Contractibility and idempotents in Banach algebras

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Abstract

Let $A$ be a Banach algebra. It is shown that a contractible ideal of a Banach algebra is complemented by its annihilator. Then, it is proved the existence of minimal central idempotents in a contractible Banach algebra with a nonzero character. Moreover, the notion of $b$-contractibility and one of its equivalent forms are introduced. Through an example, it is shown that $b$-contractibility is strictly weaker than contractibility.

Introduction

Taylor in [13, Theorem 5.11] showed that a contractible Banach algebra with bounded approximation property is finite dimensional. Johnson in [6, Proposition 8.1] showed that a contractible commutative semisimple Banach algebra is finite dimensional. Curtis and Loy [1, Theorem 6.2] extended this result by dropping the semisimplicity assumption. But the question for noncommutative case has remained open. For more results of this type see [4], [5], [8], [10], [13].

This paper is organized as follows. In the second section, we show that a contractible ideal of a Banach algebra is controlled by its commutant and annihilator. Then, we prove the existence of minimal central idempotents in a contractible Banach algebra with a nonzero character. In the third section, we introduce a weaker version of contractibility which we call $b$-contractibility. We give a characterization of $b$-contractibility analog to that of contractibility given by Taylor. Also, we show that $b$-contractibility is strictly weaker than contractibility.

2000 Mathematics Subject Classification, 46H10, 46H20.
Key words: Contractible Banach algebras, Minimal idempotent.
First we recall some terminology. Throughout this paper, \( A \) is a Banach algebra and \( A \)-module means Banach \( A \)-bimodule. For a subset \( E \) of \( A \), \( E' \) is the commutant of \( E \). If for every \( A \)-bimodule \( X \) every bounded derivation from \( A \) into \( X \) is inner, then \( A \) is called contractible. Also, the term "semisimple" means Jacobson semisimple.

An idempotent \( e \in A \) is called minimal if \( eAe \) is a division ring. If \( e \) and \( f \) are idempotents in \( A \), we write \( e \leq f \) if \( fe = ef = e \) holds. A nonzero idempotent \( e \in A \) is called primitive if \( 0 \leq f \leq e \) implies that \( f = 0 \) or \( f = e \). Also, two idempotents \( e \) and \( f \) are said to be orthogonal if they satisfy \( ef = fe = 0 \). Let \( S \) be a subset of \( A \). The right annihilator of \( S \) in \( A \) which we denote by \( \text{ran}(S) \) is the set
\[
\text{ran}(S) = \{ a \in A : ba = 0 \text{ for } b \in S \}.
\]

The left annihilator \( \text{lan}(S) \) is defined similarly. The annihilator of \( S \) is the set
\[
\text{Ann}(S) = \text{ran}(S) \cap \text{lan}(S).
\]

**Contractibility**

**Theorem 2.1.** Let \( A \) be a contractible Banach algebra which is an ideal in a Banach algebra \( B \). Then \( A + A^\perp = B \).

**Proof.** If \( A + A^\perp \neq B \), then we can choose \( b \in B - (A + A^\perp) \). Now define
\[
D : A \rightarrow A, x \mapsto xb - bx.
\]

Clearly \( D \) is a derivation on \( A \). By assumption there exists an \( a \in A \) such that \( D(x) = xa - ax \) for all \( x \in A \). The latter result implies that \( b - a \in A^\perp \) or equivalently \( b \in A + A^\perp \) which contradicts the selection of \( b \). Therefore \( A + A^\perp = B \).

**Theorem 2.2.** Let \( A \) be a contractible Banach algebra which is an ideal in a Banach algebra \( B \). Then \( B = A \oplus \text{Ann}(A) \).

**Proof.** Since \( A \) is contractible then \( M_2(A) \) with \( l^1 \)-norm is contractible, where \( M_2(A) \) is the algebra of \( 2 \times 2 \) matrices with the entries from \( A \). On the other hand \( M_2(A) \) is an ideal in \( M_2(B) \) and by Theorem 2.1 we have the equality \( M_2(B) = M_2(A) + M_2(A)^\perp \). One can easily observe that
Thus $B = A + \text{Ann}(A)$. But $A \cap \text{Ann}(A) = 0$, because $A$ is unital. Therefore the identity $B = A \oplus \text{Ann}(A)$ holds.

**Remark.** In Theorems 2.1 and 2.2, $A$ and $B$ are related only algebraically. Indeed if there exists an infinite dimensional contractible Banach algebra $A$ which is an ideal in a Banach algebra $B$, then the norm topology of $A$ could be different from the relative norm topology of $A$ which inherits from $B$.

**Theorem 2.3.** Let $A$ be a contractible Banach algebra which admits a nonzero multiplicative linear functional $f$. Then $A$ contains a central minimal idempotent.

**Proof.** Let $d = \sum_{n=1}^{\infty} a_n \otimes b_n$ be a diagonal for $A$ and define

$$T : A \to a \mapsto \sum_{n=1}^{\infty} < f, aa_n > b_n.$$ 

Since $\sum_{n} a_n b_n = 1$, then

$$< f, T(1) > = < f, \sum_{n} < f, a_n > b_n > = \sum_{n} < f, a_n > < f, b_n > = \sum_{n} < f, a_n b_n > = < f, 1 > = 1.$$

Thus $T(1) \neq 0$. Moreover for every $a \in A$ and $g, h \in A^*$ we have

$$< h, \sum_{n} < g, aa_n > b_n > = \sum_{n} < g, aa_n > < h, b_n > = < g \otimes h, \sum_{n} aa_n \otimes b_n > = < g \otimes h, \sum_{n} a_n \otimes b_n > a > = \sum_{n} < g, a_n > < h, b_n a > = < h, \sum_{n} < g, a_n > b_n a >.$$

This implies that

$$\sum_{n} < g, aa_n > b_n = \sum_{n} < g, a_n > b_n a.$$

Thus we assume that

$
T(1)=e,$

then we have $T(a) = \sum_{n} < f, aa_n > b_n = \sum_{n} < f, a_n > b_n a = ea$. On the other hand we have $T(a) = \sum_{n} < f, aa_n > b_n = < f, a > \sum_{n} < f, a_n > b_n = < f, a > e$. Hence $T$ is an operator of rank one and $e^2 = T(e) = < f, e > e = e$. Now define

$$T_1 : A \to A, a \mapsto \sum_{n} a_n < f, aa_n >.$$
With a similar argument we can show that
\[ T_i(a) = ae = \langle f, a \rangle e' = a \in A \]
where \( e' = T_i(1) \). Also we have \( e'^2 = e' \) and \( \langle f, e' \rangle = 1 \). Now the identities
\[ ee' = \langle f, e' \rangle = e = e' \]
\[ ee' = \langle f, e' \rangle = e' = e' \]
imply that \( e = e' \) and for every \( a \in A \) we have
\[ ea = \langle f, a \rangle e = \langle f, a \rangle e' = ae' = ae. \]

Therefore \( e \) is a central idempotent. In addition since \( T \) is a rank one operator and \( \text{ran} T = eAe \), then \( eA = eAe = Ce \) is a division ring. Therefore \( e \) is a minimal idempotent.

**b-Contractibility**

**Definition.** Let \( A \) be a Banach algebra and \( \pi \) be the natural map,
\[ \pi : A \otimes A \rightarrow A, \quad \pi \left( \sum_n a_n \otimes b_n \right) = \sum_n a_n b_n. \]

Let \( b \in A \) and \( X \) be an \( A \)-module. We say that a derivation \( \delta : A \rightarrow X \) is a \( b \)-derivation if there exists another derivation \( \delta' : A \rightarrow X \) such that \( \delta = b\delta' \), where \( (b\delta')(a) = b\delta'(a) \). Also we say that \( A \) is \( b \)-contractible if for every \( A \)-module \( X \), every bounded \( b \)-derivation from \( A \) into \( X \) is inner. We call \( d \in A \hat{\otimes} A \) a \( b \)-diagonal if \( \pi(d) = b \) and \( ad = da \) for all \( a \in A \).

**Theorem 3.1.** Let \( A \) be a unital Banach algebra and \( b \in A' - \{0\} \). Then \( A \) is \( b \)-contractible if and only if \( A \) has a \( b \)-diagonal.

**Proof.** First suppose \( A \) is \( b \)-contractible and \( \pi \) is defined as above. Clearly \( \ker \pi \) is an \( A \)-module and if we define
\[ \delta : A \rightarrow \ker \pi, a \mapsto ab \otimes 1 - b \otimes a \]
then it is easy to see that \( \delta \) is a \( b \)-derivation. Indeed \( \delta = b\delta' \) where
\[ \delta' : A \rightarrow \ker \pi, a \mapsto a \otimes 1 - 1 \otimes a \]
ince \( A \) is \( b \)-contractible, then there exists an element \( \sum_n c_n \otimes d_n \in \ker \pi \) such that
\[ \delta(a) = \sum_n c_n \otimes d_n - \sum_n c_n \otimes d_n a \quad a \in A. \]
Let $d = b \otimes 1 - \sum_n c_n \otimes d_n$. The above identities show that $\pi(d) = b$ and $a.d = d.a$ for all $a \in A$. Therefore, $d$ is a $b$-diagonal for $A$.

Conversely suppose $d = \sum_n a_n \otimes b_n$ is a $b$-diagonal for $A$, $X$ is an $A$-module and $\delta : A \to X$ is a bounded derivation. Clearly the map

$$\psi : A \times A \to X, (a, c) \mapsto a\delta(c)$$

is a bounded bilinear map. So by the universal property of projective tensor product there is a linear map $\varphi : A \hat{\otimes} A \to X$ such that $\varphi \circ \otimes = \psi$ that is $\varphi(a \otimes c) = a\delta(c)$. In particular using the fact that $d$ is a $b$-diagonal for $A$, we get

$$\sum_n a_n \delta(b_n) = \varphi(a.d) = \varphi(d.a) = \sum_n a_n \delta(b_n), \quad a \in A.$$

Now if $x = \sum_n a_n \delta(b_n)$, then for every $a \in A$ we have

$$ax - xa = \sum_n a_n \delta(b_n) - \sum_n a_n \delta(b_n) a = \sum_n a_n \delta(b_n) + b\delta(a) - \sum_n a_n \delta(b_n).$$

Thus the identity $ax - xa = b\delta(a)$ holds for every $a \in A$. Therefore every $b$-derivation is inner.

**Example 3.2.** Let $A$ be the Banach algebra $l_1(N)$ with pointwise multiplication and $\{e_n\}$ be the standard basis for $A$. Then for every positive integer $n$, $A$ is $e_n$-contractible. Indeed $e_n \otimes e_n$ is an $e_n$-diagonal for $A$. But $A$ is not contractible, since it is not unital.

Therefore $b$-contractibility does not imply contractibility.

**Remark.** If $A$ is contractible, then it is unital and one can easily observe that $A$ is $b$-contractible for every $b \in A - \{0\}$. However the above example shows that for non-unital Banach algebras the converse is not true. We do not know whether this is true for unital Banach algebras or not.

**Problem.** Does there exist a unital Banach algebra which is $b$-contractible for some nonzero central idempotent $b$, but is not contractible?

**References**


