21}$$

We approximate equation (20) by

$$u_k(x) = \int_0^x K(x,t)W_k(t)dt + g(x), \tag{22}$$

by substituting the approximations (14) and (15) into (21) we obtain:

$$\begin{aligned} \mathbf{w}^t\mathbf{h}(t) &= \phi\left[t, \int_0^t \mathbf{h}^t(t)\mathbf{K}\mathbf{h}(x)\mathbf{h}^t(x)\mathbf{w}dx + g(t)\right] \\ &= \phi[t, \mathbf{h}^t(t)\mathbf{K}\mathbf{A}(t)\mathbf{w} + g(t)], \end{aligned} \tag{23}$$

Where $\mathbf{A}(t) = \int_0^t \mathbf{h}(x)\mathbf{h}^t(x)dx$. In section 4, we consider evaluation of $\mathbf{A}(t)$ at the collocation points t_j using properties of Block-Pulse Functions (BPF).

4. Evaluation of A(t) at the collocation points t_j

In this section, we present the Haar coefficient matrix **H**; it is a $k \times k$ matrix with the elements

$$\mathbf{H} = [h_n(t_j)]_{0 \le n \le k-1, 1 \le j \le k},$$

where the points t_j are the collocation points

$$t_j = \frac{j-0.5}{k}, \qquad j = 1,2,\ldots,k.$$

Also, define a k-set of Block-Pulse Functions (BPF) as:

$$B_i(t) = \begin{cases} 1, & \frac{i-1}{k} \le t < \frac{i}{k}, \text{for all } i = 1,2,\ldots,k \\ 0, & elsewhere \end{cases} \tag{24}$$

The functions $B_r(t)$ are disjoint and orthogonal. That is,

$$B_i(t)B_j(t) = \begin{cases} 0, & i \ne j \\ B_i(t), & i = j \end{cases} \tag{25}$$

$$< B_i(t), B_j(t) >= \begin{cases} 0, & i \ne j \\ \frac{1}{k}, & i = j. \end{cases} \tag{26}$$

It can be shown that $\mathbf{h}(t) = \mathbf{HB}(t)$ [6], vector $\mathbf{h}(t)$ and matrix **H** are already introduced and $\mathbf{B}(t) = [B_1(t), \ldots, B_k(t)]^t$. Using (25) leads to

$$\mathbf{B}(x)\mathbf{B}^t(x) = \begin{bmatrix} B_1(x) & & \emptyset \\ & \vdots & \\ \emptyset & & B_k(x) \end{bmatrix}$$

$$= B_1(x)\begin{bmatrix} 1 & & & \emptyset \\ & 0 & & \\ & & \vdots & \\ \emptyset & & & 0 \end{bmatrix} + B_2(x)\begin{bmatrix} 0 & & & \emptyset \\ & 1 & & \\ & & 0 & \\ & & \vdots & \\ \emptyset & & & 0 \end{bmatrix} + \cdots + B_k(x)\begin{bmatrix} 0 & & & \emptyset \\ & 0 & & \\ & \vdots & & \\ & & 0 & \\ \emptyset & & & 1 \end{bmatrix}$$

$$= \sum_{i=1}^{k} B_i(x)\,\mathbf{d}^{(i)},$$

where, $\mathbf{d}^{(i)}$ is a $k \times k$ matrix with the elements

$$\mathbf{d}_{mn}^{(i)} = \begin{cases} 1, & m = n = i \\ 0, & m \neq i \text{ or } n \neq i. \end{cases}$$

Therefore we have

$$\begin{aligned} \mathbf{h}(x)\mathbf{h}^t(x) &= \mathbf{H}\mathbf{B}(x)\mathbf{B}^t(x)\mathbf{H}^t \\ &= \mathbf{H}\Big(\sum_{i=1}^{k} B_i(x)\mathbf{d}^i\Big)\mathbf{H}^t \\ &= \sum_{i=1}^{k} B_i(x)\mathbf{H}\mathbf{d}^{(i)}\mathbf{H}^t. \end{aligned} \tag{27}$$

By integrating (27) we obtain:

$$\begin{aligned} \mathbf{A}(t) &= \int_0^t \mathbf{h}(x)\mathbf{h}^t(x)\,dx \\ &= \sum_{i=1}^{k} \int_0^t B_i(x)\,dx\,\mathbf{H}\mathbf{d}^{(i)}\mathbf{H}^t \\ &= \sum_{i=1}^{k} n_i(t)\mathbf{H}\mathbf{d}^{(i)}\mathbf{H}^t, \end{aligned} \tag{28}$$

where , $n_i(t) = \int_0^t B_i(x)dx,\ t \in [0,1]$.

From (24) we have

$0 \le t < \frac{1}{k}$ implies that $B_1(t) = 1$ and $B_i(t) = 0$ for $i = 2, \ldots, k$.

$\frac{1}{k} \le t < \frac{2}{k}$ implies that $B_2(t) = 1$ and $B_i(t) = 0$ for $i = 1, \ldots, k$ and $i \neq 2$.

$\vdots$

$\frac{k-1}{k} \le t < 1$ implies that $B_k(t) = 1$ and $B_i(t) = 0$ for $i = 1, \ldots, k-1$.

Therefore,

$n_1(t_1) = \int_0^{t_1} B_1(x)dx = \frac{1}{2k}$ and $n_i(t_1) = 0$ for $i = 2, \ldots, k$.

$n_1(t_2) = \int_0^{t_2} B_1(x)dx = \frac{1}{k}$, $n_2(t_2) = \int_0^{t_2} B_2(x)dx = \frac{1}{2k}$ and $n_i(t_2) = 0$ for $i = 3, \ldots, k$.

$\vdots$

$n_1(t_k) = \int_0^{t_k} B_1(x)dx = \frac{1}{k}, \ldots, n_{k-1}(t_k) = \int_0^{t_k} B_{k-1}(x)dx = \frac{1}{k}$ and $n_k(t_k) = \int_0^{t_k} B_k(x)dx = \frac{1}{2k}$.

Evaluating (28) at the collocation points t_j leads to

$$A(t_1) = \frac{1}{2k}\mathbf{H}\mathbf{d}^{(1)}\mathbf{H}^t$$

$$A(t_2) = \frac{1}{k}\mathbf{H}\mathbf{d}^{(1)}\mathbf{H}^t + \frac{1}{2k}\mathbf{H}\mathbf{d}^{(2)}\mathbf{H}^t$$

$$\vdots$$

$$A(t_k) = \frac{1}{k}\mathbf{H}\mathbf{d}^{(1)}\mathbf{H}^t + \cdots + \frac{1}{k}\mathbf{H}\mathbf{d}^{(k-1)}\mathbf{H}^t + \frac{1}{2k}\mathbf{H}\mathbf{d}^{(k)}\mathbf{H}^t,$$

or in abstract form

$$A(t_1) = \frac{1}{2k}\mathbf{H}\mathbf{d}^{(1)}\mathbf{H}^t$$

$$A(t_j) = \frac{1}{k}\sum_{i=1}^{j-1}\mathbf{H}\mathbf{d}^{(i)}\mathbf{H}^t + \frac{1}{2k}\mathbf{H}\mathbf{d}^{(j)}\mathbf{H}^t, \text{ for } j = 2, \ldots, k.$$

By evaluating (23) at the collocation points t_j, $j = 1, 2, \ldots, k$, we have

$$\mathbf{w}^t\mathbf{h}(t_j) = \phi[t_j, \mathbf{h}^t(t_j)\mathbf{K}\mathbf{A}(t_j)\mathbf{w} + g(t_j)] \tag{29}$$

Solving nonlinear system of algebraic equations (29) gives column vector **w**. Therefore from (9) we can approximate W (t) by W_k (t) and from (22) we get desired approximation $u_k(t)$ for u(t).

5. Numerical Examples

Example 1 [9]:

$$u(x) = \int_0^1 2xte^{-u^2(t)}dt + \frac{x}{e}, \quad 0 \le x < 1,$$

with exact solution u(x) = x. Now we explain the details for solving example 1 with k =16. We have

$k(x, t) = 2xt$, $g(x) = \frac{x}{e}$ and $\phi[t, u(t)] = e^{-u^2(t)}$. From system (17) we have

$$\mathbf{w}^t\mathbf{h}(t_j) = e^{-(\mathbf{h}^t(t_j)\mathbf{K}\mathbf{w}+g(t_j))^2}, \tag{30}$$

for j = 1, 2,...,k, where $t_j = \frac{j-0.5}{k}$, $\mathbf{h}(t_j) = [h_0(t_j), h_1(t_j), \ldots, h_{15}(t_j)]^t$, $g(t_j) = \frac{t_j}{e}$ and entries of the matrix **K** are given by $K_{ij} = \langle h_i(x), \langle K(x,t), h_j(t)\rangle\rangle$. So, system (30) gives column vector **w**:

w=[.7465741616, .1759828194, 0.4041735453e-1, 0.7412321740e-1, 0.7674107908e-2, 0.2025990210e-1, 0.2630329292e-1, 0.2533820104e-1, 0.1387065386e-2, 0.4007826334e-2, 0.6266570310e-2, 0.7983160604e-2, 0.9054739168e-2, 0.9463819862e-2, 0.9270852644e-2, 0.8594532883e-2]^t .

Now by substituting $W_{15}(t) = \sum_{n=0}^{15} w_n h_n(t)$ into (13) we obtain :

$$u_{16}(x) = 2x\left(\sum_{n=0}^{15} w_n \int_0^1 t h_n(t)\,dt\right) + \frac{x}{e}$$

$$= .6321231112x + \frac{x}{e},$$

that is desired approximation for u(x) over the interval [0,1).

Example 2 [9]:

$$u(x) + \int_0^1 e^{x-2t}[u(t)]^3\,dt = e^{x+1}, \qquad 0 \le x < 1,$$

with exact solution $u(x) = e^x$.

Example 3 [9]:

$$u(x) - \int_0^1 [4tx + \pi x \sin(\pi t)]\frac{1}{u^2(t) + t^2 + 1}\,dt = \sin\left(\frac{\pi}{2}x\right) - 2x\ln 3, \qquad 0 \le x < 1,$$

with exact solution $u(x) = \sin\left(\frac{\pi}{2}x\right)$

Example 4 [10]:

$$u(x) - \int_0^1 xt[u(t)]^3\,dt = e^x - \frac{(1+2e^3)x}{9}, \qquad 0 \le x < 1,$$

with exact solution $u(x) = e^x$.

Example 5 [8]:

$$u(x) = 1 + \sin^2 x - \int_0^x 3\sin(x-t)[u(t)]^2\,dt, \qquad 0 \le x < 1,$$

with exact solution u(x) = cos x.

Example 6 [5]:

$$u(x) = x + \cos x - 1 + \int_0^x \sin[u(t)]\,dt, \qquad 0 \le x < 1,$$

with exact solution u(x) = x.

Example 7 [5]:

$$u(x) = e^x - \frac{1}{2}(e^{2x} - 1) + \int_0^x [u(t)]^2\,dt, \qquad 0 \le x < 1,$$

with exact solution $u(x) = e^x$.

Table 1 shows the computed error $\|e\| = \|u(x) - u_k(x)\|$ for the examples 1-7 with k = 16.

Conclusion

In present paper, Haar wavelets together with the collocation points are applied to solve the nonlinear Fredholm and Volterra integral equations of Hammerstein type.

Numerical examples show the accuracy of the method, therefore for better results, using a larger k is recommended. Evaluating W_k (t) at different points of the interval [0,1) shows that in the examples 1 and 3 where g is nonexponential, W_k (t) is a more accurate approximation of W (t), compared with the examples 2 and 4 whose g functions are exponential. In fact, when g is nonexponential, the vector **w** obtained as a solution to (17), is much more accurate than in the examples where g is exponential. As a result, $u_k(t)$ is a more accurate approximation of u(t).

Table 1

t	Example 1	Example 2	Example 3	Example 4	Example 5	Example 6	Example 7
.1	2E -7	8E - 3	4E - 7	1E - 3	3E - 5	5E - 4	1E - 3
.2	5E - 7	9E - 3	9E - 7	3E - 3	1E - 4	4E - 4	1E - 3
.3	7E - 7	1E - 2	1E - 6	5E - 3	3E - 4	4E - 4	2E - 3
.4	1E - 6	1E - 2	1E - 6	7E - 3	8E - 4	6E - 4	3E - 3
.5	1E - 6	1E - 2	2E - 6	9E - 3	1E - 3	3E - 4	3E - 3
.6	1E - 6	1E - 2	2E - 6	1E - 2	2E - 3	7E - 4	8E - 3
.7	1E - 6	1E - 2	3E - 6	1E - 2	2E - 3	7E - 4	2E - 2
.8	2E - 6	1E - 2	3E - 6	1E - 2	1E - 3	9E - 4	7E - 2
.9	2E - 6	1E - 2	4E - 6	1E - 2	9E - 3	1E - 3	1E - 1

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