A certain \( n \)-Generalized Principally Quasi-Baer Subring of the Matrix rings

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Abstract

For a fixed positive integer \( n \), we say a ring with identity is \emph{n-generalized right principally quasi-Baer}, if for any principal right ideal \( I \) of \( R \), the right annihilator of \( I^n \) is generated by an idempotent. This class of rings includes the right principally quasi-Baer rings and hence all prime rings. A certain \( n \)-generalized principally quasi-Baer subring of the matrix ring \( M_n(R) \) are studied, and connections to related classes of rings (e.g., p.q.-Baer rings and \( n \)-generalized p.p. rings) are considered.

1. Introduction and Preliminaries

Throughout all rings are assumed to be associative with identity. From [12, 21], a ring \( R \) is (quasi-)Baer if the right annihilator of any (right ideal) nonempty subset of \( R \) is generated, as a right ideal, by an idempotent. Moreover, in [12] Clark proved the left-right symmetry of this condition. He uses this condition to characterize when a finite dimensional algebra with unity over an algebraically closed field is isomorphic to a twisted matrix units semigroup algebra. The class of quasi-Baer rings is a nontrivial generalization of the class of Baer rings. Every prime ring is quasi-Baer, hence prime rings with nonzero right singular ideal are quasi-Baer; but not Baer [24]. For a positive integer \( n > 1 \), the \( n \times n \) matrix ring over a non-Prüfer commutative domain is a prime quasi-Baer ring which is not a Baer ring by [27] and [21, p.17]. The \( n \times n \) (\( n > 1 \)) upper triangular matrix ring over a domain which is not a division ring is quasi-Baer but not

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Baer by an example due to Cohn; see [1], [20] and [5]. The theory of Baer and quasi-
Baer rings has come to play an important role and major contributions have been made
in recent years by a number of authors, including Birkenmeier, Chatters, Khuri, Kim,
Hirano and Park (see, for example [1], [4-7], [16], [21], [26] and [28]).

A ring satisfying a generalization of Rickart’s condition [30] (i.e., right annihilator of
any element is generated (as a right ideal) by an idempotent) has a homological
characterization as a right p.p.-ring. A ring R is called a right (resp. left) p.p.-ring if
every principal right (resp. left) ideal is projective. R is called a p.p.-ring (also called a
Rickart ring [2, p.18]), if it is both right and left p.p.-ring. In [9] Chase shows the
concept of p.p.-ring is not left-right symmetric. Small [30] shows that a right p.p.-ring
R is Baer (so p.p), when R is orthogonally finite. Also it is shown by Endo [13] that a
right p.p.-ring R is p.p when R is abelian (i.e., every idempotent is central). Finally
Chatters and Xue [11] prove that in a duo (i.e., every one sided ideal is two sided) p.p.-
ring R, if l is a finitely generated right projective ideal of R, then l is left projective
and a direct summand of an invertible ideal. Following Birkenmeier et al. [7], R is
called right principally quasi-Baer (or simply right p.q.-Baer), if the right annihilator of
a principal right ideal is generated by an idempotent. Equivalently, R is right p.q.-Baer
if R modulo the right annihilator of any principal right ideal is projective. Similarly, left
p.q.-Baer rings can be defined. If R is both right and left p.q.-Baer, then it is called p.q.-
Baer. The class of p.q.-Baer rings includes all biregular rings, all quasi-Baer rings, and
all abelian p.p.-rings. A ring R is said to be p-regular, if for every x ∈ R there exists
a natural number n, depending on x, such that x^n ∈ x^n Rx^n. A ring R is called a
generalized right p.p.-ring if for any x ∈ R the right ideal x^n Rx^n is projective for some
positive integer n, depending on x, or equivalently, if for any x ∈ R the right annihilator
of x^n is generated by an idempotent for some positive integer n, depending on x. A ring
is called generalized p.p.-ring, if it is both generalized right and left p.p.-ring.

Note that Von Neumann regular rings are right (left) p.p.-rings by Goodearl [14,
Theorem 1.1], and a same argument as [14, Theorem 1.1] shows that \( p \)-regular rings are generalized p.p.-rings. Right p.p.-rings are generalized right p.p obviously. See [18] for more details.

**Definition 1.1.** Given a fixed positive integer \( n \), we say a ring \( R \) is \( n \)-generalized right principally quasi Baer (or \( n \)-generalized right p.q.-Baer), if for all principal right ideal \( I \) of \( R \), the right annihilator of \( I^n \) is generated by an idempotent. Left cases may be defined analogously.

The class of \( n \)-generalized right p.q.-Baer rings includes all right p.q.-Baer rings, (and hence all biregular rings, quasi-Baer rings, abelian p.p.-rings and semicommutative (i.e., if \( R(x) \) is an ideal for all \( x \in R \) generalized p.p rings). Theorem 2.1 in section 2, allows us to construct examples of \( n \)-generalized p.q.-Baer rings that are not p.q.-Baer. Some conditions on the equivalence of \( n \)-generalized p.q.-Baer and \( n \)-generalized p.p.-rings are discussed. However, we show by examples that the class of \( n \)-generalized p.q.-Baer rings properly extends the aforementioned classes.

In this paper, we discuss some type of matrix rings formed over p.q.-Baer or p.p. rings. We study \( n \)-generalized p.q.-Baer subrings of the matrix ring \( M_n(R) \). Theorem 2.2, enables us to generate examples of \( n \)-generalized p.q.-Baer subrings of the matrix ring \( M_n(R) \). Theorem 2.2, which extends [18, Proposition 6], enables us to provide more examples of matrix rings, that are both \( n \)-generalized p.q.-Baer and \( n \)-generalized p.p.-ring. Connections to related classes of rings are investigated. Although the class of \( n \)-generalized p.q.-Baer rings, includes all p.q.-Baer rings (and hence, all biregular rings, and all abelian p.p. rings), however we show by examples that the class of \( n \)-generalized p.q.-Baer rings properly extends the aforementioned classes.

Note that, for a reduced ring (which has no nonzero nilpotent elements), we have \( I_R(Rx) = I_R((Rx)^n) = I_R(x^n) = I_R(x) = r_R(x) = r_R((xR)^n) = r_R(xR) \), for every \( x \in R \) and every positive integer \( n \). Therefore reduced rings are semicommutative and semicommutative rings are abelian. Also for reduced rings the definitions of right p.q.-
Baer, n-generalized right p.q.-Baer, generalized p.p. and p.p.-ring are coincide. This leads one ask whether commutative reduced rings are n-generalized p.q-Baer. However, the answer is negative by the following.

**Example 1.2.** Let \( p \) be a prime number and \( R = \{(a,b) \in \mathbb{Z} \oplus \mathbb{Z} \mid a \equiv b \pmod{p}\} \), then \( R \) is a commutative reduced ring. Note that the only idempotents of \( R \) are \((0,0)\) and \((1,1)\). One can show that \( r_k((p,0)R) = (0,p)R \), so \( r_k((p,0)R) \) does not contain a nonzero idempotent of \( R \); and hence \( R \) is not n-generalized right quasi-Baer, for any positive integer \( n \).

**Lemma 1.3.** Let \( R \) be an abelian \( n \)-generalized right p.q.-Baer ring, then \( r_k(l^n) = r_k(l^m) \) for every principal right ideal \( I \) of \( R \) and each positive integer \( m \) with \( n \leq m \).

**Proof.** It is enough to show that \( r_k(l^n) = r_k(l^{n+1}) \). Let \( x \in r_k(l^{n+1}) \), then \( l^nx \subseteq r_k(l^n) = fR \) for some idempotent \( f \in R \). Hence \( l^nx = l^nxf = 0 \). Thus \( x \in r_k(l^n) \).

2. **N-generalized right principally quasi Baer subrings of the matrix rings**

In this section we discuss some type of matrix rings formed over p.q.-Baer or p.p. rings. Theorem 2.3, which extends [18, Proposition 6], enables us to provide more examples of matrix rings that are both \( n \)-generalized p.q.-Baer and \( n \)-generalized p.p.-ring. We begin with Theorem 2.2 below, which enables us to generate examples of \( n \)-generalized p.q.-Baer subrings of the matrix ring \( M_n(R) \):

**Lemma 2.1**[18, Lemma 2]. Let \( R \) be an abelian ring and define

\[
S_n := \left\{ \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} : a,a_{ij} \in R \right\},
\]

with \( n \) a positive integer \( \geq 2 \). Then every idempotent in \( S_n \) is of the form

\[
\begin{pmatrix} f & 0 & \cdots & 0 \\ 0 & f & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f \end{pmatrix}
\]

with \( f^2 = f \in R \).
We will use $S_n$. Throughout the remainder of the paper, to denote the ring indicated in Lemma 2.1.

**Theorem 2.2.** If $R$ is an abelian p.q.-Baer ring and $n \geq 2$ is a positive integer, then $S_n$ is an $n$-generalized right p.q.-Baer ring.

**Proof.** We proceed by induction on $n$. It is easy to show that $S_2$ is a 2-generalized right p.q.-Baer ring. Let $I_n$ be a principal right ideal of $S_n$. Consider $\begin{bmatrix} \epsilon_{n-1} & & \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$, and $\begin{bmatrix} \epsilon_{n-1} \\ \vdots \\ 0 \end{bmatrix}$. Let $J$ be the set of entries of the main diagonal of the elements of $I_{n-1}$. It is clear that $J$ is a principal right ideal of $S_{n-1}$. By induction hypothesis and Lemma 2.1, there are $x_i, y_i \in S_{n-1}$ for $1 \leq i \leq n$ such that $x_i y_{n-i} = y_{n-i} x_i$. So we have $x_{n-i} + y_{n-i} = 0$. Hence $x_i y_{n-i} = 0$, since $R$ is an abelian ring. Now let

$$X = \begin{bmatrix} x & x_{12} & \cdots & x_{1n} \\ 0 & x & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x \end{bmatrix} \in R_n(I_{n-1})$$

and

$$Y = \begin{bmatrix} a_1 a_2 \cdots a_n & y_{12} & \cdots & y_{1n} \\ 0 & a_2 a_3 \cdots a_n & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \cdots a_n \end{bmatrix} \in I_{n-1}$$

Since $r_{n-1}(I_{n-1}) = r_{n-1}(I_{n-2}) = e_2 S_{n-1}$, $x$ and $x_{ij}$'s are in $f_i R$ for each $i$ and $j$ except $x_{in}$. So we have $a_1 a_2 \cdots a_n x_{in} + y_{in} x = 0$. Hence $y_{in} x = 0$, since $f_i \in B(R)$. Thus $x_{in} \in f_i R$ and hence $r_{n}(I_{n}) \subseteq \epsilon S_{n}$ for

$$e = \begin{bmatrix} f_1 & 0 & \cdots & 0 \\ 0 & f_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_1 \end{bmatrix} \in S_{n}.$$ 

Since, for each $Y \in I_{n} e$, all entries of the main diagonal of $Y$ are zero and $e$ is central,

$$I_{n} e = (I_{n} e)^n = 0.$$ 

Thus $r_{n}(I_{n}) = e S_{n}$. Therefore $S_{n}$ is $n$-generalized right p.q.-Baer.

The following result, which generalizes [18, Proposition 6], provides examples of
matrix rings that are both n-generalized p.q.-Baer and n-generalized p.p.-ring:

**Theorem 2.3.** If \( R \) is an abelian p.p.-ring, then \( S_n \) is an abelian \( n \)-generalized p.p.-ring.

**Proof.** We prove by induction on \( n \). First, we show that the trivial extension \( S_2 \) of \( R \) is 2-generalized right p.p. Let \( A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in S_2 \) and \( r_n(a) = eR \), with \( e^2 = e \in R \). It is clear that, \( fR \subseteq r_n(A^2) \) with \( f = \begin{pmatrix} e \\ 0 \end{pmatrix} \). Next, let \( A^2 = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} = 0 \). Since \( R \) is reduced, \( a^2x = ax = 0 \) and \( a^2y = ay = 0 \). Hence \( ex = x \) and \( ey = y \). Thus \( A^2 = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} = f \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \).

Therefore \( S_2 \) is 2-generalized right p.p. Now assume \( B = \begin{pmatrix} a & a_{i2} & \cdots & a_{in} \\ 0 & a & \cdots & a^{2n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \) and \( B_2 = \begin{pmatrix} a & a_{23} & \cdots & a_{2n} \\ 0 & a & \cdots & a_{2n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \) in \( S_{n-1} \), then by the induction hypothesis, there exist \( e^2 = e_i \in S_{n-1} \), \( f_i = f_i \in R \), such that \( r_{n-1}(B_i^{n-1}) = e_i S_{n-1} \), \( e_i = \begin{pmatrix} f_i & 0 & \cdots & 0 \\ 0 & f_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_i \end{pmatrix} \) for \( i = 1, 2 \). By direct calculations, we have \( r_n(B^{2n-2}) = e S_n \) with \( e = \begin{pmatrix} f & 0 & \cdots & 0 \\ 0 & f & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f \end{pmatrix} \). Since \( r_n(a) = eR \), by [27, Lemma 3], \( r_n(B^n) = r_n(B^{2n-2}) = e S_n \).

**Corollary 2.4** [18, Proposition 6]. If \( R \) is a domain, then \( S_n \) is an abelian \( n \)-generalized p.p.-ring.

For a semicommutative ring, the definitions of \( n \)-generalized right p.q.-Baer and \( n \)-generalized right p.p. are coincide:

**Proposition 2.5.** Let \( R \) be a semicommutative ring. Then \( R \) is \( n \)-generalized right p.q.-Baer if and only if \( R \) is \( n \)-generalized right p.p.
Proof. Let \( R \) be \( n \)-generalized right p.q.-Baer and \( a \in R \). Then \( r_R(aR)^n = eR \) for some idempotent \( e \in R \). Let \( x \in r_R(a^n) \). Since \( R \) is semicommutative, \( Rax \subseteq r_R(a^{n-1}) \), which implies that \( r_R(aR)^n = eR \). The converse is similar.

There exists an \( n \)-generalized right p.q.-Baer ring, which is generalized p.p.-ring but is not semicommutative.

**Example 2.6.** Let \( R \) be an integral domain and \( S_4 \) be defined over \( R \). Then \( S_4 \) is abelian \( 4 \)-generalized p.p.-ring and is \( 4 \)-generalized p.q.-Baer by Corollary 2.4. By considering \( b = a = e_{22} + e_{14} + e_{34} \) and \( c = e_{23} \) in \( S_4 \), where \( e_{ij} \) denote the matrix units, we have \( ab = 0 \), and \( acb \neq 0 \), hence \( aSb \neq 0 \).

Now we conjecture that subrings of \( n \)-generalized right p.q.-Baer rings are also \( n \)-generalized right p.q.-Baer. But the answer is negative by the following.

**Example 2.7.** For a field \( F \), take \( F_n = F \) for \( n = 1, 2, \ldots \), and let \( S \) be the \( 2 \times 2 \) matrix ring over the ring \( \prod_{n=1}^{\infty} F_n \). By [7, Proposition 2.1 and Theorem 2.2] we have that \( S \) is a p.q.-Baer ring. Let

\[
R = \left( \begin{array}{cc}
\prod_{n=1}^{\infty} F_n & \bigoplus_{n=1}^{\infty} F_n \\
\bigoplus_{n=1}^{\infty} F_n & < \bigoplus_{n=1}^{\infty} F_n, 1 >
\end{array} \right),
\]

which is a subring of \( S \), where \( < \bigoplus_{n=1}^{\infty} F_n, 1 > \) is the \( F \)-algebra generated by \( \bigoplus_{n=1}^{\infty} F_n \) and 1. Then by [7, Example 1.6], \( R \) is semiprime p.p which is neither right p.q.-Baer (and hence not \( n \)-generalized right p.q.-Baer), nor left p.q.-Baer (and hence not \( n \)-generalized left p.q.-Baer).

### 3. Examples of \( n \)-generalized p.q.-Baer subrings

Although the class of \( n \)-generalized p.q.-Baer rings, includes all p.q.-Baer rings (and hence, all biregular rings, all quasi-Baer rings, and all abelian p.p. rings), however we show by examples that the class of \( n \)-generalized p.q.-Baer rings properly extends the aforementioned classes.

By the following example, there is an abelian p.q.-Baer (hence semiprime) ring \( R \),
which is not reduced, but $S_n$ is an abelian $n$-generalized right p.q.-Baer ring that is not semiprime.

**Example 3.1.** By Zalesskii and Neroslavskii [10, Example 14.17, p.179], there is a simple noetherian ring $R$ that is not a domain and in which 0 and 1 are the only idempotents. Thus $R$ is an abelian p.q.-Baer ring that is neither left nor right p.p., and hence is not reduced. By [7, Proposition 1.17] $R$ is semiprime and by Theorem 2.1, $S_n$ is abelian $n$-generalized p.q.-Baer, that is not semiprime and hence is not right p.q.-Baer.

**Example 3.2.** If $R$ is an abelian p.q.-Baer ring, then $R[x]/ (x^3)$ is an $n$-generalized p.q.-Baer ring.

**Proof.** First we note that $\Theta : T \rightarrow R[x]/ (x^3)$ defined by

$$(a_0,a_1,a_2) \rightarrow (a_0 + a_1x + a_2x^2) + (x^3)$$

is an isomorphism, where $T = \{(a,b,c) | a,b,c \in R\}$ is a ring with addition componentwise and the multiplication defined by

$$(a_1,b_1,c_1)(a_2,b_2,c_2) = (a_1a_2,b_1b_2 + b_1a_2 ,a_1c_2 + b_2a_2 + c_1a_2).$$

Let $I$ be an ideal of $T$. Suppose $I = \{(a,b,c) \in T\}$, it is clear that $I$ is an ideal of $R$. Since $R$ is p.q.-Baer, $r_R(I) = eR$ for an idempotent $e \in R$. We can show that $r(I^3) = \langle e,0,0\rangle T$, and hence, the result follows.

There exists a commutative $n$-generalized p.q.-Baer (hence $n$-generalized p.p.-) ring $R$, over which $S_n$ is not an $n$-generalized p.p.-ring.

**Example 3.3.** Let $p = 3$ be a prime integer and $Z_{p^3}$ be the ring of integers modulo $p^3$, and $S_3$ be defined over $Z_{p^3}$. Let $A = pl_3 + e_{13}$, where $l_3$ is the identity matrix and $e_i$ denote the matrix units. It is clear that $pl_3 + e_3 + e_{22} \in r_{S_n}(A^3)$ and idempotents of $S_3$ are $l_3$ and 0. Hence $r_{S_3}(A^3) = l_3S_3$ and that $S_3$ is not 3-generalized p.p.-ring, but $Z_{p^3}$ is a 3-generalized p.p.-ring.

**Example 3.4.** For every abelian quasi-Baer (resp. p.p.-) ring $R$, by Theorems 2.1 and
2.2, the ring $S_n$ is $n$-generalized right p.q.-Baer, which is not right p.q.-Baer. Therefore we are able to provide examples of $n$-generalized right p.q.-Baer rings that is not right p.q.-Baer:

Let $F$ be a field, and $R = F[x]$ be the polynomial ring where $x$ is an indeterminate. Then $S_n$ is a $n$-generalized right p.q.-Baer ring that is not right p.q.-Baer.

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**Reference**