Contractibility and idempotents in Banach algebras

R. Alizadeh: Amirkabir University of Technology

Abstract

Let \( A \) be a Banach algebra. It is shown that a contractible ideal of a Banach algebra is complemented by its annihilator. Then, it is proved the existence of minimal central idempotents in a contractible Banach algebra with a nonzero character. Moreover, the notion of b-contractibility and one of its equivalent forms are introduced. Through an example, it is shown that b-contractibility is strictly weaker than contractibility.

Introduction

Taylor in [13, Theorem 5.11] showed that a contractible Banach algebra with bounded approximation property is finite dimensional. Johnson in [6, Proposition 8.1] showed that a contractible commutative semisimple Banach algebra is finite dimensional. Curtis and Loy [1, Theorem 6.2] extended this result by dropping the semisimplicity assumption. But the question for noncommutative case has remained open. For more results of this type see [4], [5], [8], [10], [13].

This paper is organized as follows. In the second section, we show that a contractible ideal of a Banach algebra is controlled by its commutant and annihilator. Then, we prove the existence of minimal central idempotents in a contractible Banach algebra with a nonzero character. In the third section, we introduce a weaker version of contractibility which we call b-contractibility. We give a characterization of b-contractibility analog to that of contractibility given by Taylor. Also, we show that b-contractibility is strictly weaker than contractibility.

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First we recall some terminology. Throughout this paper, $A$ is a Banach algebra and $A$-module means Banach $A$-bimodule. For a subset $E$ of $A$, $E^*$ is the commutant of $E$. If for every $A$-bimodule $X$ every bounded derivation from $A$ into $X$ is inner, then $A$ is called contractible. Also, the term "semisimple" means Jacobson semisimple.

An idempotent $e \in A$ is called minimal if $eAe$ is a division ring. If $e$ and $f$ are idempotents in $A$, we write $e \leq f$ if $fe=ef=e$ holds. A nonzero idempotent $e \in A$ is called primitive if $0 \leq f \leq e$ implies that $f=0$ or $f=e$. Also, two idempotents $e$ and $f$ are said to be orthogonal if they satisfy $ef=fe=0$. Let $S$ be a subset of $A$. The right annihilator of $S$ in $A$ which we denote by $\text{ran}(S)$ is the set

$$\text{ran}(S) = \{a \in A : ba = 0 \text{ for } b \in S\}.$$ 

The left annihilator $\text{lan}(S)$ is defined similarly. The annihilator of $S$ is the set

$$\text{Ann}(S) = \text{ran}(S) \cap \text{lan}(S).$$

**Contractibility**

**Theorem 2.1.** Let $A$ be a contractible Banach algebra which is an ideal in a Banach algebra $B$. Then $A + A^* = B$.

*Proof.* If $A + A^* \neq B$, then we can choose $b \in B - (A + A^*)$. Now define

$$D : A \to A, x \mapsto xb - bx.$$ 

Clearly $D$ is a derivation on $A$. By assumption there exists an $a \in A$ such that $D(x) = xa - ax$ for all $x \in A$. The latter result implies that $b - a \in A^*$ or equivalently $b \in A + A^*$ which contradicts the selection of $b$. Therefore $A + A^* = B$.

**Theorem 2.2.** Let $A$ be a contractible Banach algebra which is an ideal in a Banach algebra $B$. Then $B = A \oplus \text{Ann}(A)$.

*Proof.* Since $A$ is contractible then $M_2(A)$ with $l^1$-norm is contractible, where $M_2(A)$ is the algebra of $2 \times 2$ matrices with the entries from $A$. On the other hand $M_2(A)$ is an ideal in $M_2(B)$ and by Theorem 2.1 we have the equality $M_2(B) = M_2(A) + M_2(A)^*$. One can easily observe that
Thus $B = A + \text{Ann}(A)$. But $A \cap \text{Ann}(A) = 0$, because $A$ is unital. Therefore the identity $B = A \oplus \text{Ann}(A)$ holds.

**Remark.** In Theorems 2.1 and 2.2, $A$ and $B$ are related only algebraically. Indeed if there exists an infinite dimensional contractible Banach algebra $A$ which is an ideal in a Banach algebra $B$, then the norm topology of $A$ could be different from the relative norm topology of $A$ which inherits from $B$.

**Theorem 2.3.** Let $A$ be a contractible Banach algebra which admits a nonzero multiplicative linear functional $f$. Then $A$ contains a central minimal idempotent.

**Proof.** Let $d = \sum_{n=1}^{\infty} a_n \otimes b_n$ be a diagonal for $A$ and define

$$T: A \rightarrow a \mapsto \sum_{n=1}^{\infty} < f, aa_n > b_n.$$ 

Since $\sum_{n} a_n b_n = 1$, then

$$< f, T(1) > = \sum_{n} < f, a_n b_n > = \sum_{n} < f, a_n > < f, b_n >$$

$$= \sum_{n} < f, a_n b_n > = \sum_{n} a_n b_n = 1.$$ 

Thus $T(1) \neq 0$. Moreover for every $a \in A$ and $g, h \in A^*$ we have

$$< h, \sum_{n} < g, aa_n > b_n > = \sum_{n} < g, aa_n > < h, b_n > = < g \otimes h, \sum_{n} a_n b_n > = \sum_{n} < g, a_n > < h, b_n > = < h, \sum_{n} < g, a_n > b_n >.$$ 

This implies that

$$\sum_{n} < g, aa_n > b_n = \sum_{n} < g, a_n > b_n a.$$ 

Thus we assume that

$$T(1) = e,$$ then we have $T(a) = \sum_{n} < f, aa_n > b_n = \sum_{n} < f, a_n > b_n a = ea$. On the other hand we have $T(a) = \sum_{n} < f, aa_n > b_n = < f, a > \sum_{n} < f, a_n > b_n = < f, a > e$. Hence $T$ is an operator of rank one and $e^2 = T(e) = < f, e > e = e$. Now define

$$T_1: A \rightarrow A, a \mapsto \sum_{n} a_n < f, aa_n >.$$
With a similar argument we can show that
\[ T_i(a) = ae^r = \langle f, a > e^r \quad a \in A \]
where \( e^r = T_i(1) \). Also we have \( e^{r2} = e^r \) and \( \langle f, e^r > = 1 \). Now the identities
\[ ee^r = \langle f, e^r > e = e, \quad ee^r = \langle f, e > e^r = e^r \]
imply that \( e = e^r \) and for every \( a \in A \) we have
\[ ea = \langle f, a > e = \langle f, a > e^r = ae^r = ae. \]

Therefore \( e \) is a central idempotent. In addition since \( T \) is a rank one operator and \( ranT = eAe \) , then \( eA = eAe = Ce \) is a division ring. Therefore \( e \) is a minimal idempotent.

**b-Contractibility**

**Definition.** Let \( A \) be a Banach algebra and \( \pi \) be the natural map,
\[ \pi : A \otimes A \rightarrow A, \quad \pi \left( \sum a_n \otimes b_n \right) = \sum a_n b_n. \]

Let \( b \in A \) and \( X \) be an \( A \)-module. We say that a derivation \( \delta : A \rightarrow X \) is a \( b \)-derivation if there exists another derivation \( \delta' : A \rightarrow X \) such that \( \delta = b\delta' \), where \( (b\delta')(a) = b\delta'(a) \). Also we say that \( A \) is \( b \)-contractible if for every \( A \)-module \( X \), every bounded \( b \)-derivation from \( A \) into \( X \) is inner. We call \( d \in \hat{A} \otimes A \) a \( b \)-diagonal if \( \pi(d) = b \) and \( a.d = d.a \) for all \( a \in A \).

**Theorem 3.1.** Let \( A \) be a unital Banach algebra and \( b \in A' - \{0\} \). Then \( A \) is \( b \)-contractible if and only if \( A \) has a \( b \)-diagonal.

**Proof.** First suppose \( A \) is \( b \)-contractible and \( \pi \) is defined as above. Clearly \( \ker \pi \) is an \( A \)-module and if we define
\[ \delta : A \rightarrow \ker \pi, a \mapsto ab \otimes 1 - b \otimes a \]
then it is easy to see that \( \delta \) is a \( b \)-derivation. Indeed \( \delta = b\delta' \) where
\[ \delta' : A \rightarrow \ker \pi, a \mapsto a \otimes 1 - 1 \otimes a \]
ince \( A \) is \( b \)-contractible, then there exists an element \( \sum c_n \otimes d_n \in \ker \pi \) such that
\[ \delta(a) = \sum a_c \otimes d_n - \sum c_n \otimes d_n a \quad a \in A. \]
Let \( d = b \otimes 1 - \sum_n e_n \otimes d_n \). The above identities show that \( \pi(d) = b \) and \( a.d = d.a \) for all \( a \in A \). Therefore, \( d \) is a \( b \)-diagonal for \( A \).

Conversely suppose \( d = \sum_n a_n \otimes b_n \) is a \( b \)-diagonal for \( A \), \( X \) is an \( A \)-module and \( \delta : A \rightarrow X \) is a bounded derivation. Clearly the map

\[ \psi : A \times A \rightarrow X, (a, c) \mapsto a\delta(c) \]

is a bounded bilinear map. So by the universal property of projective tensor product there is a linear map \( \varphi : A \hat{\otimes} A \rightarrow X \) such that \( \varphi \circ \otimes = \psi \) that is \( \varphi(a \otimes c) = a\delta(c) \). In particular using the fact that \( d \) is a \( b \)-diagonal for \( A \), we get

\[ \sum_n a_n \delta(b_n) = \varphi(a.d) = \varphi(d.a) = \sum_n a_n \delta(b_n a), \quad a \in A. \]

Now if \( x = \sum_n a_n \delta(b_n) \), then for every \( a \in A \) we have

\[ ax - xa = \sum_n a_n \delta(b_n) - \sum_n a_n \delta(b_n) a = \sum_n a_n \delta(b_n) + b\delta(a) - \sum_n a_n \delta(b_n a). \]

Thus the identity \( ax - xa = b\delta(a) \) holds for every \( a \in A \). Therefore every \( b \)-derivation is inner.

**Example 3.2.** Let \( A \) be the Banach algebra \( l_1(N) \) with pointwise multiplication and \( \{e_n\} \) be the standard basis for \( A \). Then for every positive integer \( n \), \( A \) is \( e_n \)-contractible. Indeed \( e_n \otimes e_n \) is an \( e_n \)-diagonal for \( A \). But \( A \) is not contractible, since it is not unital.

Therefore \( b \)-contractibility does not imply contractibility.

**Remark.** If \( A \) is contractible, then it is unital and one can easily observe that \( A \) is \( b \)-contractible for every \( b \in A - \{0\} \). However the above example shows that for non-unital Banach algebras the converse is not true. We do not know whether this is true for unital Banach algebras or not.

**Problem.** Does there exist a unital Banach algebra which is \( b \)-contractible for some nonzero central idempotent \( b \), but is not contractible?

**References**


