A certain \textit{N}-Generalized Principally Quasi-Baer Subring of the Matrix rings

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Abstract

For a fixed positive integer \(n\), we say a ring with identity is \textit{n-generalized right principally quasi-Baer}, if for any principal right ideal \(I\) of \(R\), the right annihilator of \(I^n\) is generated by an idempotent. This class of rings includes the right principally quasi-Baer rings and hence all prime rings. A certain \(n\)-generalized principally quasi-Baer subring of the matrix ring \(M_n(R)\) are studied, and connections to related classes of rings (e.g., p.q.-Baer rings and \(n\)-generalized p.p. rings) are considered\(^1\).

1. Introduction and Preliminaries

Throughout all rings are assumed to be associative with identity. From [12, 21], a ring \(R\) is (quasi-)Baer if the right annihilator of any (right ideal) nonempty subset of \(R\) is generated, as a right ideal, by an idempotent. Moreover, in [12] Clark proved the left-right symmetry of this condition. He uses this condition to characterize when a finite dimensional algebra with unity over an algebraically closed filed is isomorphic to a twisted matrix units semigroup algebra. The class of quasi-Baer rings is a nontrivial generalization of the class of Baer rings. Every prime ring is quasi-Baer, hence prime rings with nonzero right singular ideal are quasi-Baer; but not Baer [24]. For a positive integer \(n > 1\), the \(n \times n\) matrix ring over a non-Prüfer commutative domain is a prime quasi-Baer ring which is not a Baer ring by [27] and [21, p.17]. The \(n \times n\) (\(n > 1\)) upper triangular matrix ring over a domain which is not a division ring is quasi-Baer but not

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Baer by an example due to Cohn; see [1], [20] and [5]. The theory of Baer and quasi-Baer rings has come to play an important role and major contributions have been made in recent years by a number of authors, including Birkenmeier, Chatters, Khuri, Kim, Hirano and Park (see, for example [1], [4-7], [16], [21], [26] and [28]).

A ring satisfying a generalization of Rickart’s condition [30] (i.e., right annihilator of any element is generated (as a right ideal) by an idempotent) has a homological characterization as a right p.p.-ring. A ring $R$ is called a right (resp. left) p.p.-ring if every principal right (resp. left) ideal is projective. $R$ is called a p.p.-ring (also called a Rickart ring [2, p.18]), if it is both right and left p.p.-ring. In [9] Chase shows the concept of p.p.-ring is not left-right symmetric. Small [30] shows that a right p.p.-ring $R$ is Baer (so p.p), when $R$ is orthogonally finite. Also it is shown by Endo [13] that a right p.p.-ring $R$ is p.p when $R$ is abelian (i.e., every idempotent is central). Finally Chatters and Xue [11] prove that in a duo (i.e., every one sided ideal is two sided) p.p.-ring $R$, if $I$ is a finitely generated right projective ideal of $R$, then $I$ is left projective and a direct summand of an invertible ideal. Following Birkenmeier et al. [7], $R$ is called right principally quasi-Baer (or simply right p.q.-Baer), if the right annihilator of a principal right ideal is generated by an idempotent. Equivalently, $R$ is right p.q.-Baer if $R$ modulo the right annihilator of any principal right ideal is projective. Similarly, left p.q.-Baer rings can be defined. If $R$ is both right and left p.q.-Baer, then it is called p.q.-Baer. The class of p.q.-Baer rings includes all biregular rings, all quasi-Baer rings, and all abelian p.p.-rings. A ring $R$ is said to be $p$–regular, if for every $x \in R$ there exists a natural number $n$, depending on $x$, such that $x^n \in x^0Rx^0$. A ring $R$ is called a generalized right p.p.-ring if for any $x \in R$ the right ideal $x^0R$ is projective for some positive integer $n$, depending on $x$, or equivalently, if for any $x \in R$ the right annihilator of $x^0$ is generated by an idempotent for some positive integer $n$, depending on $x$. A ring is called generalized p.p.-ring, if it is both generalized right and left p.p.-ring.

Note that Von Neumann regular rings are right (left) p.p.-rings by Goodearl [14,
Theorem 1.1], and a same argument as [14, Theorem 1.1] shows that p - regular rings are generalized p.p.-rings. Right p.p.-rings are generalized right p.p obviously. See [18] for more details.

**Definition 1.1.** Given a fixed positive integer \( n \), we say a ring \( R \) is \( n \)-generalized right principally quasi Baer (or \( n \)-generalized right p.q.-Baer), if for all principal right ideal \( I \) of \( R \), the right annihilator of \( I^n \) is generated by an idempotent. Left cases may be defined analogously.

The class of \( n \)-generalized right p.q.-Baer rings includes all right p.q.-Baer rings, (and hence all biregular rings, quasi-Baer rings, abelian p.p.-rings and semicommutative (i.e., if \( r(x) \) is an ideal for all \( x \in R \) ) generalized p.p rings). Theorem 2.1 in section 2, allows us to construct examples of \( n \)-generalized p.q.-Baer rings that are not p.q.-Baer. Some conditions on the equivalence of \( n \)-generalized p.q.-Baer and \( n \)-generalized p.p.-rings are discussed. However, we show by examples that the class of \( n \)-generalized p.q.-Baer rings properly extends the aforementioned classes.

In this paper, we discuss some type of matrix rings formed over p.q.-Baer or p.p. rings. We study \( n \)-generalized p.q.-Baer subrings of the matrix ring \( M_n(R) \). Theorem 2.2, enables us to generate examples of \( n \)-generalized p.q.-Baer subrings of the matrix ring \( M_n(R) \). Theorem 2.2, which extends [18, Proposition 6], enables us to provide more examples of matrix rings, that are both \( n \)-generalized p.q.-Baer and \( n \)-generalized p.p.-ring. Connections to related classes of rings are investigated. Although the class of \( n \)-generalized p.q.-Baer rings, includes all p.q.-Baer rings (and hence, all biregular rings, and all abelian p.p. rings), however we show by examples that the class of \( n \)-generalized p.q.-Baer rings properly extends the aforementioned classes.

Note that, for a reduced ring (which has no nonzero nilpotent elements), we have \( l_R(Rx) = l_R((Rx)^n) = l_R(x^n) = l_R(x) = r_R(x) = r_R((xR)^n) = r_R(xR) \), for every \( x \in R \) and every positive integer \( n \). Therefore reduced rings are semicommutative and semicommutative rings are abelian. Also for reduced rings the definitions of right p.q.-
Baer, n-generalized right p.q.-Baer, generalized p.p. and p.p.-ring are coincide. This leads one ask whether commutative reduced rings are n-generalized p.q.-Baer. However, the answer is negative by the following.

**Example 1.2.** Let $p$ be a prime number and $R = \{(a, b) \in \mathbb{Z} \oplus \mathbb{Z} \mid a \equiv b \pmod{p}\}$, then $R$ is a commutative reduced ring. Note that the only idempotents of $R$ are $(0, 0)$ and $(1, 1)$. One can show that $r_R((p, 0)R) = (0, p)R$, so $r_R((p, 0)R)$ does not contain a nonzero idempotent of $R$; and hence $R$ is not n-generalized right quasi-Baer, for any positive integer $n$.

**Lemma 1.3.** Let $R$ be an abelian n-generalized right p.q.-Baer ring, then $r_R(l^n) = r_R(l^m)$ for every principal right ideal $I$ of $R$ and each positive integer $m$ with $n \leq m$.

**Proof.** It is enough to show that $r_R(l^n) = r_R(l^{n+1})$. Let $x \in r_R(l^{n+1})$, then $l^nx \subseteq r_R(l^n) = fR$ for some idempotent $f \in R$. Hence $l^nx = l^nf = 0$. Thus $x \in r_R(l^n)$.

### 2. n-generalized right principally quasi Baer subrings of the matrix rings

In this section we discuss some type of matrix rings formed over p.q.-Baer or p.p.-rings. Theorem 2.3, which extends [18, Proposition 6], enables us to provide more examples of matrix rings that are both n-generalized p.q.-Baer and n-generalized p.p.-ring. We begin with Theorem 2.2 below, which enables us to generate examples of n-generalized p.q.-Baer subrings of the matrix ring $M_n(R)$:

**Lemma 2.1** [18, Lemma 2]. Let $R$ be an abelian ring and define

$$S_n := \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} : a, a_{ij} \in R,$$

with $n$ a positive integer $\geq 2$. Then every idempotent in $S_n$ is of the form

$$\begin{pmatrix} f & 0 & \cdots & 0 \\ 0 & f & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f \end{pmatrix}$$

with $f^2 = f \in R$. 
We will use $S_n$. Throughout the remainder of the paper, to denote the ring indicated in Lemma 2.1.

**Theorem 2.2.** If $R$ is an abelian p.q. -Baer ring and $n \geq 2$ is a positive integer, then $S_n$ is an $n$-generalized right p.q.-Baer ring.

**Proof.** We proceed by induction on $n$. It is easy to show that $2S$ is a 2-generalized right p.q.-Baer ring. Let $nI$ be a principal right ideal of $nS$. Consider $\begin{bmatrix} nI & - & - \\ nI & - & - \\ \vdots & \vdots & \vdots \\ nI & - & - \end{bmatrix}$, and $\begin{bmatrix} nI & - & - \\ nI & - & - \\ \vdots & \vdots & \vdots \\ nI & - & - \end{bmatrix}$. It is clear that $nI$ and $nI$ are principal right ideals of $nS$. By induction hypothesis and Lemma 2.1, there are $\begin{bmatrix} \alpha_1 \alpha_2 \cdots \alpha_n \\ 0 \alpha_1 \alpha_2 \cdots \alpha_n \\ \vdots \vdots \vdots \\ 0 \alpha_1 \alpha_2 \cdots \alpha_n \end{bmatrix}$ for $i = 0, 1, 2, \ldots, n$ such that $\begin{bmatrix} nS & - & - \\ nS & - & - \\ \vdots & \vdots & \vdots \\ nS & - & - \end{bmatrix} = 0$, $\begin{bmatrix} nS & - & - \\ nS & - & - \\ \vdots & \vdots & \vdots \\ nS & - & - \end{bmatrix} = 0$. Let $J$ be the set of entries of the main diagonal of the elements of $\begin{bmatrix} nS & - & - \\ nS & - & - \\ \vdots & \vdots & \vdots \\ nS & - & - \end{bmatrix}$ or $\begin{bmatrix} nS & - & - \\ nS & - & - \\ \vdots & \vdots & \vdots \\ nS & - & - \end{bmatrix}$. It is clear that $J$ is a principal right ideal of $R$. Since $R$ is right p.q.-Baer, $rR = fR = fR$. Hence $f_1 = f_2$, since $R$ is an abelian ring. Now let

$$X = \begin{bmatrix} x & x_{12} & \cdots & x_{1n} \\ 0 & x & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x \end{bmatrix} \in S_n(\mathbb{R}^n)$$

and $Y = \begin{bmatrix} \alpha_1 \alpha_2 \cdots \alpha_n & y_{12} & \cdots & y_{1n} \\ 0 & \alpha_1 \alpha_2 \cdots \alpha_n & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_1 \alpha_2 \cdots \alpha_n \end{bmatrix} \in S_n(\mathbb{R}^n)$

Since $rS(l_{n-1}^{0}) = rS(l_{n-1}^{0}) = eS_{n-1}$, $x$ and $x_{ij}$’s are in $fR$ for each $i$ and $j$ except $x_{1n}$. So we have $a \alpha_2 \cdots a_n x_{1n} + y_{1n} x = 0$. Hence $y_{1n} x = 0$, since $f_1 \in B(R)$. Thus $x_{1n} \in fR$ and hence $rS(l_{n}^{0}) \subseteq eS_n$ for $e = \begin{bmatrix} f_1 & 0 & \cdots & 0 \\ 0 & f_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_1 \end{bmatrix} \in S_n$.

Since, for each $Y \in S_n e$, all entries of the main diagonal of $Y$ are zero and $e$ is central, $l_n e = (l_n e)^n = 0$. Thus $rS(l_{n}^{0}) = eS_n$. Therefore $S_n$ is $n$-generalized right p.q.-Baer.

The following result, which generalizes [18, Proposition 6], provides examples of
matrix rings that are both \( n \)-generalized p.q.-Baer and \( n \)-generalized p.p.-ring:

**Theorem 2.3.** If \( R \) is an abelian p.p.-ring, then \( S_n \) is an abelian \( n \)-generalized p.p.-ring.

**Proof.** We prove by induction on \( n \). First, we show that the trivial extension \( S_2 \) of \( R \) is 2-generalized right p.p. Let \( A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in S_2 \) and \( r_k(a) = eR \), with \( e^2 = e \in R \). It is clear that, \( eR \subseteq S_2(A^2) \) with \( f = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} \). Next, let \( A^2 \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} = 0 \). Since \( R \) is reduced, \( a^2 x = ax = 0 \) and \( a^2 y = ay = 0 \). Hence \( ex = x \) and \( ey = y \). Thus \( \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \). Therefore \( S_2 \) is 2-generalized right p.p. Now assume \( B = \begin{pmatrix} a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \in S_n \).

Consider \( B_1 = \begin{pmatrix} a & a_{12} & \cdots & a_{1n-1} \\ 0 & a & \cdots & a_{2n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \) and \( B_2 = \begin{pmatrix} a_{23} & \cdots & a_{2n} \\ 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \) in \( S_{n-1} \), then by the induction hypothesis, there exists \( e^2 = e_i \in S_{n-1} \), \( f_i = f_i \in R \), such that \( r_{n-1}(B_i^{n-1}) = e_i S_{n-1} \), \( e_i = \begin{pmatrix} f_i & 0 & \cdots & 0 \\ 0 & f_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_i \end{pmatrix} \) for \( i = 1, 2 \). By direct calculations, we have \( r_n(B^{2n-2}) = eS_n \) with \( e = \begin{pmatrix} f & 0 & \cdots & 0 \\ 0 & f & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f \end{pmatrix} \). Since \( r_1(a) = eR \), by [27, Lemma 3], \( r_n(B^n) = r_n(B^{2n-2}) = eS_n \).

**Corollary 2.4** [18, Proposition 6]. If \( R \) is a domain, then \( S_n \) is an abelian \( n \)-generalized p.p.-ring.

For a semicommutative ring, the definitions of \( n \)-generalized right p.q.-Baer and \( n \)-generalized right p.p. are coincide:

**Proposition 2.5.** Let \( R \) be a semicommutative ring. Then \( R \) is \( n \)-generalized right p.q.-Baer if and only if \( R \) is \( n \)-generalized right p.p.
Proof. Let $R$ be $n$-generalized right $p.q.$-Baer and $a \in R$. Then $r_R(aR)^n = eR$ for some idempotent $e \in R$. Let $x \in r_R(a^n)$. Since $R$ is semicommutative, $Rax \subseteq r_R(a^{n-1})$, which implies that $r_R(aR)^n = eR$. The converse is similar.

There exists an $n$-generalized right $p.q.$-Baer ring, which is generalized $p.p.$-ring but is not semicommutative.

**Example 2.6.** Let $R$ be an integral domain and $S_4$ be defined over $R$. Then $S_4$ is abelian $4$-generalized $p.p.$-ring and is $4$-generalized $p.q.$-Baer by Corollary 2.4. By considering $b = a = e_{32} + e_{14} + e_{34}$ and $c = e_{23}$ in $S_4$, where $e_{ij}$ denote the matrix units, we have $ab = 0$, and $acb \neq 0$, hence $aS_4b \neq 0$.

Now we conjecture that subrings of $n$-generalized right $p.q.$-Baer rings are also $n$-generalized right $p.q.$-Baer. But the answer is negative by the following.

**Example 2.7.** For a field $F$, take $F_n = F$ for $n = 1, 2, \ldots$, and let $S$ be the $2 \times 2$ matrix ring over the ring $\Pi_{n=1}^{\infty} F_n$. By [7, Proposition 2.1 and Theorem 2.2] we have that $S$ is a $p.q.$-Baer ring. Let $R = \left\{ \begin{array}{cc} \Pi_{n=1}^{\infty} F_n & \bigoplus_{n=1}^{\infty} F_n \\ \bigoplus_{n=1}^{\infty} F_n & < \bigoplus_{n=1}^{\infty} F_n, 1 > \end{array} \right\}$, which is a subring of $S$, where $< \bigoplus_{n=1}^{\infty} F_n, 1 >$ is the $F$-algebra generated by $\bigoplus_{n=1}^{\infty} F_n$ and $1$. Then by [7, Example 1.6], $R$ is semiprime $p.p$ which is neither right $p.q.$-Baer (and hence not $n$-generalized right $p.q.$-Baer), nor left $p.q.$-Baer (and hence not $n$-generalized left $p.q.$-Baer).

3. **Examples of $n$-generalized $p.q.$-Baer subrings**

Although the class of $n$-generalized $p.q.$-Baer rings, includes all $p.q.$-Baer rings (and hence, all biregular rings, all quasi-Baer rings, and all abelian $p.p.$ rings), however we show by examples that the class of $n$-generalized $p.q.$-Baer rings properly extends the aforementioned classes.

By the following example, there is an abelian $p.q.$-Baer (hence semiprime) ring $R$,
which is not reduced, but $S_n$ is an abelian $n$-generalized right p.q.-Baer ring that is not semiprime.

**Example 3.1.** By Zalesskii and Neroslavskii [10, Example 14.17, p.179], there is a simple noetherian ring $R$ that is not a domain and in which $0$ and $1$ are the only idempotents. Thus $R$ is an abelian p.q.-Baer ring that is neither left nor right p.p, and hence is not reduced. By [7, Proposition 1.17] $R$ is semiprime and by Theorem 2.1, $S_n$ is abelian $n$-generalized p.q.-Baer, that is not semiprime and hence is not right p.q.-Baer.

**Example 3.2.** If $R$ is an abelian p.q.-Baer ring, then $R[x]/<x^3>$ is an $n$-generalized p.q.-Baer ring.

**Proof.** First we note that $\Theta : T \rightarrow R[x]/<x^3>$ defined by

$$(a_0, a_1, a_2) \rightarrow (a_0 + a_1x + a_2x^2) + <x^3>$$

is an isomorphism, where $T = \{(a, b, c) | a, b, c \in R\}$ is a ring with addition componentwise and the multiplication defined by

$$(a_1, b_1, c_1)(b_2, b_2, c_2) = (a_1b_2 + a_2b_2 + a_1c_2 + b_1b_2 + a_1c_2).$$

Let $J$ be an ideal of $T$. Suppose $I = \{a \in R | (a, b, c) \in J\}$, it is clear that $I$ is an ideal of $R$. Since $R$ is p.q.-Baer, $r_R(I) = eR$ for an idempotent $e \in R$. We can show that $r(J^3) = (e, 0, 0)T$, and hence, the result follows.

There exists a commutative $n$-generalized p.q.-Baer (hence $n$-generalized p.p.-) ring $R$, over which $S_n$ is not an $n$-generalized p.p.-ring.

**Example 3.3.** Let $p \neq 3$ be a prime integer and $Z_{p^3}$ be the ring of integers modulo $p^3$, and $S_3$ be defined over $Z_{p^3}$. Let $A = pl_3 + e_{13}$, where $l_3$ is the identity matrix and $e_{ij}$ denote the matrix units. It is clear that $pl_3 + e_{13} + e_{23} \in r_{S_3}(A^3)$ and idempotents of $S_3$ are $l_3$ and $0$. Hence $r_{S_3}(A^3) = l_3S_3$ and that $S_3$ is not 3-generalized p.p.-ring, but $Z_{p^3}$ is a 3-generalized p.p.-ring.

**Example 3.4.** For every abelian quasi-Baer (resp. p.p.-) ring $R$, by Theorems 2.1 and
2.2, the ring $S_n$ is $n$-generalized right p.q.-Baer, which is not right p.q.-Baer. Therefore we are able to provide examples of $n$-generalized right p.q.-Baer rings that is not right p.q.-Baer:

Let $F$ be a field, and $R = F[x]$ be the polynomial ring where $x$ is an indeterminate. Then $S_n$ is a $n$-generalized right p.q.-Baer ring that is not right p.q.-Baer.

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Reference