Contractibility and idempotents in Banach algebras

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Abstract

Let \( A \) be a Banach algebra. It is shown that a contractible ideal of a Banach algebra is complemented by its annihilator. Then, it is proved the existence of minimal central idempotents in a contractible Banach algebra with a nonzero character. Moreover, the notion of b-contractibility and one of its equivalent forms are introduced. Through an example, it is shown that b-contractibility is strictly weaker than contractibility.

Introduction

Taylor in [13, Theorem 5.11] showed that a contractible Banach algebra with bounded approximation property is finite dimensional. Johnson in [6, Proposition 8.1] showed that a contractible commutative semisimple Banach algebra is finite dimensional. Curtis and Loy [1, Theorem 6.2] extended this result by dropping the semisimplicity assumption. But the question for noncommutative case has remained open. For more results of this type see [4], [5], [8], [10], [13].

This paper is organized as follows. In the second section, we show that a contractible ideal of a Banach algebra is controlled by its commutant and annihilator. Then, we prove the existence of minimal central idempotents in a contractible Banach algebra with a nonzero character. In the third section, we introduce a weaker version of contractibility which we call b-contractibility. We give a characterization of b-contractibility analog to that of contractibility given by Taylor. Also, we show that b-contractibility is strictly weaker than contractibility.

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First we recall some terminology. Throughout this paper, $A$ is a Banach algebra and $A$-module means Banach $A$-bimodule. For a subset $E$ of $A$, $E'$ is the commutant of $E$. If for every $A$-bimodule $X$ every bounded derivation from $A$ into $X$ is inner, then $A$ is called contractible. Also, the term "semisimple" means Jacobson semisimple.

An idempotent $e \in A$ is called minimal if $eAe$ is a division ring. If $e$ and $f$ are idempotents in $A$, we write $e \leq f$ if $fe = ef = e$ holds. A nonzero idempotent $e \in A$ is called primitive if $0 \leq f \leq e$ implies that $f = 0$ or $f = e$. Also, two idempotents $e$ and $f$ are said to be orthogonal if they satisfy $ef = fe = 0$. Let $S$ be a subset of $A$. The right annihilator of $S$ in $A$ which we denote by $\text{ran}(S)$ is the set

$$\text{ran}(S) = \{a \in A : ba = 0 \text{ for } b \in S\}.$$ 

The left annihilator $\text{lan}(S)$ is defined similarly. The annihilator of $S$ is the set \( \text{Ann}(S) = \text{ran}(S) \cap \text{lan}(S) \).

**Contractibility**

**Theorem 2.1.** Let $A$ be a contractible Banach algebra which is an ideal in a Banach algebra $B$. Then $A + A' = B$.

**Proof.** If $A + A' \neq B$, then we can choose $b \in B - (A + A')$. Now define

$$D : A \to A, x \mapsto xb - bx.$$ 

Clearly $D$ is a derivation on $A$. By assumption there exists an $a \in A$ such that $D(x) = xa - ax$ for all $x \in A$. The latter result implies that $b - a \in A'$ or equivalently $b \in A + A'$ which contradicts the selection of $b$. Therefore $A + A' = B$.

**Theorem 2.2.** Let $A$ be a contractible Banach algebra which is an ideal in a Banach algebra $B$. Then $B = A \oplus \text{Ann}(A)$.

**Proof.** Since $A$ is contractible then $M_2(A)$ with $l^1$-norm is contractible, where $M_2(A)$ is the algebra of $2 \times 2$ matrices with the entries from $A$. On the other hand $M_2(A)$ is an ideal in $M_2(B)$ and by Theorem 2.1 we have the equality $M_2(B) = M_2(A) + M_2(A)'$. One can easily observe that
\[ M_2(A)' = \begin{bmatrix} A' & \text{Ann}(A) \\ \text{Ann}(A) & A' \end{bmatrix}. \]

Thus \( B = A + \text{Ann}(A) \). But \( A \cap \text{Ann}(A) = 0 \), because \( A \) is unital. Therefore the identity \( B = A \oplus \text{Ann}(A) \) holds.

**Remark.** In Theorems 2.1 and 2.2, \( A \) and \( B \) are related only algebrically. Indeed if there exists an infinite dimensional contractible Banach algebra \( A \) which is an ideal in a Banach algebra \( B \), then the norm topology of \( A \) could be different from the relative norm topology of \( A \) which inherits from \( B \).

**Theorem 2.3.** Let \( A \) be a contractible Banach algebra which admits a nonzero multiplicative linear functional \( f \). Then \( A \) contains a central minimal idempotent.

**Proof.** Let \( d = \sum_{n=1}^{\infty} a_n \otimes b_n \) be a diagonal for \( A \) and define

\[ T : A \to a \mapsto \sum_{n=1}^{\infty} < f, aa_n > b_n. \]

Since \( \sum_n a_n b_n = 1 \), then

\[ < f, T(1) >= < f, \sum_n a_n > b_n >= \sum_n < f, a_n > < f, b_n > = \sum_n < f, a_n b_n >= < f, 1 > = 1. \]

Thus \( T(1) \neq 0 \). Moreover for every \( a \in A \) and \( g, h \in A^* \) we have

\[ < h, \sum_n < g, aa_n > b_n > = \sum_n < g, aa_n > < h, b_n > = \sum_n < g \otimes h, \sum_n aa_n \otimes b_n > = \sum_n < g \otimes h, \sum_n a_n \otimes b_n a > = \sum_n < g, a_n > < h, b_n a > = < h, \sum_n < g, a_n > b_n >. \]

This implies that

\[ \sum_n < g, aa_n > b_n = \sum_n < g, a_n > b_n. \]

Thus we assume that \( T(1) = e \), then we have \( T(a) = \sum_n < f, aa_n > b_n = \sum_n < f, a_n > b_n a = ea \). On the other hand we have \( T(a) = \sum_n < f, aa_n > b_n = < f, a > \sum_n < f, a_n > b_n = < f, a > e \). Hence \( T \) is an operator of rank one and \( e^2 = T(e) = < f, e > e = e \). Now define

\[ T_1 : A \to A, a \mapsto \sum_n a_n < f, aa_n >. \]
With a similar argument we can show that
\[ T_1(a) = ae^\prime = < f, a > e^\prime \quad a \in A \]
where \( e^\prime = T_1(1) \). Also we have \( e^\prime 2 = e^\prime \) and \( < f, e^\prime >= 1 \). Now the identities
\[ ee^\prime = < f, e^\prime > e = e, \quad ee^\prime = < f, e > e^\prime = e^\prime \]
imply that \( e = e^\prime \) and for every \( a \in A \) we have
\[ ea = < f, a > e = < f, a > e^\prime = ae^\prime = ae. \]

Therefore \( e \) is a central idempotent. In addition since \( T \) is a rank one operator and \( \text{ran} T = eAe \), then \( eA = eAe = Ce \) is a division ring. Therefore \( e \) is a minimal idempotent.

**b-Contractibility**

**Definition.** Let \( A \) be a Banach algebra and \( \pi \) be the natural map,
\[ \pi : A \otimes A \rightarrow A, \quad \pi \left( \sum_n a_n \otimes b_n \right) = \sum_n a_n b_n. \]

Let \( b \in A \) and \( X \) be an \( A \)-module. We say that a derivation \( \delta : A \rightarrow X \) is a \( b \)-derivation if there exists another derivation \( \delta : A \rightarrow X \) such that \( \delta = b\delta \) \( \dot{\prime} \), where \( (b\delta \dot{\prime})(a) = b\delta \dot{\prime}(a) \). Also we say that \( A \) is \( b \)-contractible if for every \( A \)-module \( X \), every bounded \( b \)-derivation from \( A \) into \( X \) is inner. We call \( d \in A \otimes A \) a \( b \)-diagonal if \( \pi(d) = b \) and \( a.d = d.a \) for all \( a \in A \).

**Theorem 3.1.** Let \( A \) be a unital Banach algebra and \( b \in A^\dagger \setminus \{0\} \). Then \( A \) is \( b \)-contractible if and only if \( A \) has a \( b \)-diagonal.

**Proof.** First suppose \( A \) is \( b \)-contractible and \( \pi \) is defined as above. Clearly \( \ker \pi \) is an \( A \)-module and if we define
\[ \delta : A \rightarrow \ker \pi, a \mapsto ab \otimes 1 - b \otimes a \]
then it is easy to see that \( \delta \) is a \( b \)-derivation. Indeed \( \delta = b\delta \dot{\prime} \) where
\[ \delta : A \rightarrow \ker \pi, a \mapsto a \otimes 1 - 1 \otimes a \]
ince \( A \) is \( b \)-contractible, then there exists an element \( \sum_n c_n \otimes d_n \in \ker \pi \) such that
\[ \delta(a) = \sum_n c_n \otimes d_n - \sum_n c_n \otimes d_a a \quad a \in A. \]
Let $d = b \otimes 1 - \sum_n c_n \otimes d_n$. The above identities show that $\pi(d) = b$ and $a.d = d.a$ for all $a \in A$. Therefore, $d$ is a $b$-diagonal for $A$.

Conversely suppose $d = \sum a_n \otimes b_n$ is a $b$-diagonal for $A$, $X$ is an $A$-module and $\delta : A \rightarrow X$ is a bounded derivation. Clearly the map

$$\psi : A \times A \rightarrow X, (a, c) \mapsto a \delta(c)$$

is a bounded bilinear map. So by the universal property of projective tensor product there is a linear map $\varphi : A \hat{\otimes} A \rightarrow X$ such that $\varphi \circ \otimes = \psi$ that is $\varphi(a \otimes c) = a \delta(c)$. In particular using the fact that $d$ is a $b$-diagonal for $A$, we get

$$\sum_n a_n \delta(b_n) = \varphi(a.d) = \varphi(d.a) = \sum_n a_n \delta(b_n)a, \quad a \in A.$$

Now if $x = \sum_n a_n \delta(b_n)$, then for every $a \in A$ we have

$$ax - xa = \sum_n aa_n \delta(b_n) - \sum_n a_n \delta(b_n)a = \sum_n aa_n \delta(b_n) + b \delta(a) - \sum_n a_n \delta(b_n)a.$$

Thus the identity $ax - xa = b \delta(a)$ holds for every $a \in A$. Therefore every $b$-derivation is inner.

**Example 3.2.** Let $A$ be the Banach algebra $l_1(N)$ with pointwise multiplication and $\{e_n\}$ be the standard basis for $A$. Then for every positive integer $n$, $A$ is $e_n$-contractible.

Indeed $e_n \otimes e_n$ is an $e_n$-diagonal for $A$. But $A$ is not contractible, since it is not unital. Therefore $b$-contractibility does not imply contractibility.

**Remark.** If $A$ is contractible, then it is unital and one can easily observe that $A$ is $b$-contractible for every $b \in A - \{0\}$. However the above example shows that for non-unital Banach algebras the converse is not true. We do not know whether this is true for unital Banach algebras or not.

**Problem.** Does there exist a unital Banach algebra which is $b$-contractible for some nonzero central idempotent $b$, but is not contractible?

**References**


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