A certain $n$-Generalized Principally Quasi-Baer Subring of the Matrix rings

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Abstract

For a fixed positive integer $n$, we say a ring with identity is $n$-generalized right principally quasi-Baer, if for any principal right ideal $I$ of $R$, the right annihilator of $I^n$ is generated by an idempotent. This class of rings includes the right principally quasi-Baer rings and hence all prime rings. A certain $n$-generalized principally quasi-Baer subring of the matrix ring $M_n(R)$ are studied, and connections to related classes of rings (e.g., p.q.-Baer rings and $n$-generalized p.p. rings) are considered.

1. Introduction and Preliminaries

Throughout all rings are assumed to be associative with identity. From [12, 21], a ring $R$ is (quasi-)Baer if the right annihilator of any (right ideal) nonempty subset of $R$ is generated, as a right ideal, by an idempotent. Moreover, in [12] Clark proved the left-right symmetry of this condition. He uses this condition to characterize when a finite dimensional algebra with unity over an algebraically closed field is isomorphic to a twisted matrix units semigroup algebra. The class of quasi-Baer rings is a nontrivial generalization of the class of Baer rings. Every prime ring is quasi-Baer, hence prime rings with nonzero right singular ideal are quasi-Baer; but not Baer [24]. For a positive integer $n > 1$, the $n \times n$ matrix ring over a non-Prüfer commutative domain is a prime quasi-Baer ring which is not a Baer ring by [27] and [21, p.17]. The $n \times n$ ($n > 1$) upper triangular matrix ring over a domain which is not a division ring is quasi-Baer but not

\footnote{1. 2000 Mathematical Subject Classification. 16D15; 16D40; 16D70. Keywords and phrases. n-Generalized p.q.-Baer ring; p.p.-ring; Annihilators; triangular matrix ring}
Baer by an example due to Cohn; see [1], [20] and [5]. The theory of Baer and quasi-Baer rings has come to play an important role and major contributions have been made in recent years by a number of authors, including Birkenmeier, Chatters, Khuri, Kim, Hirano and Park (see, for example [1], [4-7], [16], [21], [26] and [28]).

A ring satisfying a generalization of Rickart’s condition [30] (i.e., right annihilator of any element is generated (as a right ideal) by an idempotent) has a homological characterization as a right p.p.-ring. A ring $R$ is called a right (resp. left) p.p.-ring if every principal right (resp. left) ideal is projective. $R$ is called a p.p.-ring (also called a Rickart ring [2, p.18]), if it is both right and left p.p.-ring. In [9] Chase shows the concept of p.p.-ring is not left-right symmetric. Small [30] shows that a right p.p.-ring $R$ is Baer (so p.p), when $R$ is orthogonally finite. Also it is shown by Endo [13] that a right p.p.-ring $R$ is p.p when $R$ is abelian (i.e., every idempotent is central). Finally Chatters and Xue [11] prove that in a duo (i.e., every one sided ideal is two sided) p.p.-ring $R$, if $1$ is a finitely generated right projective ideal of $R$, then $1$ is left projective and a direct summand of an invertible ideal. Following Birkenmeier et al. [7], $R$ is called right principally quasi-Baer (or simply right p.q.-Baer), if the right annihilator of a principal right ideal is generated by an idempotent. Equivalently, $R$ is right p.q.-Baer if $R$ modulo the right annihilator of any principal right ideal is projective. Similarly, left p.q.-Baer rings can be defined. If $R$ is both right and left p.q.-Baer, then it is called p.q.-Baer. The class of p.q.-Baer rings includes all biregular rings, all quasi-Baer rings, and all abelian p.p.-rings. A ring $R$ is said to be $p$-regular, if for every $x \in R$ there exists a natural number $n$, depending on $x$, such that $x^n \in x^0 Rx^n$. A ring $R$ is called a generalized right p.p.-ring if for any $X \in R$ the right ideal $x^0 R$ is projective for some positive integer $n$, depending on $x$, or equivalently, if for any $x \in R$ the right annihilator of $x^0$ is generated by an idempotent for some positive integer $n$, depending on $x$. A ring is called generalized p.p.-ring, if it is both generalized right and left p.p.-ring.

Note that Von Neumann regular rings are right (left) p.p.-rings by Goodearl [14,
Theorem 1.1], and a same argument as [14, Theorem 1.1] shows that p - regular rings are generalized p.p.-rings. Right p.p.-rings are generalized right p.p obviously. See [18] for more details.

**Definition 1.1.** Given a fixed positive integer n, we say a ring $R$ is $n$-generalized right principally quasi Baer (or $n$-generalized right p.q.-Baer), if for all principal right ideal $I$ of $R$, the right annihilator of $I^n$ is generated by an idempotent. Left cases may be defined analogously.

The class of $n$-generalized right p.q.-Baer rings includes all right p.q.-Baer rings, (and hence all biregular rings, quasi-Baer rings, abelian p.p.-rings and semicommutative (i.e., if $r(x)$ is an ideal for all $x \in R$) generalized p.p rings). Theorem 2.1 in section 2, allows us to construct examples of $n$-generalized p.q.-Baer rings that are not p.q.-Baer. Some conditions on the equivalence of $n$-generalized p.q.-Baer and $n$-generalized p.p.-rings are discussed. However, we show by examples that the class of $n$-generalized p.q.-Baer rings properly extends the aforementioned classes.

In this paper, we discuss some type of matrix rings formed over p.q.-Baer or p.p. rings. We study $n$-generalized p.q.-Baer subrings of the matrix ring $M_n(R)$. Theorem 2.2, enables us to generate examples of $n$-generalized p.q.-Baer subrings of the matrix ring $M_n(R)$. Theorem 2.2, which extends [18, Proposition 6], enables us to provide more examples of matrix rings, that are both $n$-generalized p.q.-Baer and $n$-generalized p.p.-ring. Connections to related classes of rings are investigated. Although the class of $n$-generalized p.q.-Baer rings, includes all p.q.-Baer rings (and hence, all biregular rings, and all abelian p.p. rings), however we show by examples that the class of $n$-generalized p.q.-Baer rings properly extends the aforementioned classes.

Note that, for a reduced ring (which has no nonzero nilpotent elements), we have $l_R(Rx) = l_R((Rx)^n) = l_R(x^n) = l_R(x) = r_R(x) = r_R((xR)^n) = r_R(xR)$, for every $x \in R$ and every positive integer $n$. Therefore reduced rings are semicommutative and semicommutative rings are abelian. Also for reduced rings the definitions of right p.q.-
Baer, n-generalized right p.q.-Baer, generalized p.p. and p.p.-ring are coincide. This leads one ask whether commutative reduced rings are n-generalized p.q.-Baer. However, the answer is negative by the following.

**Example 1.2.** Let \( p \) be a prime number and \( R = \{(a, b) \in \mathbb{Z} \oplus \mathbb{Z} \mid a \equiv b \ (\text{mod} \ p)\} \), then \( R \) is a commutative reduced ring. Note that the only idempotents of \( R \) are \((0, 0)\) and \((1, 1)\). One can show that \( r_R((p, 0)R) = (0, p)R \), so \( r_R((p, 0)R) \) does not contain a nonzero idempotent of \( R \); and hence \( R \) is not n-generalized right quasi-Baer, for any positive integer \( n \).

**Lemma 1.3.** Let \( R \) be an abelian \( n \)-generalized right p.q.-Baer ring, then \( r_R(l^n) = r_R(l^m) \) for every principal right ideal \( l \) of \( R \) and each positive integer \( m \) with \( n \leq m \).

**Proof.** It is enough to show that \( r_R(l^n) = r_R(l^{n+1}) \). Let \( x \in r_R(l^{n+1}) \), then \( lx \in r_R(l^n) = fR \) for some idempotent \( f \in R \). Hence \( lx = lxdf = 0 \). Thus \( x \in r_R(l^n) \).

### 2. \( N \)-generalized right principally quasi Baer subrings of the matrix rings

In this section we discuss some type of matrix rings formed over p.q.-Baer or p.p. rings. Theorem 2.3, which extends [18, Proposition 6], enables us to provide more examples of matrix rings that are both \( n \)-generalized p.q.-Baer and \( n \)-generalized p.p.-ring. We begin with Theorem 2.2 below, which enables us to generate examples of \( n \)-generalized p.q.-Baer subrings of the matrix ring \( M_n(R) \):

**Lemma 2.1**[18, Lemma 2]. Let \( R \) be an abelian ring and define

\[
S_n := \left\{ \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} : a_{ij} \in R \right\},
\]

with \( n \) a positive integer \( \geq 2 \). Then every idempotent in \( S_n \) is of the form

\[
\begin{pmatrix} f & 0 & \cdots & 0 \\ 0 & f & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f \end{pmatrix}
\]

with \( f^2 = f \in R \).
We will use $S_n$. Throughout the remainder of the paper, to denote the ring indicated in Lemma 2.1.

**Theorem 2.2.** If $R$ is an abelian p.q.-Baer ring and $n \geq 2$ is a positive integer, then $S_n$ is an $n$-generalized right p.q.-Baer ring.

**Proof.** We proceed by induction on $n$. It is easy to show that $S_2$ is a 2-generalized right p.q.-Baer ring. Let $I_n$ be a principal right ideal of $S_n$. Consider $\{ \begin{array}{c} n \text{th row and } n \text{th column of a matrix in } I_n \\ \end{array} \}$. It is clear that $I_n$ and $I_n$ are principal right ideals of $S_n$. By induction hypothesis and Lemma 2.1, there are $\{ \begin{array}{c} x_{11} \ldots x_{1n} \\ x \end{array} \}$ and $\{ \begin{array}{c} a_{12}a_{23}\ldots a_n \\ 0 \\ \end{array} \}$ such that $x_{11} \ldots x_{1n} + y_{1n}x = 0$. Hence $x_{1n} \in f_1R$ and hence $r_{S_n}(l_0^n) \subseteq eS_n$ for all $e \in S_n$. Since, for each $Y \in I_n e$, all entries of the main diagonal of $Y$ are zero and $e$ is central, $l_0^n e = (l_0 e)^n = 0$. Thus $r_{S_n}(l_0^n) = eS_n$. Therefore $S_n$ is $n$-generalized right p.q.-Baer.

The following result, which generalizes [18, Proposition 6], provides examples of...
matrix rings that are both n-generalized p.q.-Baer and n-generalized p.p.-ring:

**Theorem 2.3.** If $R$ is an abelian p.p.-ring, then $S_n$ is an abelian $n$-generalized p.p.-ring.

**Proof.** We prove by induction on $n$. First, we show that the trivial extension $S_2$ of $R$ is 2-generalized right p.p. Let $A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in S_2$ and $r_R(a) = eR$, with $e = e^2 \in R$. It is clear that, $fR \subseteq r_{S_2}(A^2)$ with $f = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}$. Next, let $A^2 = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} = 0$. Since $R$ is reduced, $a^2x = ax = 0$ and $a^2y = ay = 0$. Hence $eR = x$ and $y \in eR$. Thus $\begin{pmatrix} x & y \\ 0 & x \end{pmatrix} = f \begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$.

Therefore $S_2$ is 2-generalized right p.p. Now assume $B = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \in S_n$. Consider $B_1 = \begin{pmatrix} a & a_{12} & \cdots & a_{1n-1} \\ 0 & a & \cdots & a_{2n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix}$ and $B_2 = \begin{pmatrix} a_{21} & \cdots & a_{2n} \\ 0 & \cdots & a_{3n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a \end{pmatrix} \in S_{n-1}$, then by the induction hypothesis, there exist $e_i = e_i \in S_{n-1}, f_i = f_i \in R$, such that $r_{S_{n-1}}(B_1^{n-1}) = e_i S_{n-1}$, $r_{S_{n-1}}(B_2^{n-2}) = eS_n$ with $\begin{pmatrix} f_1 & 0 & \cdots & 0 \\ 0 & f_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_1 \end{pmatrix}$ for $i = 1, 2$. By direct calculations, we have $r_{S_n}(B^{n-1}) = eS_n$ with $\begin{pmatrix} f & 0 & \cdots & 0 \\ 0 & f & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f \end{pmatrix}$. Since $r_R(a) = eR$, by [27, Lemma 3], $r_{S_n}(B^n) = r_{S_n}(B^{2n-2}) = eS_n$.

**Corollary 2.4** [18, Proposition 6]. If $R$ is a domain, then $S_n$ is an abelian $n$-generalized p.p.-ring.

For a semicommutative ring, the definitions of $n$-generalized right p.q.-Baer and $n$-generalized right p.p. are coincide:

**Proposition 2.5.** Let $R$ be a semicommutative ring. Then $R$ is $n$-generalized right p.q.-Baer if and only if $R$ is $n$-generalized right p.p.
Proof. Let $R$ be $n$-generalized right p.q.-Baer and $a \in R$. Then $r_R(aR)^n = eR$ for some idempotent $e \in R$. Let $x \in r_R(a^n)$. Since $R$ is semicommutative, $Rax \subseteq r_R(a^{n-1})$, which implies that $r_R(aR)^n = eR$. The converse is similar.

There exists an $n$-generalized right p.q.-Baer ring, which is generalized p.p.-ring but is not semicommutative.

Example 2.6. Let $R$ be an integral domain and $S_4$ be defined over $R$. Then $S_4$ is abelian 4-generalized p.p.-ring and is 4-generalized p.q.-Baer by Corollary 2.4. By considering $b = a = e_{22} + e_{34} + e_{34}$ and $c = e_{23}$ in $S_4$, where $e_{ij}$ denote the matrix units, we have $ab = 0$, and $acb \neq 0$, hence $aS_4b \neq 0$.

Now we conjecture that subrings of $n$-generalized right p.q.-Baer rings are also $n$-generalized right p.q.-Baer. But the answer is negative by the following.

Example 2.7. For a field $F$, take $F_n = F$ for $n = 1, 2, \ldots$, and let $S$ be the $2 \times 2$ matrix ring over the ring $\prod_{n=1}^\infty F_n$. By [7, Proposition 2.1 and Theorem 2.2] we have that $S$ is a p.q.-Baer ring. Let

$$R = \begin{pmatrix} \prod_{n=1}^\infty F_n & \bigoplus_{n=1}^\infty F_n \\ \bigoplus_{n=1}^\infty F_n & < \bigoplus_{n=1}^\infty F_n, 1 > \end{pmatrix},$$

which is a subring of $S$, where $< \bigoplus_{n=1}^\infty F_n, 1 >$ is the $F$-algebra generated by $\bigoplus_{n=1}^\infty F_n$ and 1. Then by [7, Example 1.6], $R$ is semiprime p.p which is neither right p.q.-Baer (and hence not $n$-generalized right p.q.-Baer), nor left p.q.-Baer (and hence not $n$-generalized left p.q.-Baer).

3. Examples of $n$-generalized p.q.-Baer subrings

Although the class of $n$-generalized p.q.-Baer rings, includes all p.q.-Baer rings (and hence, all biregular rings, all quasi-Baer rings, and all abelian p.p. rings), however we show by examples that the class of $n$-generalized p.q.-Baer rings properly extends the aforementioned classes.

By the following example, there is an abelian p.q.-Baer (hence semiprime) ring $R$,.
which is not reduced, but \( S_n \) is an abelian \( n \)-generalized right p.q.-Baer ring that is not semiprime.

**Example 3.1.** By Zalesskii and Neroslavskii [10, Example 14.17, p.179], there is a simple noetherian ring \( R \) that is not a domain and in which 0 and 1 are the only idempotents. Thus \( R \) is an abelian p.q.-Baer ring that is neither left nor right p.p, and hence is not reduced. By [7, Proposition 1.17] \( R \) is semiprime and by Theorem 2.1, \( S_n \) is abelian \( n \)-generalized p.q.-Baer, that is not semiprime and hence is not right p.q.-Baer.

**Example 3.2.** If \( R \) is an abelian p.q-Baer ring, then \( R[x]/<x^3> \) is an \( n \)-generalized p.q.-Baer ring.

**Proof.** First we note that \( \Theta : T \rightarrow R[x]/<x^3> \) defined by

\[(a_0,a_1,a_2) \rightarrow (a_0 + a_1x + a_2x^2) + <x^3>\]

is an isomorphism, where \( T = \{ (a,b,c) \mid a,b,c \in R \} \) is a ring with addition componentwise and the multiplication defined by

\[(a_1,b_1,c_1)(b_2,b_2,c_2) = (a_1b_2,a_1b_2 + b_2a_1 + c_1a_2 + c_2a_1,a_1c_2 + b_2c_1 + c_2b_1,c_1c_2 + b_2b_2 + c_1c_1)\,.

Let \( J \) be an ideal of \( T \). Suppose \( l = \{ a \in R \mid (a,b,c) \in J \} \), it is clear that \( l \) is an ideal of \( R \). Since \( R \) is p.q.-Baer, \( r_k(l) = eR \) for an idempotent \( e \in R \). We can show that \( r(l^3) = (e,0,0)T \), and hence, the result follows.

There exists a commutative \( n \)-generalized p.q.-Baer (hence \( n \)-generalized p.p.-) ring \( R \), over which \( S_n \) is not an \( n \)-generalized p.p.-ring.

**Example 3.3.** Let \( p \neq 3 \) be a prime integer and \( Z_{p^3} \) be the ring of integers modulo \( p^3 \), and \( S_3 \) be defined over \( Z_{p^3} \). Let \( A = pI_3 + e_{13} \), where \( I_3 \) is the identity matrix and \( e_{ij} \) denote the matrix units. It is clear that \( pI_3 + e_{31} + e_{12} \in r_{S_3}(A^3) \) and idempotents of \( S_3 \) are \( I_3 \) and 0. Hence \( r_{S_3}(A^3) = I_3S_3 \) and that \( S_3 \) is not \( 3 \)-generalized p.p.-ring, but \( Z_{p^3} \) is a \( 3 \)-generalized p.p.-ring.

**Example 3.4.** For every abelian quasi-Baer (resp. p.p.-) ring \( R \), by Theorems 2.1 and
2.2, the ring $S_n$ is $n$-generalized right p.q.-Baer, which is not right p.q.-Baer. Therefore we are able to provide examples of $n$-generalized right p.q.-Baer rings that is not right p.q.-Baer:

Let $F$ be a field, and $R = F[x]$ be the polynomial ring where $x$ is an indeterminate. Then $S_n$ is a $n$-generalized right p.q.-Baer ring that is not right p.q.-Baer.

**Acknowledgement**

The authors are deeply indebted to the referee for many helpful comments and suggestions for the improvement of this paper.

**Reference**