A certain $N$-Generalized Principally Quasi-Baer Subring of the Matrix rings

H. Haj Seyyed Javadi  Amirkabir University
A. Moussavi, E. Hashemi: University of Tarbiat Modarres

Abstract

For a fixed positive integer $n$, we say a ring with identity is $n$-generalized right principally quasi-Baer, if for any principal right ideal $I$ of $R$, the right annihilator of $I^n$ is generated by an idempotent. This class of rings includes the right principally quasi-Baer rings and hence all prime rings. A certain $n$-generalized principally quasi-Baer subring of the matrix ring $M_n(R)$ are studied, and connections to related classes of rings (e.g., $p.q.$-Baer rings and $n$-generalized p.p. rings) are considered.

1. Introduction and Preliminaries

Throughout all rings are assumed to be associative with identity. From [12, 21], a ring $R$ is (quasi-)Baer if the right annihilator of any (right ideal) nonempty subset of $R$ is generated, as a right ideal, by an idempotent. Moreover, in [12] Clark proved the left-right symmetry of this condition. He uses this condition to characterize when a finite dimensional algebra with unity over an algebraically closed filed is isomorphic to a twisted matrix units semigroup algebra. The class of quasi-Baer rings is a nontrivial generalization of the class of Baer rings. Every prime ring is quasi-Baer, hence prime rings with nonzero right singular ideal are quasi-Baer; but not Baer [24]. For a positive integer $n > 1$, the $n \times n$ matrix ring over a non-Prüfer commutative domain is a prime quasi-Baer ring which is not a Baer ring by [27] and [21, p.17]. The $n \times n$ ($n > 1$) upper triangular matrix ring over a domain which is not a division ring is quasi-Baer but not

1. 2000 Mathematical Subject Classification. 16D15; 16D40; 16D70.

Keywords and phrases. n-Generalized p.q.-Baer ring; p.p.-ring; Annihilators; triangular matrix ring

89
Baer by an example due to Cohn; see [1], [20] and [5]. The theory of Baer and quasi-Baer rings has come to play an important role and major contributions have been made in recent years by a number of authors, including Birkenmeier, Chatters, Khuri, Kim, Hirano and Park (see, for example [1], [4-7], [16], [21], [26] and [28]).

A ring satisfying a generalization of Rickart’s condition [30] (i.e., right annihilator of any element is generated (as a right ideal) by an idempotent) has a homological characterization as a right p.p.-ring. A ring \( R \) is called a right (resp. left) p.p.-ring if every principal right (resp. left) ideal is projective. \( R \) is called a p.p.-ring (also called a Rickart ring [2, p.18]), if it is both right and left p.p.-ring. In [9] Chase shows the concept of p.p.-ring is not left-right symmetric. Small [30] shows that a right p.p.-ring \( R \) is Baer (so p.p), when \( R \) is orthogonally finite. Also it is shown by Endo [13] that a right p.p.-ring \( R \) is p.p when \( R \) is abelian (i.e., every idempotent is central). Finally Chatters and Xue [11] prove that in a duo (i.e., every one sided ideal is two sided) p.p.-ring \( R \), if \( I \) is a finitely generated right projective ideal of \( R \), then \( I \) is left projective and a direct summand of an invertible ideal. Following Birkenmeier et al. [7], \( R \) is called right principally quasi-Baer (or simply right p.q.-Baer), if the right annihilator of a principal right ideal is generated by an idempotent. Equivalently, \( R \) is right p.q.-Baer if \( R \) modulo the right annihilator of any principal right ideal is projective. Similarly, left p.q.-Baer rings can be defined. If \( R \) is both right and left p.q.-Baer, then it is called p.q.-Baer. The class of p.q.-Baer rings includes all biregular rings, all quasi-Baer rings, and all abelian p.p.-rings. A ring \( R \) is said to be \( p \)-regular, if for every \( x \in R \) there exists a natural number \( n \), depending on \( x \), such that \( x^n \in x^0 Rx^0 \). A ring \( R \) is called a generalized right p.p.-ring if for any \( x \in R \) the right ideal \( x^0 R \) is projective for some positive integer \( n \), depending on \( x \), or equivalently, if for any \( x \in R \) the right annihilator of \( x^0 \) is generated by an idempotent for some positive integer \( n \), depending on \( x \). A ring is called generalized p.p.-ring, if it is both generalized right and left p.p.-ring.

Note that Von Neumann regular rings are right (left) p.p.-rings by Goodearl [14,
Theorem 1.1], and a same argument as [14, Theorem 1.1] shows that \( p \)-regular rings are generalized p.p.-rings. Right p.p.-rings are generalized right p.p obviously. See [18] for more details.

**Definition 1.1.** Given a fixed positive integer \( n \), we say a ring \( R \) is \( n \)-generalized right principally quasi Baer (or \( n \)-generalized right p.q.-Baer), if for all principal right ideal \( I \) of \( R \), the right annihilator of \( I^n \) is generated by an idempotent. Left cases may be defined analogously.

The class of \( n \)-generalized right p.q.-Baer rings includes all right p.q.-Baer rings, (and hence all biregular rings, quasi-Baer rings, abelian p.p.-rings and semicommutative (i.e., if \( r(x) \) is an ideal for all \( x \in R \) ) generalized p.p rings). Theorem 2.1 in section 2, allows us to construct examples of \( n \)-generalized p.q.-Baer rings that are not p.q.-Baer. Some conditions on the equivalence of \( n \)-generalized p.q.-Baer and \( n \)-generalized p.p.-rings are discussed. However, we show by examples that the class of \( n \)-generalized p.q.-Baer rings properly extends the aforementioned classes.

In this paper, we discuss some type of matrix rings formed over p.q.-Baer or p.p. rings. We study \( n \)-generalized p.q.-Baer subrings of the matrix ring \( M_n(R) \). Theorem 2.2, enables us to generate examples of \( n \)-generalized p.q.-Baer subrings of the matrix ring \( M_n(R) \). Theorem 2.2, which extends [18, Proposition 6], enables us to provide more examples of matrix rings, that are both \( n \)-generalized p.q.-Baer and \( n \)-generalized p.p.-ring. Connections to related classes of rings are investigated. Although the class of \( n \)-generalized p.q.-Baer rings, includes all p.q.-Baer rings (and hence, all biregular rings, and all abelian p.p. rings), however we show by examples that the class of \( n \)-generalized p.q.-Baer rings properly extends the aforementioned classes.

Note that, for a reduced ring (which has no nonzero nilpotent elements), we have \( I_R(Rx) = I_R((Rx)^n) = I_R(x^n) = I_R(x) = r_R(x) = r_R((xR)^n) = r_R(xR) \), for every \( x \in R \) and every positive integer \( n \). Therefore reduced rings are semicommutative and semicommutative rings are abelian. Also for reduced rings the definitions of right p.q.-
Baer, n-generalized right p.q.-Baer, generalized p.p. and p.p.-ring are coincide. This leads one ask whether commutative reduced rings are n-generalized p.q-Baer. However, the answer is negative by the following.

**Example 1.2.** Let \( p \) be a prime number and \( R = (a,b) \in \mathbb{Z} \oplus \mathbb{Z} \mid a \equiv b \pmod{p} \), then \( R \) is a commutative reduced ring. Note that the only idempotents of \( R \) are \((0,0)\) and \((1,1)\). One can show that \( r_R((p,0)R) = (0,p)R \), so \( r_R((p,0)R) \) does not contain a nonzero idempotent of \( R \); and hence \( R \) is not n-generalized right quasi-Baer, for any positive integer \( n \).

**Lemma 1.3.** Let \( R \) be an abelian \( n \)-generalized right p.q.-Baer ring, then \( r_R(l^n) = r_R(l^m) \) for every principal right ideal \( I \) of \( R \) and each positive integer \( m \) with \( n \leq m \).

**Proof.** It is enough to show that \( r_R(l^n) = r_R(l^{n+1}) \). Let \( x \in r_R(l^{n+1}) \), then \( l^nx \leq r_R(l^n) = fR \) for some idempotent \( f \in R \). Hence \( l^nx = l^nxf = 0 \). Thus \( x \in r_R(l^n) \).

## 2. N -generalized right principally quasi Baer subrings of the matrix rings

In this section we discuss some type of matrix rings formed over p.q.-Baer or p.p. rings. Theorem 2.3, which extends [18, Proposition 6], enables us to provide more examples of matrix rings that are both \( n \)-generalized p.q.-Baer and \( n \)-generalized p.p.-ring. We begin with Theorem 2.2 below, which enables us to generate examples of \( n \)-generalized p.q.-Baer subrings of the matrix ring \( M_n(R) \):

**Lemma 2.1**[18, Lemma 2]. Let \( R \) be an abelian ring and define

\[
S_n := \begin{bmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{bmatrix} : a, a_{ij} \in R,
\]

with \( n \) a positive integer \( \geq 2 \). Then every idempotent in \( S_n \) is of the form

\[
\begin{bmatrix} f & 0 & \cdots & 0 \\ 0 & f & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f \end{bmatrix} \text{ with } f^2 = f \in R.
\]
We will use $S_n$. Throughout the remainder of the paper, to denote the ring indicated in Lemma 2.1.

**Theorem 2.2.** If $R$ is an abelian p.q.-Baer ring and $n$ ($\geq 2$) is a positive integer, then $S_n$ is an $n$-generalized right p.q.-Baer ring.

**Proof.** We proceed by induction on $n$. It is easy to show that $S_2$ is a 2-generalized right p.q.-Baer ring. Let $l_n$ be a principal right ideal of $S_n$. Consider

$$\begin{aligned}
\{ & nS_xl_xS \subseteq l_n \mid \textbf{there are } e \in S_n : e^2 = e, f^2 = f \text{ for } i, j = 1, 2 \text{ such that } r_{S_n} (l_{n-1}^{n-1}) = eS_{n-1}, e = 0 \begin{pmatrix}
0 & f & \cdots & 0 \\
& \ddots & \ddots & \\
& & \ddots & \ddots \\
0 & \cdots & \cdots & 0
\end{pmatrix}
\}
\end{aligned}$$

Let $J$ be the set of entries of the main diagonal of the elements of $\{ l_{n-1}^{n-1} \text{ or } l_n \}$. It is clear that $J$ is a principal right ideal of $R$. Since $R$ is right p.q.-Baer, $r_{S_n} (J) = f_1R = f_2R$. Hence $f_1 = f_2$, since $R$ is an abelian ring. Now let

$$X = \begin{pmatrix}
x & x_{12} & \cdots & x_{1n} \\
0 & x & \cdots & x_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & x
\end{pmatrix} \in r_{S_n} (l_n^n) \text{ and } Y = \begin{pmatrix}
a_1a_2a_3 \cdots a_n & y_{12} & \cdots & y_{1n} \\
0 & a_1a_2a_3 \cdots a_n & \cdots & y_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_1a_2a_3 \cdots a_n
\end{pmatrix} \in l_n^n$$

Since $r_{S_n} (l_{n-1}^{n-1}) = r_{S_n} (l_{n-2}^{n-2}) = eS_{n-1}$, $x$ and $x_{ij}$'s are in $f_1R$ for each $i$ and $j$ except $x_{in}$. So we have $a_1a_2 \cdots a_n x_{in} + y_{in}x = 0$. Hence $y_{in}x = 0$, since $f_1 \in B(R)$. Thus $x_{in} \in f_1R$ and hence $r_{S_n} (l_n^n) \subseteq eS_n$ for

$$e = \begin{pmatrix}
f_1 & 0 & \cdots & 0 \\
0 & f_1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & f_1
\end{pmatrix} \in S_n.$$ 

Since, for each $Y \in l_n e$, all entries of the main diagonal of $Y$ are zero and $e$ is central, $l_n^ne = (l_ne)^n = 0$. Thus $r_{S_n} (l_n^n) = eS_n$. Therefore $S_n$ is $n$-generalized right p.q.-Baer.

The following result, which generalizes [18, Proposition 6], provides examples of
matrix rings that are both n-generalized p.q.-Baer and n-generalized p.p.-ring:

**Theorem 2.3.** If $R$ is an abelian p.p.-ring, then $S_n$ is an abelian $n$-generalized p.p.-ring.

**Proof.** We prove by induction on $n$. First, we show that the trivial extension $S_2$ of $R$ is 2-generalized right p.p. Let $A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in S_2$ and $r_n(a) = eR$, with $e = e^2 \in R$. It is clear that, $A^2 = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} = 0$. Since $R$ is reduced, $a^2x = ax = 0$ and $a^2y = ay = 0$. Hence $ex = x$ and $ey = y$. Thus $A^2 = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} = f \begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$.

Therefore $S_2$ is 2-generalized right p.p. Now assume $B = \begin{pmatrix} a_{12} & \ldots & a_{1n-1} \\ \vdots & \ddots & \vdots \\ 0 & \ldots & a \\ 0 & \ldots & a \end{pmatrix} \in S_n$.

Consider $B_1 = \begin{pmatrix} a & a_{12} & \ldots & a_{1n-1} \\ 0 & a & \ldots & a_{2n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a \end{pmatrix}$ and $B_2 = \begin{pmatrix} a & a_{23} & \ldots & a_{2n} \\ 0 & a & \ldots & a_{3n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a \end{pmatrix}$ in $S_{n-1}$, then by the induction hypothesis, there exists $e^2_i = e_i \in S_{n-1}$, $f_i = f_i \in R$, such that $r_{n-1}(B_i^{n-1}) = e_iS_{n-1}$, $r_{n-1}(B_i^{n-2}) = eS_n$ with $r_n(A^2) = eR$, by [27, Lemma 3], $r_{n-1}(B^{n-1}) = eS_n$.

**Corollary 2.4** [18, Proposition 6]. If $R$ is a domain, then $S_n$ is an abelian $n$-generalized p.p.-ring.

For a semicommutative ring, the definitions of $n$-generalized right p.q.-Baer and $n$-generalized right p.p. are coincide:

**Proposition 2.5.** Let $R$ be a semicommutative ring. Then $R$ is $n$-generalized right p.q.-Baer if and only if $R$ is $n$-generalized right p.p.
Proof. Let \( R \) be \( n \)-generalized right p.q.-Baer and \( a \in R \). Then \( r_R(aR)^n = eR \) for some idempotent \( e \in R \). Let \( x \in r_R(a^n) \). Since \( R \) is semicommutative, \( raRx \subseteq r_R(a^{n-1}) \), which implies that \( r_R(aR)^n = eR \). The converse is similar.

There exists an \( n \)-generalized right p.q.-Baer ring, which is generalized p.p.-ring but is not semicommutative.

Example 2.6. Let \( R \) be an integral domain and \( S_4 \) be defined over \( R \). Then \( S_4 \) is abelian \( 4 \)-generalized p.p.-ring and is \( 4 \)-generalized p.q.-Baer by Corollary 2.4. By considering \( b = a = e_{32} + e_{14} + e_{34} \) and \( c = e_{23} \) in \( S_4 \), where \( e_{ij} \) denote the matrix units, we have \( ab = 0 \), and \( acb \neq 0 \), hence \( aS_4b \neq 0 \).

Now we conjecture that subrings of \( n \)-generalized right p.q.-Baer rings are also \( n \)-generalized right p.q.-Baer. But the answer is negative by the following.

Example 2.7. For a field \( F \), take \( F_n = F \) for \( n = 1, 2, \ldots \), and let \( S \) be the \( 2 \times 2 \) matrix ring over the ring \( \prod_{n=1}^{\infty} F_n \). By [7, Proposition 2.1 and Theorem 2.2] we have that \( S \) is a p.q.-Baer ring. Let

\[
R = \left( \prod_{n=1}^{\infty} F_n \oplus \bigoplus_{n=1}^{\infty} F_n < \bigoplus_{n=1}^{\infty} F_n, 1 > \right),
\]

which is a subring of \( S \), where \( < \bigoplus_{n=1}^{\infty} F_n, 1 > \) is the \( F \)-algebra generated by \( \bigoplus_{n=1}^{\infty} F_n \) and 1. Then by [7, Example 1.6], \( R \) is semiprime p.p which is neither right p.q.-Baer (and hence not \( n \)-generalized right p.q.-Baer), nor left p.q.-Baer (and hence not \( n \)-generalized left p.q.-Baer).

3. Examples of \( n \)-generalized p.q.-Baer subrings

Although the class of \( n \)-generalized p.q.-Baer rings, includes all p.q.-Baer rings (and hence, all biregular rings, all quasi-Baer rings, and all abelian p.p. rings), however we show by examples that the class of \( n \)-generalized p.q.-Baer rings properly extends the aforementioned classes.

By the following example, there is an abelian p.q.-Baer (hence semiprime) ring \( R \),
which is not reduced, but $S_n$ is an abelian $n$-generalized right p.q.-Baer ring that is not semiprime.

**Example 3.1.** By Zalesskii and Neroslavskii [10, Example 14.17, p.179], there is a simple noetherian ring $R$ that is not a domain and in which 0 and 1 are the only idempotents. Thus $R$ is an abelian p.q.-Baer ring that is neither left nor right p.p, and hence is not reduced. By [7, Proposition 1.17] $R$ is semiprime and by Theorem 2.1, $S_n$ is abelian $n$-generalized p.q.-Baer, that is not semiprime and hence is not right p.q.-Baer.

**Example 3.2.** If $R$ is an abelian p.q.-Baer ring, then $R[x]/<x^3>$ is an $n$-generalized p.q.-Baer ring.

**Proof.** First we note that $\Theta:T \rightarrow R[x]/<x^3>$ defined by

$$(a_0,a_1,a_2) \rightarrow (a_0 + a_1x + a_2x^2)+<x^3>$$

is an isomorphism, where $T = \{(a,b,c) \mid a,b,c \in R\}$ is a ring with addition componentwise and the multiplication defined by

$$(a_1,b_1,c_1)(a_2,b_2,c_2) = (a_1a_2,a_1b_2 + b_2a_1,a_1c_2 + b_2c_1 + c_1a_2).$$

Let $J$ be an ideal of $T$. Suppose $I = \{a \in R \mid (a,b,c) \in J\}$, it is clear that $I$ is an ideal of $R$. Since $R$ is p.q.-Baer, $r_R(I) = eR$ for an idempotent $e \in R$. We can show that $r(I^3) = (e,0,0)T$, and hence, the result follows.

There exists a commutative $n$-generalized p.q.-Baer (hence $n$-generalized p.p.) ring $R$, over which $S_n$ is not an $n$-generalized p.p.-ring.

**Example 3.3.** Let $p = 3$ be a prime integer and $Z_{p^3}$ be the ring of integers modulo $p^3$, and $S_3$ be defined over $Z_{p^3}$. Let $A = pI_3 + e_{13}$, where $I_3$ is the identity matrix and $e_i$ denote the matrix units. It is clear that $pI_3 + e_{31} + e_{12} \in r_{S_3}(A^3)$ and idempotents of $S_3$ are $I_3$ and 0. Hence $r_{S_3}(A^3) = I_3S_3$ and that $S_3$ is not 3-generalized p.p.-ring, but $Z_{p^3}$ is a 3-generalized p.p.-ring.

**Example 3.4.** For every abelian quasi-Baer (resp. p.p.-) ring $R$, by Theorems 2.1 and
2.2, the ring $S_n$ is $n$-generalized right p.q.-Baer, which is not right p.q.-Baer. Therefore we are able to provide examples of $n$-generalized right p.q.-Baer rings that is not right p.q.-Baer:

Let $F$ be a field, and $R = F[x]$ be the polynomial ring where $x$ is an indeterminate. Then $S_n$ is a $n$-generalized right p.q.-Baer ring that is not right p.q.-Baer.

**Acknowledgement**

The authors are deeply indebted to the referee for many helpful comments and suggestions for the improvement of this paper.

**Reference**