A certain $N$-Generalized Principally Quasi-Baer Subring of the Matrix rings

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Abstract

For a fixed positive integer $n$, we say a ring with identity is $n$-generalized right principally quasi-Baer, if for any principal right ideal $I$ of $R$, the right annihilator of $I^n$ is generated by an idempotent. This class of rings includes the right principally quasi-Baer rings and hence all prime rings. A certain $n$-generalized principally quasi-Baer subring of the matrix ring $M_n(R)$ are studied, and connections to related classes of rings (e.g., p.q.-Baer rings and $n$-generalized p.p. rings) are considered.

1. Introduction and Preliminaries

Throughout all rings are assumed to be associative with identity. From [12, 21], a ring $R$ is (quasi-)Baer if the right annihilator of any (right ideal) nonempty subset of $R$ is generated, as a right ideal, by an idempotent. Moreover, in [12] Clark proved the left-right symmetry of this condition. He uses this condition to characterize when a finite dimensional algebra with unity over an algebraically closed field is isomorphic to a twisted matrix units semigroup algebra. The class of quasi-Baer rings is a nontrivial generalization of the class of Baer rings. Every prime ring is quasi-Baer, hence prime rings with nonzero right singular ideal are quasi-Baer; but not Baer [24]. For a positive integer $n > 1$, the $n \times n$ matrix ring over a non-Prüfer commutative domain is a prime quasi-Baer ring which is not a Baer ring by [27] and [21, p.17]. The $n \times n$ ($n > 1$) upper triangular matrix ring over a domain which is not a division ring is quasi-Baer but not

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Baer by an example due to Cohn; see [1], [20] and [5]. The theory of Baer and quasi-Baer rings has come to play an important role and major contributions have been made in recent years by a number of authors, including Birkenmeier, Chatters, Khuri, Kim, Hirano and Park (see, for example [1], [4-7], [16], [21], [26] and [28]).

A ring satisfying a generalization of Rickart’s condition [30] (i.e., right annihilator of any element is generated (as a right ideal) by an idempotent) has a homological characterization as a right p.p.-ring. A ring \( R \) is called a right (resp. left) p.p.-ring if every principal right (resp. left) ideal is projective. \( R \) is called a p.p.-ring (also called a Rickart ring [2, p.18]), if it is both right and left p.p.-ring. In [9] Chase shows the concept of p.p.-ring is not left-right symmetric. Small [30] shows that a right p.p.-ring \( R \) is Baer (so p.p), when \( R \) is orthogonally finite. Also it is shown by Endo [13] that a right p.p.-ring \( R \) is p.p when \( R \) is abelian (i.e., every idempotent is central). Finally Chatters and Xue [11] prove that in a duo (i.e., every one sided ideal is two sided) p.p.-ring \( R \), if \( I \) is a finitely generated right projective ideal of \( R \), then \( I \) is left projective and a direct summand of an invertible ideal. Following Birkenmeier et al. [7], \( R \) is called right principally quasi-Baer (or simply right p.q.-Baer), if the right annihilator of a principal right ideal is generated by an idempotent. Equivalently, \( R \) is right p.q.-Baer if \( R \) modulo the right annihilator of any principal right ideal is projective. Similarly, left p.q.-Baer rings can be defined. If \( R \) is both right and left p.q.-Baer, then it is called p.q.-Baer. The class of p.q.-Baer rings includes all biregular rings, all quasi-Baer rings, and all abelian p.p.-rings. A ring \( R \) is said to be \( p \) - regular, if for every \( x \in R \) there exists a natural number \( n \), depending on \( x \), such that \( x^n \in x^nRx^n \). A ring \( R \) is called a generalized right p.p.-ring if for any \( x \in R \) the right ideal \( x^nR \) is projective for some positive integer \( n \), depending on \( x \), or equivalently, if for any \( x \in R \) the right annihilator of \( x^n \) is generated by an idempotent for some positive integer \( n \), depending on \( x \). A ring is called generalized p.p.-ring, if it is both generalized right and left p.p.-ring.

Note that Von Neumann regular rings are right (left) p.p.-rings by Goodearl [14,
Theorem 1.1], and a same argument as [14, Theorem 1.1] shows that $p$-regular rings are generalized p.p.-rings. Right p.p.-rings are generalized right p.p obviously. See [18] for more details.

**Definition 1.1.** Given a fixed positive integer $n$, we say a ring $R$ is $n$-generalized right principally quasi Baer (or $n$-generalized right p.q.-Baer), if for all principal right ideal $l$ of $R$, the right annihilator of $l^n$ is generated by an idempotent. Left cases may be defined analogously.

The class of $n$-generalized right p.q.-Baer rings includes all right p.q.-Baer rings, (and hence all biregular rings, quasi-Baer rings, abelian p.p.-rings and semicommutative (i.e., if $r(x)$ is an ideal for all $x \in R$) generalized p.p rings). Theorem 2.1 in section 2, allows us to construct examples of $n$-generalized p.q.-Baer rings that are not p.q.-Baer. Some conditions on the equivalence of $n$-generalized p.q.-Baer and $n$-generalized p.p.-rings are discussed. However, we show by examples that the class of $n$-generalized p.q.-Baer rings properly extends the aforementioned classes.

In this paper, we discuss some type of matrix rings formed over p.q.-Baer or p.p. rings. We study $n$-generalized p.q.-Baer subrings of the matrix ring $M_n(R)$. Theorem 2.2, enables us to generate examples of $n$-generalized p.q.-Baer subrings of the matrix ring $M_n(R)$. Theorem 2.2, which extends [18, Proposition 6], enables us to provide more examples of matrix rings, that are both $n$-generalized p.q.-Baer and $n$-generalized p.p.-ring. Connections to related classes of rings are investigated. Although the class of $n$-generalized p.q.-Baer rings, includes all p.q.-Baer rings (and hence, all biregular rings, and all abelian p.p. rings), however we show by examples that the class of $n$-generalized p.q.-Baer rings properly extends the aforementioned classes.

Note that, for a reduced ring (which has no nonzero nilpotent elements), we have $l_R(R x) = l_R((R x)^n) = l_R(x^n) = l_R(x) = r_R(x) = r_R((x R)^n) = r_R(x R)$, for every $x \in R$ and every positive integer $n$. Therefore reduced rings are semicommutative and semicommutative rings are abelian. Also for reduced rings the definitions of right p.q.-
Baer, \(n\)-generalized right p.q.-Baer, generalized p.p. and p.p.-ring are coincide. This leads one ask whether commutative reduced rings are \(n\)-generalized p.q-Baer. However, the answer is negative by the following.

**Example 1.2.** Let \(p\) be a prime number and \(R = \{(a, b) \in \mathbb{Z} \oplus \mathbb{Z} \mid a \equiv b \pmod{p}\}\), then \(R\) is a commutative reduced ring. Note that the only idempotents of \(R\) are \((0, 0)\) and \((1, 1)\). One can show that \(r_R((p, 0)R) = (0, p)R\), so \(r_R((p, 0)R)\) does not contain a nonzero idempotent of \(R\); and hence \(R\) is not \(n\)-generalized right quasi-Baer, for any positive integer \(n\).

**Lemma 1.3.** Let \(R\) be an abelian \(n\)-generalized right p.q.-Baer ring, then \(r_R(l^n) = r_R(l^m)\) for every principal right ideal \(I\) of \(R\) and each positive integer \(m\) with \(n \leq m\).

**Proof.** It is enough to show that \(r_R(l^n) = r_R(l^{n+1})\). Let \(x \in r_R(l^{n+1})\), then \(lx \subseteq r_R(l^n) = fR\) for some idempotent \(f \in R\). Hence \(l^n x = l^n x f = 0\). Thus \(x \in r_R(l^n)\).

## 2. \(N\)-generalized right principally quasi Baer subrings of the matrix rings

In this section we discuss some type of matrix rings formed over p.q.-Baer or p.p. rings. Theorem 2.3, which extends [18, Proposition 6], enables us to provide more examples of matrix rings that are both \(n\)-generalized p.q.-Baer and \(n\)-generalized p.p.-ring. We begin with Theorem 2.2 below, which enables us to generate examples of \(n\)-generalized p.q.-Baer subrings of the matrix ring \(M_n(R)\):

**Lemma 2.1**[18, Lemma 2]. Let \(R\) be an abelian ring and define
\[
S_n := \begin{bmatrix}
    a & a_{12} & \cdots & a_{1n} \\
    0 & a & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & a
\end{bmatrix} : a, a_{ij} \in R,
\]

with \(n\) a positive integer \(\geq 2\). Then every idempotent in \(S_n\) is of the form
\[
\begin{bmatrix}
    f & 0 & \cdots & 0 \\
    0 & f & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & f
\end{bmatrix}
\]
with \(f^2 = f \in R\).
We will use $S_n$ Throughout the remainder of the paper, to denote the ring indicated in Lemma 2.1.

**Theorem 2.2.** If $R$ is an abelian p.q.-Baer ring and $n \ (\geq 2)$ is a positive integer, then $S_n$ is an $n$-generalized right p.q.-Baer ring.

**Proof.** We proceed by induction on $n$. It is easy to show that $S_2$ is a 2-generalized right p.q.-Baer ring. Let $I_n$ be a principal right ideal of $S_n$. Consider $I_{n-1} = \{ B \in S_{n-1} \mid B \text{ is obtained by deleting } n \text{-th row and } n \text{-th column of a matrix in } I_n \}$, and $I_{n-2} = \{ B \in S_{n-1} \mid B \text{ is obtained by deleting } (n-1) \text{-th row and } 1 \text{-th column of a matrix in } I_n \}$. It is clear that $I_{n-1}$ and $I_{n-2}$ are principal right ideals of $S_{n-1}$. By induction hypothesis and Lemma 2.1, there are $e_1, e_2 \in S_{n-1}$ such that $e_1 R = e_2 R = f_1 R = f_2 R$. Hence $f_1 = f_2$, since $R$ is an abelian ring. Now let

$$X = \begin{pmatrix} x & x_2 & \ldots & x_{1n} \\ 0 & x & \ldots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & x \end{pmatrix} \in r_{S_n}(I_n) \text{ and } Y = \begin{pmatrix} a_1 a_2 a_3 \ldots a_n & y_{12} & \ldots & y_{1n} \\ 0 & a_3 a_4 a_5 \ldots a_n & \ldots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_{n-1} a_n a_1 \ldots a_2 \end{pmatrix} \in I_n$$

Since $r_{S_{n-1}}(I_{n-1}) = r_{S_{n-1}}(I_{n-2}) = e_1 S_{n-1}$, $x$ and $x_{ij}$'s are in $f_1 R$ for each $i$ and $j$ except $x_{1n}$. So we have $a_1 a_2 a_3 \ldots a_n x_{1n} + y_{1n} x = 0$. Hence $y_{1n} x = 0$, since $f_1 \in B(R)$. Thus $x_{1n} \in f_1 R$ and hence $r_{S_n}(I_n) \subseteq e S_n$ for

$$e = \begin{pmatrix} f_1 & 0 & \ldots & 0 \\ 0 & f_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & f_1 \end{pmatrix} \in S_n.$$ 

Since, for each $Y \in I_n e$, all entries of the main diagonal of $Y$ are zero and $e$ is central, $l_n e = (l_n e)^n = 0$. Thus $r_{S_n}(I_n) = e S_n$. Therefore $S_n$ is $n$-generalized right p.q.-Baer.

The following result, which generalizes [18, Proposition 6], provides examples of
matrix rings that are both \( n \)-generalized p.q.-Baer and \( n \)-generalized p.p.-ring:

**Theorem 2.3.** If \( R \) is an abelian p.p.-ring, then \( S_n \) is an abelian \( n \)-generalized p.p.-ring.

**Proof.** We prove by induction on \( n \). First, we show that the trivial extension \( S_2 \) of \( R \) is \( 2 \)-generalized right p.p. Let \( A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in S_2 \) and \( r_n(a) = eR \), with \( e = e^2 \in R \). It is clear that, \( fR \subseteq S_2 \), and \( A^2 = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} = 0 \). Hence \( ex = x \) and \( ey = y \). Thus \( \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} = f \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \).

Therefore \( S_2 \) is \( 2 \)-generalized right p.p. Now assume \( B = \begin{pmatrix} a & 0 & \ldots & a_n \\ 0 & a & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a \end{pmatrix} \in S_n \). Consider \( B_1 = \begin{pmatrix} a & a_{12} & \ldots & a_{1n-1} \\ 0 & a & \ldots & a_{2n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a \end{pmatrix} \) and \( B_2 = \begin{pmatrix} a & a_{23} & \ldots & a_{2n} \\ 0 & a & \ldots & a_{3n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a \end{pmatrix} \). By the induction hypothesis, there exists \( e^2 = e_i \in S_{n-1} \), \( f_i = f_i \in R \), such that \( r_{n-1}(B_i^{n-1}) = e_i S_{n-1} \), \( e_i = \begin{pmatrix} f_i & 0 & \ldots & 0 \\ 0 & f_i & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & f_i \end{pmatrix} \). By direct calculations, we have \( r_n(B^{n-2}) = eS_n \) with \( e = \begin{pmatrix} f & 0 & \ldots & 0 \\ 0 & f & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & f \end{pmatrix} \). Since \( r_n(a) = eR \), by [27, Lemma 3], \( r_n(B^n) = r_n(B^{2n-2}) = eS_n \).

**Corollary 2.4** [18, Proposition 6]. If \( R \) is a domain, then \( S_n \) is an abelian \( n \)-generalized p.p.-ring.

For a semicommutative ring, the definitions of \( n \)-generalized right p.q.-Baer and \( n \)-generalized right p.p. are coincide:

**Proposition 2.5.** Let \( R \) be a semicommutative ring. Then \( R \) is \( n \)-generalized right p.q.-Baer if and only if \( R \) is \( n \)-generalized right p.p.
Proof. Let $R$ be $n$-generalized right p.q.-Baer and $a \in R$. Then $r_R(aR)^n = eR$ for some idempotent $e \in R$. Let $x \in r_R(a^n)$. Since $R$ is semicommutative, $RaRx \subseteq r_R(a^{n-1})$, which implies that $r_R(aR)^n = eR$. The converse is similar.

There exists an $n$-generalized right p.q.-Baer ring, which is generalized p.p.-ring but is not semicommutative.

Example 2.6. Let $R$ be an integral domain and $S_4$ be defined over $R$. Then $S_4$ is abelian 4-generalized p.p.-ring and is 4-generalized p.q.-Baer by Corollary 2.4. By considering $b = a = e_{22} + e_{34} + e_{34}$ and $c = e_{23}$ in $S_4$, where $e_{ij}$ denote the matrix units, we have $ab = 0$, and $ac = 0$, hence $aSb \neq 0$.

Now we conjecture that subrings of $n$-generalized right p.q.-Baer rings are also $n$-generalized right p.q.-Baer. But the answer is negative by the following.

Example 2.7. For a field $F$, take $F_n = F$ for $n = 1, 2, \ldots$, and let $S$ be the $2 \times 2$ matrix ring over the ring $\Pi_{n=1}^{\infty}F_n$. By [7, Proposition 2.1 and Theorem 2.2] we have that $S$ is a p.q.-Baer ring. Let $R = \left( \begin{array}{cc} \Pi_{n=1}^{\infty}F_n & \odot_{n=1}^{\infty}F_n \\ \odot_{n=1}^{\infty}F_n & < \odot_{n=1}^{\infty}F_n, 1 > \end{array} \right)$, which is a subring of $S$, where $< \odot_{n=1}^{\infty}F_n, 1 >$ is the $F$-algebra generated by $\odot_{n=1}^{\infty}F_n$ and 1. Then by [7, Example 1.6], $R$ is semiprime p.p which is neither right p.q.-Baer (and hence not $n$-generalized right p.q.-Baer), nor left p.q.-Baer (and hence not $n$-generalized left p.q.-Baer).

3. Examples of $n$-generalized p.q.-Baer subrings

Although the class of $n$-generalized p.q.-Baer rings, includes all p.q.-Baer rings (and hence, all biregular rings, all quasi-Baer rings, and all abelian p.p. rings), however we show by examples that the class of $n$-generalized p.q.-Baer rings properly extends the aforementioned classes.

By the following example, there is an abelian p.q.-Baer (hence semiprime) ring $R$,
which is not reduced, but $S_n$ is an abelian $n$-generalized right p.q.-Baer ring that is not semiprime.

**Example 3.1.** By Zalesskii and Neroslavskii [10, Example 14.17, p.179], there is a simple noetherian ring $R$ that is not a domain and in which 0 and 1 are the only idempotents. Thus $R$ is an abelian p.q.-Baer ring that is neither left nor right p.p, and hence is not reduced. By [7, Proposition 1.17] $R$ is semiprime and by Theorem 2.1, $S_n$ is abelian $n$-generalized p.q.-Baer, that is not semiprime and hence is not right p.q.-Baer.

**Example 3.2.** If $R$ is an abelian p.q.-Baer ring, then $R[x]/<x^3>$ is an $n$-generalized p.q.-Baer ring.

**Proof.** First we note that $\Theta : T \to R[x]/<x^3>$ defined by

$$ (a_0,a_1,a_2) \to (a_0 + a_1x + a_2x^2) + <x^3> $$

is an isomorphism, where $T = \{(a,b,c) \mid a,b,c \in R\}$ is a ring with addition componentwise and the multiplication defined by

$$ (a_1,b_1,c_1)(a_2,b_2,c_2) = (a_1a_2,a_1b_2 + b_1a_2,a_1c_2 + b_1b_2 + c_1a_2). $$

Let $J$ be an ideal of $T$. Suppose $I = \{a \in R \mid (a,b,c) \in J\}$, it is clear that $I$ is an ideal of $R$. Since $R$ is p.q.-Baer, $r_R(I) = eR$ for an idempotent $e \in R$. We can show that $r(I^3) = (e,0,0)T$, and hence, the result follows.

There exists a commutative $n$-generalized p.q.-Baer (hence $n$-generalized p.p.-) ring $R$, over which $S_n$ is not an $n$-generalized p.p.-ring.

**Example 3.3.** Let $p = 3$ be a prime integer and $Z_{p^3}$ be the ring of integers modulo $p^3$, and $S_3$ be defined over $Z_{p^3}$. Let $A = pI_3 + e_{13}$, where $I_3$ is the identity matrix and $e_{ij}$ denote the matrix units. It is clear that $pI_3 + e_{13} + e_{22} \in r_{S_3}(A^3)$ and idempotents of $S_3$ are $I_3$ and 0. Hence $r_{S_3}(A^3) = I_3S_3$ and that $S_3$ is not 3-generalized p.p.-ring, but $Z_{p^3}$ is a 3-generalized p.p.-ring.

**Example 3.4.** For every abelian quasi-Baer (resp. p.p.-) ring $R$, by Theorems 2.1 and
2.2, the ring $S_n$ is $n$-generalized right p.q.-Baer, which is not right p.q.-Baer. Therefore we are able to provide examples of $n$-generalized right p.q.-Baer rings that is not right p.q.-Baer:

Let $F$ be a field, and $R = F[x]$ be the polynomial ring where $x$ is an indeterminate. Then $S_n$ is an $n$-generalized right p.q.-Baer ring that is not right p.q.-Baer.

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**Reference**