Hypergroup Structures with Regular Multiplications

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Abstract

In Banach algebras, the group algebra $L(G)$ is Arens regular if and only if $G$ is finite.

In this paper, the researcher has obtained a hypergroup structure (in the sense of Dunkl) whose measure algebra has regular multiplication. The most interesting result was that if $L(X)$ is Arens regular then the convolution is Arens regular as a bilinear map. The condition obtained gives regularity of multiplication in the Hypergroup, which $X$ is not finite.

Introduction

The regularity of a bounded bilinear mapping was defined by Arens (see [1]). For some important Banach algebras, the first and the second Arens product, on their second dual, are different. Therefore, these algebras are not Arens regular. A number of Banach algebras, commonly occurring in functional and harmonic analysis, are not Arens regular. The group algebra $L(G)$ of a locally compact Hausdorff group is Arens regular if and only if $G$ is finite (see [4], [13], [14]). In [9], the researcher has shown that the hypergroup algebra $L(X)$, where $X$ is a locally compact Hausdorff space, can be Arens regular without $X$ being finite. Also, in [10], for a general measure algebra $£$, in $M(X)$, if $e \in X$ is not isolated in $\text{supp} \ £$, and that $\delta_e$ acts as an identity for $£$, then $£$, is not Arens regular. These are not the only ways to construct the regular or irregular multiplications. In [8], for the circle group $T$, two multiplications have been constructed on $M(T)$, one of which is regular and the other is irregular. In the present note, we obtain a hypergroup structure whose measure algebra has regular multiplication. Some related results can be

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found in [8],[9],[10]. Also, basic facts about measure algebras on hypergroups can be found in [6], [7], [11].

The researcher will begin with the Arens multiplications and the hypergroup structures.

1- Other Versions of Arens Multiplications

In [1], Arens showed how the multiplication of a Banach algebra could be extended to a multiplication on the second dual. His method was essentially algebraic, and this is indeed the easiest way to prove that the construction works. However, we shall describe the results. See [12] for details.

Let $A$ be a set with a multiplication $(x, y) \mapsto x \cdot y$. Let $B$ be a set with $A \subseteq B$, under its topology, $A$ is dense in $B$. For $x, y$ in $B$, take $(\tau_\alpha), (s_\beta)$ in $A$ such that

$$\lim_{\alpha} \tau_\alpha = x, \lim_{\beta} s_\beta = y.$$ Then, the first extension of multiplication is given by

$$\tau_\alpha \cdot y = \lim_{\beta} \tau_\alpha \cdot s_\beta, \quad x, y = \lim_{\alpha, \beta} \tau_\alpha \cdot s_\beta,$$

while the second extension is given by

$$x \circ s_\beta = \lim_{\alpha} \tau_\alpha \cdot s_\beta, \quad x \circ y = \lim_{\alpha} \tau_\alpha \cdot s_\beta.$$

The set $A$ is Arens regular (or, briefly, regular) if $x, y = x \circ y$. By the above definition for $a$ and $b$ in $B$, the product $a \cdot b$ (resp. $a \circ b$) is continuous in the $a$ (resp. $b$) variable for each fixed $b$ (resp. $a$) in $B$. Generally, $a \cdot b$ (resp. $a \circ b$) is not continuous in $b$ (resp. $a$) when $a \notin A$ (resp. $b \notin A$). This suggests the first result about regularity, which is entirely elementary.

**Proposition 1.1.** Let $A \subseteq B$ and let $A$ be dense in $B$. Then.

(i) if $A$ is commutative then $A$ is Arens regular if and only if $B$ is commutative.

(ii) $A$ is Arens regular if and only if the multiplication in $B$ is continuous in each variable (without any restriction on the other).

(iii) Let $B$ be compact. Then, $A$ is Arens regular if and only if the multiplication in $B$ separately sequentially continuous.

**Proof.** See [5] and [12].
Regularity of Banach algebras 1.2. Let $A$ be a normed space over $K$ ($K=\mathbb{R}$ or $K=\mathbb{C}$). The dual space $A^*$ is the vector space $\Lambda(A^*, K)$ equipped with the norm \[ \|f\| = \sup\{|f(x)| : x \in A, \|X\| \leq 1\}. \]

Thus, $A^*$ is a Banach space. Let $(A^*)^*$ be the dual space of $A^*$, $(A^*)^* = \Lambda(A^*, K)$. Since $A^*$ is itself a Banach space, it is susceptible to the same construct; i.e. one can form $(A^*)^* = A^{**}$; this is also a Banach space, called the dual or bidual of $A$, and denoted by $A^{**}$. This can go on.

For each $x \in A$, the value of an element $x^{**} \in A^{**}$ is defined by $x^{**}(f) = f(x)$ for all $f \in A^*$. So $x^{**}$ is linear on $A^*$, and \[ \|x^{**}\| = \|x\|. \] Thus, the canonical embedding mapping $x \rightarrow x^{**}$ preserves norms and an isometric from $A$ into its second dual $A^{**}$. Therefore, we can regard $A$ as a subspace of $A^{**}$.

Let $\sigma(A^{**}, A^*)$ be the weak* topology on $A^{**}$. By [3], $A$ is weak*-dense in $A^{**}$ So, for $F \in A^{**}$, $G \in A^{**}$, we can find two bounded nets $(\mu_\alpha), (\nu_\beta)$ in $A$ with

$$F = \omega^* - \lim_\alpha \mu_\alpha, \quad G = \omega^* - \lim_\beta \nu_\beta.$$ 

The topological extension of first and second Arens product are given by

$$FG = \omega^* - \lim_\alpha \mu_\alpha \nu_\beta, \quad F \circ G = \omega^* - \lim_\beta \nu_\beta \mu_\alpha.$$ 

Thus, the order in which the limits are taken distinguishes between the extensions. Moreover, the first Arens product is characterized by the two properties:

(i) for each $G \in A^{**}$, the map $F \rightarrow FG$ is weak*-continuous on $A^{**}$.

(ii) For each $\mu \in A$, the map $G \rightarrow \mu G$ is weak*-continuous on $A^{**}$.

The second Arens product is defined similarly. Therefore, the second dual $A^{**}$ of $A$ can be given the Banach algebra structure by means of the first (or second) Arens product.

Now, we want to describe Arens products as an algebraic extension. Indeed, for $F, G \in A^{**}$, $f \in A^*$, and $\mu, \nu \in A$, one can find $FG, FoG$ successively as follows:

$$<FG, f> = <G, Ff>, <Gf, \mu> = <G, f\mu>, <f, \mu\nu> = <f, \mu\nu>,$$

$$<F \circ G, f> = <G, f \circ F>, <f \circ F, \mu> = <F, \mu \circ f>, <\mu \circ f, \nu> = <f, \nu \mu>.$$ 

So, a Banach algebra is said to have regular multiplication if $FG = F \circ G$.

$A^{**}$ is not compact in the weak* topology. But the closed until ball of $A^{**}$ is weak-compact [3]. So, by definition or [5], we have:

**Proposition 1.3.** Let $A$ be commutative. $A$ is Arens regular if and only if the first or second Arens product in $A^{**}$ is weak*-continuous in each variable.
Proof. For $F, G \in A^{**}$, there are two nets $(\mu_\alpha)$ and $(\nu_\beta)$ in $A$ which weak*-converge to $F$ and $G$. So, $A$ is Arens regular if and only if $FG = F \circ G$. It is equivalent to this fact; for all $f \in A^*$,
\[
\lim_{\alpha} \lim_{\beta} f(\mu_\alpha \nu_\beta) = \lim_{\beta} \lim_{\alpha} f(\mu_\alpha \nu_\beta).
\]

2- Hypergroup Structures

Let $X$ be a locally compact Hausdorff space and $M(X)$ denotes the set of all bounded, regular, complex Borel measures on $X$. For each $\mu$ and $\nu$ in $M(X)$, $\mu^* \nu$ denotes the convolution of $\mu$ and $\nu$. Let $\delta_r$ be the unit mass at $r$. The product formulas of the type.
\[
\mu^* \nu(f) = \int_X \int_X (\delta_x^* \delta_y)(f) d\mu(x) d\nu(y)
\]
On $M(X)$ becomes a Banach algebra. Dunkl (1972) and Jewett (1975) have shown how one defines a product on $M(X)$ which makes it a Banach algebra. In some cases, an investigation begins with a convolution algebra of measures as the primitive object, upon which to build a theory; this is the case of the analysis of the objects called hypergroups which are generalizations of the convolution algebra of Borel measures on a group. One of the objects of this paper will be the introduction of a large class of new convolution structures, many of which are new hypergroups.

Let $C_b(X), C_0(X)$ and $C_c(X)$ denote the spaces of continuous functions on $X$ which are bounded, those which vanish at infinity and those having compact support respectively. By $M(X)$ and $\mathcal{M}(X)$, we abbreviate the space of Radon measures and probability measures on $X$.

Definition 2.1. A hypergroup $(X, \ast)$ is a Banach algebra of the Borel measures $M(X)$ on a locally compact Hausdorff space $X$ with product $\ast$ called convolution it satisfies the following axioms:

(i) There is a map $\lambda : X \times X \to M_p(X)$ with for every $x, y \in X$, the measures $\lambda_{(x,y)}$ have compact supports and $\lambda_{(x,y)} = \lambda_{(y,x)}$. 

\[
\lambda_{(x,y)}(f) = \int_X \int_X (\delta_x^* \delta_y)(f) d\mu(x) d\nu(y)
\]
Hypergroup Structures with Regular Multiplications  

J. Laali

(ii) for each \( f \in C_c(X) \), the map \( (x, y) \to \lambda_{(x,y)}(\int) \) is in \( C_b(X \times X) \) and \( r \to \lambda_{(x,y)}(\int) \) is in \( C_c(X) \), for every \( y \in x \);

(iii) the convolution \( (\mu, \nu) \to \mu * \nu \) of measures defined by

\[
\mu * \nu(f) = \int_X \int_X \lambda_{(x,y)}(f) d\mu(x) d\nu(y), \quad (\mu, \nu \in M(X), \int \in C_\sigma(X))
\]

is associative (and clearly \( \lambda_{(x,y)} = \delta_x * \delta_y \))

(iv) there is a unique \( e \in X \) such that \( \lambda_{(x,y)} = \delta_x \) for all \( x \in X \).

In [7], Ghahramani and Medghalchi have constructed and studied the subalgebra of \( M(X) \) which is determined in the following way:

\[
L(X) = \{ \mu \in M(X) : \lambda_{(x,y)}(\int) \text{ are norm-continuous} \}.
\]

This algebra generalizes the algebra \( L(G) \) of \( M(G) \) for the locally compact topological groups \( G \). They have shown that \( L(X) \) is a Banach subalgebra of \( M(X) \) and it has a bounded approximate identity of norm 1. Therefore, \( L(X) \) can be regarded as a subspace of \( L(X)^{**} \) and then \( L(X) \) is weak* dual of \( L(X)^* \) in [11], Medghalchi studied the second dual of \( L(X) \).

Let \( A, B, C, \) be disjoint. And \( e, z \) be single points not in \( A \cup B \cup C \). Write \( X = \{ e \} \cup A \cup B \cup C \cup \{ z \} \). Let \( X \) be a compact Hausdorff space with \( e \) and \( z \) as isolated points. Each \( \mu \in M(X) \) can be written in the unique form

\[
\mu_x \delta_e + \mu_y \delta_f + \mu_z \delta_e
\]

where, \( \mu_A, \mu_B, \mu_C \) are the restriction of \( \mu \) to \( A, B, C \), respectively, and \( \mu_e \) and \( \mu_z \) are scalars.

The following Theorem gives the structure of a hypergroup on the locally compact space \( X \).

**Theorem 2.2.** Let \( \lambda : A \times B \rightarrow (M_y(C), \text{weak}^*) \) be continuous and the map \( x \to \lambda_{(x,y)}(f) \) is in \( C_c(C) \) for every \( x, y \in A \cup B, f \in C(C) \) and \( \lambda(a, b) = \lambda(b, a) \) (\( a \in A, b \in B \)). There is a hypergroup structure on \( X \) such that \( M(X-\{e\}) = L(X) \) if and only if \( M(A \cup B) \subseteq L(X) \).

**Proof.** We define a multiplication on \( M(X) \) by the following way:

1. take a map \( \tilde{\lambda} : X \times X \rightarrow M_y(X) \) with two properties.

   \( \tilde{\lambda}_{(x,x)} = \tilde{\lambda}_{(e,e)} = \delta_x (x \in X) \);
(ii) \( \tilde{\lambda}_{(x,y)} = \begin{cases} \lambda_{(x,y)} & (x \in A, y \in B) \\ \delta_z & \text{otherwise} \end{cases} \)

It is clear that \( \|\tilde{\lambda}_{(x,y)}\| = 1 \) for all \( x, y \in X \).

Then, we define a multiplication \( \mu \nu \), for \( \mu, \nu \in M(X) \), by

\[
\mu \nu = \int_X \int_X \tilde{\lambda}_{(x,y)} \, d\mu(x) \, d\nu(y)
\]

It is easy to show that the multiplications is commutative and is the identity. Now, we prove that it is associative.

If one of the \( x, y, z, (\text{in} X) \) is equal to \( e \), then

\[
(\delta_x \ast \delta_y) \ast \delta_z = \delta_x \ast (\delta_y \ast \delta_z)
\]

Now, suppose that \( \mu, \nu, \xi \) are in \( M(X - \{e\}) \). Therefore,

\[
\mu = \mu_A + \mu_B + \mu_C + \mu_z \delta_z,
\nu = \nu_A + \nu_B + \nu_C + \nu_z \delta_z,
\xi = \xi_A + \xi_B + \xi_C + \xi_z \delta_z.
\]

Suppose \( Y = A \cup C \cup \{z\} \). We have

\[
\mu_A \nu_Y = \int_X \int_X \tilde{\lambda}(x,y) \, d\mu_A(x) \, d\nu_Y(y)
= \int_X \int_X \delta_z \, d\mu_A(x) \, d\nu_Y(y)
= \mu_A(1) \nu_Y(1) \delta_z.
\]

Similarly, \( \mu_Y \nu_A = \mu_T(1) \nu_A(1) \delta_z \) and if \( T = B \cup C \cup \{z\} \) then

\[
\mu_Y \nu_B = \mu_Y(1) \nu_B(1) \delta_z, \mu_B \nu_T = \mu_B(1) \nu_T(1) \delta_z.
\]

Let \( **\) be the convolution arising from the multiplication in \( A \) and \( B \) (or in \( B \) and \( A \)) i.e.,

\[
\mu_A \ast \nu_B = \mu_A \nu_B = \int_X \int_X \tilde{\lambda}(x,y) \, d\mu_A(y) \, d\nu_B(x),
\mu_A \ast \nu_B(f) = \int_A \int_C f(t) \, d\tilde{\lambda}(x,y) \, \nu_B(x) \, d\mu_A(y).
\]

Hence, \( \text{supp}(\mu_A \ast \nu_B) \subseteq C \). Therefore, we have

\[
\mu \nu = (\mu_A + \mu_B + \mu_C + \mu_z \delta_z)(\nu_A + \nu_B + \nu_C + \nu_z \delta_z)
= \mu_A \nu_Y + \mu_A \ast \nu_B + \mu_B \ast \nu_Y + \mu_B \ast \nu_A + \mu \ast \nu_T
= \mu_A(1) \nu_Y(1) \delta_z + \mu_B(1) \nu_T(1) \delta_z + \mu_z \delta_z(1) \nu_z(1) \delta_z + \mu_A \ast \nu_B + \mu_B \ast \nu_A
= [\mu(1) \nu(1) - \mu_A(1) \nu_B(1) - \mu_B(1) \nu_A(1) \delta_z + \mu_A \ast \nu_B + \mu_B \ast \nu_A]
\]

On the other hand,
Hypergroup Structures with Regular Multiplications  

J. Laali

\[ \mu_A \ast v_B (1) = \int_{A} \int_{B} d\lambda (x,y) (t)d\nu_B (x)d\mu_A (y). \]

\[ = \int_{A} d\nu_B (x)d\mu_A (y) = \mu_A (1)v_B (1). \]

Hence,

\[ (\mu_1 \nu)_\ast \mu = [\mu_1 (1)\nu (1) - \mu_A (1)v_B (1) - \mu_B (1)v_A (1)] \delta_z \nu (1) + (\mu_A \ast v_A + \mu_B \ast v_B) \nu \]

\[ = \mu (1)v (1) \nu (1) \delta_z. \]

Similarly \( \mu (v \nu) = \mu (1)v (1) \nu (1) \delta_z \). So that, the multiplication is associative.

Now, let \( f \in C_c(X) \), then we have

\[ \int_X f (t)d \lambda (x,t) (t) = f (x), \]

\[ \int_X f (t)d \lambda (x,t) (t) = \begin{cases} \int d \lambda (x,y) (f) & (x \in A, y \in B) \\ f (z) & otherwise. \end{cases} \]

Then the map \((x, y) \mapsto \lambda (x,y) (f) \) is in \( C_b(X \times X) \) and the map \( x \mapsto \lambda (x,y) (f) \) is in \( C_c(X) \) for all \( y \in X \). Therefore, \( X \) is a hypergroup.

We now prove that \( L (X) = M (X-\{e\}) \). Since \( X \) is not discrete, \( \delta_z \notin L (X) \) (Theorem 2, [9]). Let \( \mu \in M (X-\{e\}) \). Then

\[ \delta_z [\mu] = \begin{cases} [\mu] & (x = c) \\ [\mu] (1) \delta_z & (x = z). \end{cases} \]

Now, if \( x \in X - \{e, z\} \) then,

\[ \delta_z [\mu] = \begin{cases} [\mu] (1) - [\mu] (1) \delta_z + [\delta_z \ast [\mu] ] & (x = A) \\ [\mu] (1) - [\mu] (1) \delta_z + [\delta_z \ast [\mu] ] & (x = B). \end{cases} \]

Hence, for \( x = y = z \) or \( x = y = e \),

\[ \|\delta_z [\mu] - \delta_y [\mu]\| = 0 \]

Otherwise,

\[ \|\delta_z [\mu] - \delta_y [\mu]\| = \begin{cases} \|\delta_z [\mu] - \delta_y [\mu]\| & (x, y \in A) \\ \|\delta_z [\mu] - \delta_y [\mu]\| & (x, y \in B). \end{cases} \]

Therefore, \( \mu \in L (X) \) if and only if \( \mu_A \in L (X) \), \( \mu_B \in L (X) \). This statement is equivalent to, \( \mu \in L (X) \) if and only if \( M (A \cup B) \subseteq L (X) \). So, the conclusion holds.

Let \( M (X) \) be the space of bounded regular Borel measures. We shall say that \( M (X) \) has a general measure multiplication, if there exists a bilinear associative map.
\( \phi : M(X) \times M(X) \to M(x) \) such that
\[ \phi(M_p(X) \times M_p)) \subseteq M_p(X). \]

Also, we shall say that \( \phi \) is Arens regular, if for every two nets \((\mu_\alpha), (v_\beta)\) in \(M(X)\),
\[ w^* - \lim_{\alpha} w^* - \lim_{\beta} \phi(\mu_\alpha, v_\beta) = w^* - \lim_{\beta} \phi(\mu_\alpha, v_\beta), \]
when, both exist (see introduction [10]).

**Theorem 2.3.** Let \( M(A \cup B) \subseteq L(X) \). Then, \( L(X) \) is Arens regular if and only if there is a bilinear map \( \phi : M(A) \times M(B) \to M(C) \) which is Arens regular.

**Proof.** By theorem 1, \( L(X) = M(X - \{e\}) \) and (by the lemma, 4, [9]),
\[ L(X) = M(A) \oplus M(B) \oplus M(C) \oplus \delta_x. \]
\[ L(X)^{**} = M(A)^{**} \oplus M(B)^{**} \oplus M(C)^{**} \oplus \delta_x. \]

So, each \( \mu \in L(X), F \in L(X)^{**} \), can be written uniquely in the form
\[ \mu = \mu_A + \mu_B + \mu_C + \mu_x \delta_x. \]
\[ F = F_A + F_B + F_C + \mu_x \delta_x. \]

where, \( \mu_A \) is the restriction of \( \mu \) to \( A \) and \( F_A \) is the restriction of \( F \) to \( M(A) \) and so on.

Let \( \mu, \nu \in M(X) \). We define \( \Phi : M(X) \times M(X) \to M(X) \) by
\[ \Phi(\mu, \nu) = \mu \nu = \int_X \int_X \lambda(x, y) d\mu(x) d\nu(y). \]

If \( \mu, \nu \in M_p(X) \) then \( \phi(\mu, \nu) \in M_p(X) \). Thus, the multiplication \( \Phi \) maps probability measures to probability measures.

First, suppose that \( L(X) \) is Arens regular. So, for every nets \((\mu_\alpha) \subseteq M(A), (\nu_\beta) \subseteq M(B)\),
\[ w^* - \lim_{\alpha} w^* - \lim_{\beta} \Phi(\mu_\alpha, \nu_\beta), w^* - \lim_{\beta} \Phi(\mu_\alpha, \nu_\beta), \]
exist, and then they are equal (by Theorem 1, [5]).

Conversely, let \( \Phi \) be Arens regular and \( F, G \in L(X)^{**} \). There are two nets \((\mu_\alpha)\) and \((\nu_\beta)\) in \(L(X)\) whit
\[ w^* - \lim_{\alpha} \mu_\alpha = F, w^* - \lim_{\beta} \mu_\beta = G, \]
\[ \text{supp} \mu_\alpha \subseteq X - \{e\}, \text{supp} \mu_\beta \subseteq X - \{e\} \]

The multiplication \( \Phi \) is Arens regular. Therefore,
\[ w^* - \lim_{\alpha} w^* - \lim_{\beta} \Phi((\mu_\alpha)_A (v_\beta)_B) = w^* - \lim_{\beta} \Phi((\mu_\alpha)_A (v_\beta)_B), \]
\[ w^* - \lim_{\alpha} w^* - \lim_{\beta} \Phi((v_\beta)_A (\mu_\alpha)_A) = w^* - \lim_{\beta} \Phi((v_\beta)_A (\mu_\alpha)_A). \]
Combining the above equalities, we have
\[ FG = w^* - \lim_{\alpha} w^* - \lim_{\beta} \mu_\alpha v_\beta, \]
\[ = w^* - \lim_{\alpha} w^* - \lim_{\beta} (\mu_\alpha v_\beta) - (\mu_\alpha v_\beta) + (\mu_\alpha v_\beta) \delta_z + \phi((\mu_\alpha v_\beta) + \phi((\mu_\beta v_\beta)) = w^* - \lim_{\alpha} w^* - \lim_{\beta} \mu_\alpha v_\beta = GF. \]

Thus, FG=GF. By (Proposition 1, [5]), L(X) is Arens regular.

Now, let A and C be disjoint, X={e} \cup A \cup C \cup \{z\} and e, z \in A \cup C. The topology of X, A and C are compact subspaces of X and e, z are isolated points.

**Theorem 2.4.** If \( \lambda: A \times A \to M_p(X) \) is weak*-continuous and symmetric, then there exists a hypergroup structure on X so that

(i) \( M(A) \subseteq L(X) \) if and only if \( M(X-\{e\})=L(X) \);

(ii) Let \( M(A) \subseteq L(X) \). Then there exists a bilinear associative map

\[ \Phi: M(A) \subseteq M(A) \to M(C) \] which maps probability measures to probability measures and L(X) is Arens regular if and only if \( \Phi \) is Arens regular.

**Proof.** Define \( \tilde{\lambda}: X \times X \to M_p(X) \) with the following equations:

(i) \( \tilde{\lambda}_{(x,e)} = \tilde{\lambda}_{(e,x)} = \delta_z; \)

(ii) \( \tilde{\lambda}_{(x,y)} = \begin{cases} \tilde{\lambda}(x,y) & \text{(x, y} \in A) \\ \delta_z & \text{(otherwise).} \end{cases} \)

Then, we define a convolution \( \mu \nu \) for \( \mu, \nu \in M(X) \) by

\[ \mu \nu = \int_X \int_X \tilde{\lambda}_{(x,y)} d\mu(x)d\nu(y). \]

It is clear that, if \( \mu, \nu \in M(X-\{e\}) \) and then

\[ \mu \nu = (\mu(l) \nu(l) - \mu_A(l) \nu_A(l) \delta_z) + \mu_A \ast \nu_A. \]

The rest of the proof is the same as the proof of last theorem.

Let G be an arbitrary locally compact Housdorff group and \( \mu \) be a right invariant Haar measure on G. The space \( L^1(G) \) of integrable functions on G, with the convolution taken as product, is a Banach Algebra. P. Civin and B. Yood [4] have shown that; if G is commutative and infinite set then the Banach Algebra \( L^1(G) \) is not Arens regular. N. Young [14] has extended this result to non-commutative case. A. Ulger [13] has
presented a very simple proof of the Theorem, which says that the group algebra \( L^1(G) \) is Arens regular if and only if \( G \) is finite. In this paper we present a Theorem, which shows that the Young’s result does not hold in a hypergroup. This theorem is an application of Theorem 2.2.

**Theorem 2.4.** There is a hypergroup algebra \( M(X) \) which has regular multiplication and \( L(X) \) is Arens regular but \( X \) is not finite.

**Construction.** Let \( A=[a_1,b_1] \), \( B=[a_2,b_2] \), \( C=\{a,b\} \) be subsets of an ordered set in the ordered topology and \( A,B,C \) are disjoint. Let \( X=\{e\} \cup A \cup B \cup C \cup \{z\} \), with the topology in which \( e, a, b, z \) are isolated points and supposed \( \phi:X \to [0,1] \) is a continuous function with supp \( \phi=X \). Define \( \lambda:A \times B \to M_p(C) \) by

\[
\lambda_{(x,y)}(f) = \phi(x)\phi(y)d\delta + (1-\phi(x)\phi(y))d\delta.
\]

So, \( \lambda_{(x,y)} = \lambda_{(y,x)} \). If \( f \in C_o(X)(-C_c(X)) \) then,

\[
\lambda_{(x,y)}(f) = \int_X f(t)d\lambda_{(x,y)}(t) = \phi(x)\phi(y)f(a) + (1-\phi(x)\phi(y))f(b)
\]

Hence, if \( \{(x_n, y_n)\} \) is sequence converge to \( (x, y) \) then

\[
l_i{(x_n,y_n)}(f) = \lambda_{(x,y)}(f).
\]

Therefore, \( \lambda:A \times B \to (M_p(C), \text{Weak}^*) \) is continuous. By Theorem 2.2 \( X \) is a hypergroup.

To prove \( M(A \cup B) \subseteq L(X) \), let \( \mu \in M(A \cup B) \). Then, \( \mu = \mu_A + \mu_B \). Suppose that \( x \in B \), then,

\[
\delta_x |\mu_A| = \int_X \int_X \lambda_{(u,v)} d\delta(v)d|\mu_A|(u)
= \int_X \lambda_{(u,x)} d|\mu_A|(u),
\]

So, for \( x, y \in B \),

\[
\|\delta_x |\mu_A| - \delta_y |\mu_A| \| = \|\int_X \lambda_{(u,x)} - \lambda_{(u,y)} d|\mu_A|(u)\|
= |\phi(x) - \phi(y)|\|\int_X \phi(u)d|\mu_A|(u)\|\delta_x - \int_X \phi(u)d|\mu_A|(u)\|\delta_y\|
\leq 2|\phi(x) - \phi(y)||\mu_A| \| \phi \|
\]

\( \phi \) is continuous, so \( \mu_A \in L(X) \). Similarly, \( \mu_B \in L(X) \). Then \( L(X)=M(X-\{e\}) \).
We now prove that $\phi: M(A) \times M(B) \to M(C)$ is Arens regular. First suppose that $\mu_A \in M(A)$, $\nu_B \in M(B)$. So,

$$\phi(\mu_A, \nu_B) = \mu_A^* \nu_B = \int_X \int_X \lambda_{x,y} d\mu_A(x) d\nu_B(y)$$

$$= \int_X \int_X [\phi(x) \phi(y)] \delta_a + (1 - \phi(x) \phi(y)) \delta_b ] d\mu_A(x) d\nu_B(y)$$

$$= \mu_A(\phi) \nu_B(\phi) \delta_a + \mu_A(1) \nu_B(1) - \mu_A(\phi) \nu_B(\phi)] \delta_b.$$ 

Now, suppose $\{ (\mu_n) \}, \{ (\nu_n) \}$ are two sequences such that $w^* - \lim w^* - \lim \phi((\mu_n)_{\nu_m}(v_n))$, $w^* - \lim w^* - \lim \phi((\mu_n)_{\nu_m}(v_n))$ exist, then

$$w^* - \lim w^* - \lim \phi((\mu_n)_{\nu_m}(v_n)) = w^* - \lim w^* - \lim (\mu_n)_{\nu_m}(\phi(v_n) m(\phi) \delta_a$$

$$= \lim w^* - \lim \phi((\mu_n)_{\nu_m}(v_n)).$$

Hence, $\phi$ is Arens regular. By Theorem 2.3, $L(X)$ is Arens regular, but $X$ is not finite.

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**References**

8. Laali J. Regularity and Irregularity of Multiplication In the Circle Algebra, 28’’ Annual Iranian Mathematics Conference, Tabriz University, Tabriz (1997).


