Formal Local Cohomology Modules and Serre Subcategories

A. Kianezhad; Science and Research Branch, Islamic Azad University
A. J. Taherizadeh*; Kharazmi University
A. Tehranian; Science and Research Branch, Islamic Azad University

Abstract

Let $(R,\boldsymbol{m})$ be a Noetherian local ring, $\boldsymbol{a}$ an ideal of R and M a finitely generated R-module. We investigate some properties of formal local cohomology modules with respect to a Serre subcategory. We provide a common language to indicate some properties of formal local cohomology modules.

1. Introduction

Throughout this paper $(R,\boldsymbol{m})$ is a commutative Noetherian local ring, $\boldsymbol{a}$ an ideal of R and M is a finitely generated R- module. For an integer $i \in \mathbb{N}_0, H^i_{\boldsymbol{a}}(N)$ denotes the i-th local cohomology module of M with respect to $\boldsymbol{a}$ as introduced by Grothendieck (cf. [1], [2].

We shall consider the family of local cohomology modules $\{H^i_{\boldsymbol{m}}\left(\frac{M}{a^nM}\right)\}_{n\in\mathbb{N}}$ for a non-negative integer $i \in \mathbb{N}_0$. With natural homomorphisms; this family forms an inverse system. Schenzel introduced the i-th formal local cohomology of M with respect to $\boldsymbol{a}$ in the form of $f^i_{\boldsymbol{a}}(M) := \frac{lim}{n\in\mathbb{N}} H^i_{\boldsymbol{m}}\left(\frac{M}{a^nM}\right)$, which is the i-th cohomology module of the $\boldsymbol{a}$-adic completion of the Čech complex $\check{c}_x \otimes_R M$, where $\underline{x}$ denotes a system of elements of R such that $Rad(\underline{x},R) = \boldsymbol{m}$ (see [3, Definition 3.1]). He defines the formal grade as $f.grade(\boldsymbol{a},M) = \inf\{i \in \mathbb{N}_0 \mid f^i_{\boldsymbol{a}}(M) \neq 0\}$. For any ideal $\boldsymbol{a}$ of R and finitely generated R- module M the following statements hold:

(i) (See [3, Theorem 3.11]). If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of finitely generated R- modules, then there is the following long exact sequence:

$$\cdots \to f^i_{\boldsymbol{a}}(M') \to f^i_{\boldsymbol{a}}(M) \to f^i_{\boldsymbol{a}}(M'') \to \cdots .$$

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*Corresponding author: taheri@khu.ac.ir

(ii) (See [3, Theorem 1.3]). $f.grade(\boldsymbol{a}, M) \leq \dim(M) - cd(\boldsymbol{a}, M)$; some properties of formal local cohomology have been presented in [3, 4, 5 and 6].

Throughout this paper $\mathcal{S}$ denotes a Serre subcategory of the category of R- modules and $R-$homomorphisms (we recall that a class $\mathcal{S}$of R-modules is a Serre subcategory of the category of R- modules and R-homomorphisms if $\mathcal{S}$ is closed under taking submodules, quotients and extensions).

Our paper contains three sections. In Section 2, we shall define the formal grade of $\boldsymbol{a}$ with respect to M in $\mathcal{S}$ as the infimum of the integers i such that $f_{\boldsymbol{a}}^{i}(M) \notin \mathcal{S}$ and is denoted by $f.grade_{\mathcal{S}}(\boldsymbol{a}, M)$. (See definition 2.1). Then we shall obtain some properties of this notion. We show that if $\Gamma_{\boldsymbol{a}}(M)$ is a pure submodule of M, then $Hom_R(\frac{R}{\boldsymbol{m}}, f_{\boldsymbol{a}}^{t}(\Gamma_{\boldsymbol{a}}(M))$ and $Hom_R(\frac{R}{\boldsymbol{m}}, f_{\boldsymbol{a}}^{t-1}\left(\frac{\mathrm{M}}{\Gamma_{\boldsymbol{a}}(M)}\right))$ belong to $\mathcal{S}$,where $t = f.grade_{\mathcal{S}}(\boldsymbol{a}, M)$.

In Section 3, we shall define the formal cohomological dimension of $\boldsymbol{a}$ with respect to M in $\mathcal{S}$ as the supremum of the integers i such that $f_{\boldsymbol{a}}^{i}(M) \notin \mathcal{S}$ and is denoted by $f.cd_{\mathcal{S}}(\boldsymbol{a}, M)$. (See definition 3.1). The main result of this section is that if $f_{\boldsymbol{a}}^{i}(M) \in \mathcal{S}$ and $H_{\boldsymbol{m}}^{i}(M) \in \mathcal{S}$ for all $i > t$, then $\frac{R}{\boldsymbol{a}} \otimes_R f_{\boldsymbol{a}}^{t}(M)$ belongs to $\mathcal{S}$.

2. The formal grade of a module in a Serre subcategory

Definition 2.1. The formal grade of $\boldsymbol{a}$ with respect to M in $\mathcal{S}$ is the infimum of the integers i such that $f_{\boldsymbol{a}}^{i}(M) \notin \mathcal{S}$ and is denoted by $f.grade_{\mathcal{S}}(\boldsymbol{a}, M)$.

Proposition 2.2. Let $(R, \boldsymbol{m})$ be a local ring and $\boldsymbol{a}$ be an ideal of R. If $0 \to L \to M \to N \to 0$ is an exact sequence of finitely generated R-modules, then the following statements hold.

(a) $f.grade_{\mathcal{S}}(\boldsymbol{a}, M) \geq \min\{f.grade_{\mathcal{S}}(\boldsymbol{a}, L), f.grade_{\mathcal{S}}(\boldsymbol{a}, N)\}$.

(b) $f.grade_{\mathcal{S}}(\boldsymbol{a}, L) \geq \min\{f.grade_{\mathcal{S}}(\boldsymbol{a}, M), f.grade_{\mathcal{S}}(\boldsymbol{a}, N) + 1\}$.

(c) $f.grade_{\mathcal{S}}(\boldsymbol{a}, N) \geq \min\{f.grade_{\mathcal{S}}(\boldsymbol{a}, L) - 1, f.grade_{\mathcal{S}}(\boldsymbol{a}, M)\}$.

Proof. According to [3, Theorem 3.11], the above short exact sequence induces the following long exact sequence.

$$\cdots \to f_{\boldsymbol{a}}^{i-1}(N) \to f_{\boldsymbol{a}}^{i}(L) \to f_{\boldsymbol{a}}^{i}(M) \to f_{\boldsymbol{a}}^{i}(N) \to f_{\boldsymbol{a}}^{i+1}(L) \to \cdots.$$

So, the result follows.

Corollary 2.3. If $\underline{x} = x_1, \dots, x_n$ is a regular M-sequence, then $f.grade_{\mathcal{S}} \left(\boldsymbol{a}, \frac{M}{\underline{x}M}\right) \geq f.grade_{\mathcal{S}} \,(\boldsymbol{a}, M\,) - n.$

Proof. Consider the following exact sequence $(n \in \mathbb{N})$

$$0 \to \frac{M}{(x_1, \dots, x_{n-1})M} \xrightarrow{x_n} \frac{M}{(x_1, \dots, x_{n-1})M} \xrightarrow{nat.} \frac{M}{(x_1, \dots, x_n)M} \to 0$$

whenever $n = 1$ by $(x_1, \dots, x_{n-1})M$ we means 0.

Corollary 2.4. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be ideals of R. Then

(a) $f.grade_{\mathcal{S}} \,(\boldsymbol{a} \cap \boldsymbol{b}\,, M\,) \geq$
$\min\{f.grade_{\mathcal{S}} \,(\boldsymbol{a}, M\,), f.grade_{\mathcal{S}} \,(\boldsymbol{b}, M\,), f.grade_{\mathcal{S}} \,((\boldsymbol{a}, \boldsymbol{b}), M) + 1\,)\}.$

(b) $f.grade_{\mathcal{S}} \,((\boldsymbol{a}, \boldsymbol{b}), M\,) \geq$
$\min\{f.grade_{\mathcal{S}} \,(\boldsymbol{a} \cap \boldsymbol{b}\,, M\,) - 1, f.grade_{\mathcal{S}} \,(\boldsymbol{a}, M\,), f.grade_{\mathcal{S}} \,(\boldsymbol{b}, M\,)\}\,.$

Proof. For all $n \in \mathbb{N}$ there is a short exact sequence as follows:

$$0 \to \frac{M}{\boldsymbol{a}^n M \cap \boldsymbol{b}^n M} \to \frac{M}{\boldsymbol{a}^n M} \oplus \frac{M}{\boldsymbol{b}^n M} \to \frac{M}{(\boldsymbol{a}^n, \boldsymbol{b}^n)M} \to 0.$$

By using [3,Theorem 5.1], the above exact sequence induces the following long exact sequence.

$$\cdots \to \varprojlim_{n\in\mathbb{N}} H^i_{\boldsymbol{m}}\left(\frac{M}{(\boldsymbol{a}\cap\boldsymbol{b})^n M}\right) \to \varprojlim_{n\in\mathbb{N}} H^i_{\boldsymbol{m}}\left(\frac{M}{\boldsymbol{a}^n M}\right) \oplus \varprojlim_{n\in\mathbb{N}} H^i_{\boldsymbol{m}}\left(\frac{M}{\boldsymbol{b}^n M}\right) \to \varprojlim_{n\in\mathbb{N}} H^i_{\boldsymbol{m}}\left(\frac{M}{(\boldsymbol{a},\boldsymbol{b})^n M}\right) \to \cdots.$$

So by using an argument similar to that of Proposition 2.2, the result follows.

Corollary 2.5. Assume that M is a finitely generated R- module and N_1 and N_2 are submodules of M. Then considering the exact sequence $0 \to \frac{M}{N_1 \cap N_2} \to \frac{M}{N_1} \oplus \frac{M}{N_2} \to \frac{M}{N_1 + N_2} \to 0$ we shall have

(a) $f.grade_{\mathcal{S}} \left(\boldsymbol{a}\ ,\ \frac{M}{N_1 \cap N_2}\right) \geq \min\{f.grade_{\mathcal{S}} \left(\boldsymbol{a}\,,\ \frac{M}{N_1}\right), f.grade_{\mathcal{S}} \Big(\boldsymbol{a}\ ,$
$MN2\,, f.grade\mathcal{S}\,\boldsymbol{a}\ ,\ MN1+\ N2\ +1\}.$

(b) $f.grade_{\mathcal{S}} \left(\boldsymbol{a}\ ,\ \frac{M}{N_1 + N_2}\right) \geq \min\Big\{f.grade_{\mathcal{S}} \left(\frac{M}{N_1 \cap N_2}\right) - 1, f.grade_{\mathcal{S}} \Big(\boldsymbol{a}\ ,$
$MN1\,, f.grade\mathcal{S}\boldsymbol{a}\ ,\ MN2\,.$

Theorem 2.6. Let $\boldsymbol{a}$ be an ideal of a local ring $(\,R, \boldsymbol{m}\,)$, M be a finitely generated R- module and L be a pure submodule of M. Then $f.grade_{\mathcal{S}}(\boldsymbol{a}\,, L) \geq f.grade_{\mathcal{S}}(\boldsymbol{a}\,, M)$ where $\mathcal{S}$ is a Serre subcategory of the category of R- modules and R-homomorphisms. In particular, $\inf\,\{i | H^i_{\boldsymbol{m}}(L) \notin \mathcal{S}\} \geq \inf\,\{i | H^i_{\boldsymbol{m}}(M) \notin \mathcal{S}\}\,.$

Proof. Let L be a pure submodule of M. So $\frac{L}{\boldsymbol{a}^n L} \to \frac{M}{\boldsymbol{a}^n M}$ is pure for each $n \in \mathbb{N}$. Now according to [8, Corollary 3.2 (a)] , $H^i_{\boldsymbol{m}}\left(\frac{L}{\boldsymbol{a}^n L}\right) \to H^i_{\boldsymbol{m}}\left(\frac{M}{\boldsymbol{a}^n M}\right)$ is injective. Since inverse limit is a left exact functor, $f^i_{\boldsymbol{a}}(L)$ is isomorphic to a submodule of $f^i_{\boldsymbol{a}}(M)$. Consequently ,$f.grade_{\mathcal{S}}(\boldsymbol{a}, L) \geq f.grade_{\mathcal{S}}(\boldsymbol{a}, M)$. If $\boldsymbol{a} = 0$ then, $f.grade_{\mathcal{S}}(0, M) = \inf\{i | H^i_{\boldsymbol{m}}(M) \notin \mathcal{S}\}$ and the result follows.

Corollary 2.7. If $0 \to L \to M \to N \to 0$ is a pure exact sequence of finitely generated R- modules, then $\min\{f.grade_{\mathcal{S}}(\boldsymbol{a}, L), f.grade_{\mathcal{S}}(\boldsymbol{a}, N) + 1\} \geq f.grade_{\mathcal{S}}(\boldsymbol{a}, M)$.

Proof. Since L is a pure submodules of M, as a result of the previous theorem, $f.grade_{\mathcal{S}}(\boldsymbol{a}, L) \geq f.grade_{\mathcal{S}}(\boldsymbol{a}, M)$. Hence we must prove that $f.grade_{\mathcal{S}}(\boldsymbol{a}, N) + 1 \geq f.grade_{\mathcal{S}}(\boldsymbol{a}, M)$. We assume that $i < f.grade_{\mathcal{S}}(\boldsymbol{a}, M)$ and we show that $i < f.grade_{\mathcal{S}}(\boldsymbol{a}, N) + 1$. Consider the following long exact sequence.

$$\cdots \to f^{i-1}_{\boldsymbol{a}}(M) \to f^{i-1}_{\boldsymbol{a}}(N) \to f^i_{\boldsymbol{a}}(L) \to f^i_{\boldsymbol{a}}(M) \to f^i_{\boldsymbol{a}}(N) \to \cdots . \ (**)$$

If $i < f.grade_{\mathcal{S}}(\boldsymbol{a}, M)$, then $f^0_{\boldsymbol{a}}(M), f^1_{\boldsymbol{a}}(M), \ldots, f^{i-1}_{\boldsymbol{a}}(M), f^i_{\boldsymbol{a}}(M) \in \mathcal{S}$. On the other hand, since $i < f.grade_{\mathcal{S}}(\boldsymbol{a}, M) \leq f.grade_{\mathcal{S}}(\boldsymbol{a}, L), f^0_{\boldsymbol{a}}(L), \ldots, f^i_{\boldsymbol{a}}(L) \in \mathcal{S}$. Hence, it follows from $(**)$ that $f^0_{\boldsymbol{a}}(N), \ldots, f^{i-1}_{\boldsymbol{a}}(N) \in \mathcal{S}\ and\ so\ i - 1 < f.grade_{\mathcal{S}}(\boldsymbol{a}, N)$.

Theorem 2.8. Let $(R, \boldsymbol{m})$ be a local ring, $\boldsymbol{a}$ be an ideal of R, $\mathcal{S}$ be a Serre subcategory of the category of R- modules and R_homomorphisms and $M \in \mathcal{S}$ be a finitely generated R- module such that $\Gamma_{\boldsymbol{a}}(M)$ is a pure submodule of M. Then $Hom_R\left(\frac{R}{\boldsymbol{a}}, f^t_{\boldsymbol{a}}(\Gamma_{\boldsymbol{a}}(M))\right) \in \mathcal{S}$, where $t = f.grade_{\mathcal{S}}(\boldsymbol{a}, M)$.

Proof. Due to the previous theorem, $f.grade_{\mathcal{S}}(\boldsymbol{a}, \Gamma_{\boldsymbol{a}}(M)) \geq f.grade_{\mathcal{S}}(\boldsymbol{a}, M)$. If $f.grade_{\mathcal{S}}(\boldsymbol{a}, \Gamma_{\boldsymbol{a}}(M)) > f.grade_{\mathcal{S}}(\boldsymbol{a}, M)$, then the result is obvious. Accordingly, we assume that $f.grade_{\mathcal{S}}(\boldsymbol{a}, \Gamma_{\boldsymbol{a}}(M)) = f.grade_{\mathcal{S}}(\boldsymbol{a}, M)$. We know that $Supp(\Gamma_{\boldsymbol{a}}(M)) \subseteq Var(\boldsymbol{a})$. By using [4, Lemma 2.3], $f^i_{\boldsymbol{a}}(\Gamma_{\boldsymbol{a}}(M)) \cong H^i_{\boldsymbol{m}}(\Gamma_{\boldsymbol{a}}(M))$ for all $i \geq 0$. So, if $j < f.grade_{\mathcal{S}}(\boldsymbol{a}, M)$, then $f^j_{\boldsymbol{a}}(\Gamma_{\boldsymbol{a}}(M)) \cong H^j_{\boldsymbol{m}}(\Gamma_{\boldsymbol{a}}(M)) \in \mathcal{S}$ and $Ext^k_R(\frac{R}{\boldsymbol{m}}, H^j_{\boldsymbol{m}}(\Gamma_{\boldsymbol{a}}(M)) \in \mathcal{S}$ for all $k \geq 0$ and $j < f.grade_{\mathcal{S}}(\boldsymbol{a}, M)$. Moreover $Ext^t_R\left(\frac{R}{\boldsymbol{m}}, \Gamma_{\boldsymbol{a}}(M)\right) \in \mathcal{S}$, because $\Gamma_{\boldsymbol{a}}(M) \in \mathcal{S}$. Consequently, according to [7, Theorem 2.2],

$$Hom_R(\frac{R}{\boldsymbol{m}}, H^t_{\boldsymbol{m}}(\Gamma_{\boldsymbol{a}}(M)) \in \mathcal{S}, \text{ where } t = f.grade_{\mathcal{S}}(\boldsymbol{a}, M).$$

Corollary 2.9 With the same notations as Theorem 2.8, let $X \in \mathcal{S}$ be a submodule of $f^t_{\boldsymbol{a}}(\Gamma_{\boldsymbol{a}}(M))$, where $t = f.grade_{\mathcal{S}}(\boldsymbol{a}, M)$. Then $Hom_R(\frac{R}{\boldsymbol{m}}, \frac{f^t_{\boldsymbol{a}}(\Gamma_{\boldsymbol{a}}(M))}{X}) \in \mathcal{S}$.

Proof. Consider the long exact sequence:

$$Hom_R\left(\frac{R}{\boldsymbol{m}}, f_{\boldsymbol{a}}^t(\Gamma_{\boldsymbol{a}}(M))\right) \to Hom_R\left(\frac{R}{\boldsymbol{m}}, \frac{f_{\boldsymbol{a}}^t(\Gamma_{\boldsymbol{a}}(M))}{X}\right) \to Ext_R^1\left(\frac{R}{\boldsymbol{m}}, X\right). \ (*)$$

In accordance with the previous theorem $Hom_R(\frac{R}{\boldsymbol{m}}, f_{\boldsymbol{a}}^t(\Gamma_{\boldsymbol{a}}(M))) \in \mathcal{S}$. Moreover $Ext_R^1(\frac{R}{\boldsymbol{m}}, X) \in \mathcal{S}$. It follows from the exact sequence $(*)$ that $Hom_R\left(\frac{R}{\boldsymbol{m}}, \frac{f_{\boldsymbol{a}}^t(\Gamma_{\boldsymbol{a}}(M))}{X}\right) \in \mathcal{S}$.

Theorem 2.10. Suppose that $\boldsymbol{a}$ is an ideal of $(R, \boldsymbol{m})$ and $M \in \mathcal{S}$ is a finitely generated R-module such that $\Gamma_{\boldsymbol{a}}(M)$ is a pure submodule of M. Then $Hom_R\left(\frac{R}{\boldsymbol{m}}, f_{\boldsymbol{a}}^{t-1}\left(\frac{M}{\Gamma_{\boldsymbol{a}}(M)}\right)\right) \in \mathcal{S}$, where $t = f.grade_{\mathcal{S}}(\boldsymbol{a}, M)$.

Proof. One has $f.grade_{\mathcal{S}}(\boldsymbol{a}, \Gamma_{\boldsymbol{a}}(M)) \geq f.grade_{\mathcal{S}}(\boldsymbol{a}, M)$, by Theorem 2.6. Now, the exact sequence $0 \to \Gamma_{\boldsymbol{a}}(M) \to M \to \frac{M}{\Gamma_{\boldsymbol{a}}(M)} \to 0$ induces the following long exact sequence:

$$\cdots \xrightarrow{\alpha} f_{\boldsymbol{a}}^{t-1}(\Gamma_{\boldsymbol{a}}(M)) \xrightarrow{\beta} f_{\boldsymbol{a}}^{t-1}(M) \xrightarrow{\gamma} f_{\boldsymbol{a}}^{t-1}\left(\frac{M}{\Gamma_{\boldsymbol{a}}(M)}\right) \xrightarrow{\xi} f_{\boldsymbol{a}}^t(\Gamma_{\boldsymbol{a}}(M)) \xrightarrow{\varphi} \cdots. \ (*)$$

Using the exact sequence $(*)$, we obtain the short exact sequence $0 \to \mathrm{Im}(\beta) \to f_{\boldsymbol{a}}^{t-1}(M) \to Im(\gamma) \to 0$. Since $f_{\boldsymbol{a}}^{t-1}(M) \in \mathcal{S}$, $\mathrm{Im}(\beta) \in \mathcal{S}$ and $Im(\gamma) \in \mathcal{S}$. Furthermore, we have the exact sequence $0 \to \mathrm{Im}(\xi) \to H_{\boldsymbol{m}}^t(\Gamma_{\boldsymbol{a}}(M)) \to Im(\varphi) \to 0$ which induces the following long exact sequence:

$$0 \to Hom_R(\frac{R}{\boldsymbol{m}}, \mathrm{Im}(\xi)) \to Hom_R(\frac{R}{\boldsymbol{m}}, H_{\boldsymbol{m}}^t(\Gamma_{\boldsymbol{a}}(M))) \to \cdots.$$

Thus $Hom_R(\frac{R}{\boldsymbol{m}}, \mathrm{Im}(\xi)) \in \mathcal{S}$. Finally, by considering the short exact sequence $0 \to \mathrm{Im}(\gamma) \to f_{\boldsymbol{a}}^{t-1}\left(\frac{M}{\Gamma_{\boldsymbol{a}}(M)}\right) \to Im(\xi) \to 0$ we can conclude that $Hom_R\left(\frac{R}{\boldsymbol{m}}, f_{\boldsymbol{a}}^{t-1}\left(\frac{M}{\Gamma_{\boldsymbol{a}}(M)}\right)\right) \in \mathcal{S}$.

Theorem 2.11. Suppose that R is complete with respect to the $\underline{a}$–adic topology and $M \in \mathcal{S}$ be a finitely generated R-module and t a positive integer such that $f_{\boldsymbol{a}}^i(M) \in \mathcal{S}$ for all $i < t$. Then $Hom_R\left(\frac{R}{\boldsymbol{m}}, f_{\boldsymbol{a}}^t(M)\right) \in \mathcal{S}$.

Proof. We use induction on t. Let t=0. Consider the following isomorphisms.

$$Hom_R(\frac{R}{\underline{m}}, f_{\underline{a}}^0(M)) \cong \varprojlim_{n\in\mathbb{N}} Hom_R(\frac{R}{\underline{m}}, H_{\underline{m}}^0(\frac{M}{\underline{a}^n M})) \cong \varprojlim_{n\in\mathbb{N}} Hom_R(\frac{R}{\underline{m}}, \frac{M}{\underline{a}^n M})$$
$$\cong Hom_R(\frac{R}{\underline{m}}, \varprojlim_{n\in\mathbb{N}} (\frac{M}{\underline{a}^n M})) \cong Hom_R(\frac{R}{\underline{m}}, \hat{M}^{\underline{a}}) \cong Hom_R(\frac{R}{\underline{m}}, M)$$

It is clear that $Hom_R(\frac{R}{\underline{m}}, M) \in S$. So by the above isomorphisms, we deduce that $Hom_R(\frac{R}{\underline{m}}, f_{\underline{a}}^0(M)) \in S$.

Suppose that t>0 and the result is true for all integer i less than t. Set N:= $\Gamma_{\mathbf{m}}(M)$. Then $f_{\boldsymbol{a}}^i(M) \cong f_{\boldsymbol{a}}^i\left(\frac{M}{N}\right)$ $for\ all\ i > 0$, and so we may assume that $depth_R(M) > 0$. There is an $M - regular$ element $x \in \boldsymbol{m}$. The exact sequence $0 \to M \xrightarrow{x} M \to \frac{M}{xM} \to 0$ induces the following long exact sequence:

$$\cdots \to f_{\boldsymbol{a}}^{t-2}(M) \xrightarrow{x} f_{\boldsymbol{a}}^{t-2}(M) \xrightarrow{f} f_{\boldsymbol{a}}^{t-2}\left(\frac{M}{xM}\right)$$

$$\to f_{\boldsymbol{a}}^{t-1}(M) \xrightarrow{x} f_{\boldsymbol{a}}^{t-1}(M) \xrightarrow{g} f_{\boldsymbol{a}}^{t-1}\left(\frac{M}{xM}\right)$$

$$\to f_{\boldsymbol{a}}^{t}(M) \xrightarrow{x} f_{\boldsymbol{a}}^{t}(M) \xrightarrow{h} \cdots. \quad (*)$$

Using the exact sequence $(*)$ we obtain the short exact sequence

$$0 \to \frac{f_{\boldsymbol{a}}^{t-1}(M)}{xf_{\boldsymbol{a}}^{t-1}(M)} \to f_{\boldsymbol{a}}^{t-1}\left(\frac{M}{xM}\right) \to (0 \underset{f_{\boldsymbol{a}}^{t}(M)}{:} x) \to 0.$$

Now, this exact sequence induces the following long exact sequence:

$$0 \to Hom_R\left(\frac{R}{\boldsymbol{m}}, \frac{f_{\boldsymbol{a}}^{t-1}(M)}{xf_{\boldsymbol{a}}^{t-1}(M)}\right) \to Hom_R\left(\frac{R}{\boldsymbol{m}}, f_{\boldsymbol{a}}^{t-1}\left(\frac{M}{xM}\right)\right) \to Hom_R\left(\frac{R}{\boldsymbol{m}}, (0 \underset{f_{\boldsymbol{a}}^{t}(M)}{:} x)\right) \to$$
$$Ext_R^1\left(\frac{R}{\boldsymbol{m}}, \frac{f_{\boldsymbol{a}}^{t-1}(M)}{xf_{\boldsymbol{a}}^{t-1}(M)}\right) \to \cdots. (**)$$

By using $(*)$, $f_{\boldsymbol{a}}^i\left(\frac{M}{xM}\right) \in \mathcal{S}$ for all $i < t-1$. Therefore by the induction hypothesis $Hom_R\left(\frac{R}{\boldsymbol{m}}, f_{\boldsymbol{a}}^{t-1}\left(\frac{M}{xM}\right)\right) \in \mathcal{S}$. Furthermore $Ext_R^1\left(\frac{R}{\boldsymbol{m}}, \frac{f_{\boldsymbol{a}}^{t-1}(M)}{xf_{\boldsymbol{a}}^{t-1}(M)}\right) \in \mathcal{S}$ because $f_{\boldsymbol{a}}^{t-1}(M) \in \mathcal{S}$. Thus in accordance with $(**)$, $\underset{f_{a}^{t}(M)}{Hom_R}\left(\frac{R}{\boldsymbol{m}}, (0 : x)\right) \in \mathcal{S}$. Since $x \in \boldsymbol{m}$ according to [9,10.86] we have the following isomorphisms.

$$\underset{f_{a}^{t}(M)}{Hom_R}\left(\frac{R}{\boldsymbol{m}}, (0 : x)\right) \cong Hom_R\left(\frac{R}{\boldsymbol{m}}, Hom_R\left(\frac{R}{xR}, f_{\boldsymbol{a}}^{t}(M)\right)\right) \cong$$

$$Hom_R\left(\frac{R}{\boldsymbol{m}} \otimes_R \frac{R}{xR}, f_{\boldsymbol{a}}^{t}(M)\right) \cong Hom_R\left(\frac{R}{\boldsymbol{m}}, f_{\boldsymbol{a}}^{t}(M)\right).$$

Consequently $Hom_R\left(\frac{R}{\boldsymbol{m}}, f_{\boldsymbol{a}}^{t}(M)\right) \in \mathcal{S}$.

3. The formal cohomological dimension in a Serre subcategory

We recall from [3,Theorem 1.1] that for a finitely generated R- module M, $\sup\{i \in \mathbb{N}_0 | f_{\boldsymbol{a}}^i(M) \neq 0\} = \dim(\frac{M}{\boldsymbol{a}M})$.

Definition 3.1. The formal cohomological dimension of M with respect to a in $\mathcal{S}$ is The supremum of the integers i such that $f_{\boldsymbol{a}}^i(M) \notin \mathcal{S}$ and is denoted by $f.cd_{\mathcal{S}}(\boldsymbol{a}, M)$.

Theorem 3.2. Suppose that $\mathcal{S}$ is a Serre subcategory of the category of R- modules and $R-$homomorphisms and L and N are two finitely generated R- modules such that $Supp_R(L) \subseteq Supp_R(N)$. Then $f.cd_{\mathcal{S}}(\boldsymbol{a}, L) \leq f.cd_{\mathcal{S}}(\boldsymbol{a}, N)$.

Proof. It is enough to prove that $f_{\boldsymbol{a}}^i(L) \in \mathcal{S}$ for all $i > f.cd_{\mathcal{S}}(\boldsymbol{a}, N)$ and all finitely generated R- module L such that $Supp_R(L) \subseteq Supp_R(N)$. We use descending induction on i.For all i $>$ dim $(\frac{L}{\boldsymbol{a}L})$+ $f.cd_{\mathcal{S}}(\boldsymbol{a}, N)$, $f_{\boldsymbol{a}}^i(L) = 0 \in \mathcal{S}$. Let $i > f.cd_{\mathcal{S}}(\boldsymbol{a}, N)$ and the result is proved for $i+1$. By Gruson's theorem, there is a chain $0 = L_0 \subset L_1 \subset \cdots \subset L_l = L$ of submodules of L such that $\frac{L_i}{L_{i-1}}$ is a homomorfic image of a direct sum of finitely many copies of N. Consider the exact sequence $0 \to L_{i-1} \to L_i \to \frac{L_i}{L_{i-1}} \to 0$ $(i = 0,1,\dots,l)$. We may assume that $l = 1$. The exact sequence $0 \to K \to \bigoplus_{j=1}^t N \to L \to 0$ where K is a finitely generated R- module iduces the following long exact sequence:

$$\cdots \to f_{\boldsymbol{a}}^i\left(\bigoplus_{j=1}^t N\right) \to f_{\boldsymbol{a}}^i(L) \to f_{\boldsymbol{a}}^{i+1}(K) \to \cdots. (*)$$

Based on the induction hypothesis $f_{\boldsymbol{a}}^{i+1}(K) \in \mathcal{S}$. Moreover $f_{\boldsymbol{a}}^i\left(\bigoplus_{j=1}^t N\right) = \bigoplus_{j=1}^t f_{\boldsymbol{a}}^i(N) \in \mathcal{S}$ for all $i > f.cd_{\mathcal{S}}(\boldsymbol{a}, N)$. Hence it follows from the exact sequence $(*)$ that $f_{\boldsymbol{a}}^i(L) \in \mathcal{S}$.

The next example shows that even if $Supp_R(M) = Supp_R(N)$, then it may not true that $f.grade_{\mathcal{S}}(\boldsymbol{a}, M) = f.grade_{\mathcal{S}}(\boldsymbol{a}, N)$.

Example 3.3. (See [4, Example 4.3 (i)]) Let $(R, \boldsymbol{m})$ be a 2 dimensional complete regular local ring, $\mathcal{S} = 0$ and $\boldsymbol{a}$ be an ideal of R with $\dim\left(\frac{R}{\boldsymbol{a}}\right) = 1$. Then by using [5,Theorem 1.1], $f.grade_{\mathcal{S}}(\boldsymbol{a}, R) = 1$ and $f.grade_{\mathcal{S}}\left(\boldsymbol{a}, \frac{R}{\boldsymbol{m}}\right) = 0$. Set $M := R \oplus \frac{R}{\boldsymbol{m}}$. Then $Supp_R(M) = Supp_R(R)$. But

$$f.grade_{\mathcal{S}}(\boldsymbol{a}, M) = \inf\left\{f.grade_{\mathcal{S}}(\boldsymbol{a}, R), f.grade_{\mathcal{S}}\left(\boldsymbol{a}, \frac{R}{\boldsymbol{m}}\right)\right\} = 0.$$

Corollary3.4. For all $x \in \boldsymbol{a}$, $f.cd_{\mathcal{S}}(\boldsymbol{a}, M) \geq f.cd_{\mathcal{S}}\left(\boldsymbol{a}, \frac{M}{xM}\right)$.

Corollary3.5. Suppose that $0 \to L \to M \to N \to 0$ is an exact sequence of finitely generated R- modules. Then $f.cd_{\mathcal{S}}(\boldsymbol{a}, M) = \max\{f.cd_{\mathcal{S}}(\boldsymbol{a}, L), f.cd_{\mathcal{S}}(\boldsymbol{a}, N)\}$.

Proof. Since $Supp_R(M) = Supp_R(L) \cup Supp_R(N)$ by referring to Theorem 3.2 we deduce that $f.cd_{\mathcal{S}}(\boldsymbol{a}, M) \geq f.cd_{\mathcal{S}}(\boldsymbol{a}, L)$ and $f.cd_{\mathcal{S}}(\boldsymbol{a}, M) \geq f.cd_{\mathcal{S}}(\boldsymbol{a}, N)$. Therefore $f.cd_{\mathcal{S}}(\boldsymbol{a}, M) \geq max\,\{f.cd_{\mathcal{S}}(\boldsymbol{a}, L), f.cd_{\mathcal{S}}(\boldsymbol{a}, N)\}$.
Next we prove that $max\,\{f.cd_{\mathcal{S}}(\boldsymbol{a}, L), f.cd_{\mathcal{S}}(\boldsymbol{a}, N)\} \geq f.cd_{\mathcal{S}}(\boldsymbol{a}, M)$.
Let $i > max\,\{f.cd_{\mathcal{S}}(\boldsymbol{a}, L), f.cd_{\mathcal{S}}(\boldsymbol{a}, N)\}$. Then $f_{\boldsymbol{a}}^i(N), f_{\boldsymbol{a}}^i(L) \in \mathcal{S}$ and from the exact sequence $f_{\boldsymbol{a}}^i(L) \to f_{\boldsymbol{a}}^i(M) \to f_{\boldsymbol{a}}^i(N)$ we conclude that $f_{\boldsymbol{a}}^i(M) \in \mathcal{S}$. Thus,
$ma\,x\{f.cd_{\mathcal{S}}(\boldsymbol{a}, L), f.cd_{\mathcal{S}}(\boldsymbol{a}, N)\} \geq f.cd_{\mathcal{S}}(\boldsymbol{a}, M)$.

We recall that the cohomological dimension of an R- module M with respect to an ideal $\boldsymbol{a}$ of R in $\mathcal{S}$ is defind as

$$cd_{\mathcal{S}}(\boldsymbol{a}, M) := \sup\{i \in \mathbb{N}_0 \mid H_{\boldsymbol{a}}^i(M) \notin \mathcal{S}\}.$$

The following lemma shows that when we considering the Artinianness of $f_{\boldsymbol{a}}^i(M)$, we can assume that M is $\boldsymbol{a}$-$torsion$-$free$.

Lemma 3.6. Suppose that $\boldsymbol{a}$ is an ideal of a local ring $(R, \boldsymbol{m})$ and t be a non-negative integer. If $H_{\boldsymbol{m}}^{\,i}(M) \in \mathcal{S}$ for all $i \geq t$, then the following are equivalent:

(a) $f_{\boldsymbol{a}}^i(M) \in \mathcal{S}$ for all $i \geq t$.

(b) $f_{\boldsymbol{a}}^i\left(\frac{M}{\Gamma_{\boldsymbol{a}}(M)}\right) \in \mathcal{S}$ for all $i \geq t$.

Proof. According to the hypothesis $t > cd_{\mathcal{S}}(\boldsymbol{m}, M)$. On the other hand $Supp_R(\Gamma_{\boldsymbol{a}}(M)) \subseteq Supp_R(M)$. So by referring to [7,Theorem 3.5], $cd_{\mathcal{S}}(\boldsymbol{m}, \Gamma_{\boldsymbol{a}}(M)) \leq cd_{\mathcal{S}}(\boldsymbol{m}, M)$. Thus, $t > cd_{\mathcal{S}}(\boldsymbol{m}, \Gamma_{\boldsymbol{a}}(M))$ and $H_{\boldsymbol{m}}^i(\Gamma_{\boldsymbol{a}}(M)) \in \mathcal{S}$ for all $i \geq t$. Now, consider the following long exact sequence:

$$\cdots \to f_{\boldsymbol{a}}^i(\Gamma_{\boldsymbol{a}}(M)) \to f_{\boldsymbol{a}}^i(M) \to f_{\boldsymbol{a}}^i\left(\frac{M}{\Gamma_{\boldsymbol{a}}(M)}\right) \to f_{\boldsymbol{a}}^{i+1}(\Gamma_{\boldsymbol{a}}(M)) \to \cdots . (*)$$

According to [4,Lemma 2.3] $f_{\boldsymbol{a}}^i(\Gamma_{\boldsymbol{a}}(M)) \cong H_{\boldsymbol{m}}^i(\Gamma_{\boldsymbol{a}}(M))$. By using the hypothesis $f_{\boldsymbol{a}}^i(\Gamma_{\boldsymbol{a}}(M)) \in \mathcal{S}$ for all $i \geq t$. So it follows from the exact sequence $(*)$ that $f_{\boldsymbol{a}}^i(M) \in \mathcal{S}$ if and only if $f_{\boldsymbol{a}}^i\left(\frac{M}{\Gamma_{\boldsymbol{a}}(M)}\right) \in \mathcal{S}$ for all $i \geq t$.

Theorem 3.7. Let $(R, \boldsymbol{m})$ be a local ring and $M \in \mathcal{S}$ be a finitely generated R- module of dimension d such that $cd_{\mathcal{S}}(\boldsymbol{m}, M) \leq f.cd_{\mathcal{S}}(\boldsymbol{a}, M)$. Then $\frac{f_{\boldsymbol{a}}^t(M)}{\boldsymbol{a} f_{\boldsymbol{a}}^t(M)} \in \mathcal{S}$ where $t = f.cd_{\mathcal{S}}(\boldsymbol{a}, M)$.

Proof. We use induction on $d = \dim(M)$. If $d = 0$, then $\dim\left(\frac{M}{\boldsymbol{a}M}\right) = 0$. Accordingly to [3, Theorem 1.1], $f_{\boldsymbol{a}}^i(M) = 0$ for all $i > 0$.

Moreover $f_{\boldsymbol{a}}^{0}(M) \cong M \in \mathcal{S}$. By definition $H_{\boldsymbol{m}}^{i}(M) \in \mathcal{S}$ for all $i > t$. Therefore from the above lemma we can assume that M is $\boldsymbol{a}$-$torsion$-$free$ and there is an M -$regular$ element $x \in \boldsymbol{a}$. Consider the long exact sequence :

$$\cdots \to f_{\boldsymbol{a}}^{i}(M) \xrightarrow{x} f_{\boldsymbol{a}}^{i}(M) \xrightarrow{f} f_{\boldsymbol{a}}^{i}\left(\frac{M}{\mathrm{x}M}\right) \xrightarrow{g} f_{\boldsymbol{a}}^{i+1}(M) \xrightarrow{h} \cdots . (*)$$

By using the hypothesis $f_{\boldsymbol{a}}^{i}(M) \in \mathcal{S}$ for all $i > t$ (because $t = f.cd_{\mathcal{S}}(\boldsymbol{a}, M)$). So using the above long exact sequence $f_{\boldsymbol{a}}^{i}\left(\frac{M}{\mathrm{x}M}\right) \in \mathcal{S}$ for all $i > t$. By induction hypothesis, $\frac{f_{\boldsymbol{a}}^{t}\left(\frac{M}{\mathrm{x}M}\right)}{\boldsymbol{a}f_{\boldsymbol{a}}^{t}\left(\frac{M}{\mathrm{x}M}\right)} \in \mathcal{S}$ because $\dim\left(\frac{M}{xM}\right) = \dim(M) - 1$.

Afterwards from the exact sequence $(*)$ we get the following short exact sequence.

$$0 \to Im(f) \to f_{\boldsymbol{a}}^{t}\left(\frac{M}{\mathrm{x}M}\right) \to Im(g) \to 0$$

So we obtain the following long exact sequence.

$$\ldots \to Tor_{1}^{R}\left(\frac{R}{\boldsymbol{a}}, Im(g)\right) \to \frac{Im(f)}{\boldsymbol{a}Im(f)} \to \frac{f_{\boldsymbol{a}}^{t}\left(\frac{M}{\mathrm{x}M}\right)}{\boldsymbol{a}f_{\boldsymbol{a}}^{t}\left(\frac{M}{\mathrm{x}M}\right)} \to \frac{Im(g)}{\boldsymbol{a}Im(g)} \to 0.$$

Since $f_{\boldsymbol{a}}^{t}(M) \in \mathcal{S}$ and $Im(g)$ is a submodule of $f_{\boldsymbol{a}}^{t+1}(M)$, we deduce that $Tor_{1}^{R}(\frac{R}{\boldsymbol{a}}, Im(g)) \in \mathcal{S}$. On the other hand, $\frac{f_{\boldsymbol{a}}^{t}\left(\frac{M}{\mathrm{x}M}\right)}{\boldsymbol{a}f_{\boldsymbol{a}}^{t}\left(\frac{M}{\mathrm{x}M}\right)} \in \mathcal{S}$. Therefore, $\frac{Im(f)}{\boldsymbol{a}Im(f)} \in \mathcal{S}$ by the above long exact sequence.

Now, consider the following long exact sequence.

$$\frac{f_{\boldsymbol{a}}^{t}(M)}{\boldsymbol{a}f_{\boldsymbol{a}}^{t}(M)} \xrightarrow{x} \frac{f_{\boldsymbol{a}}^{t}(M)}{\boldsymbol{a}f_{\boldsymbol{a}}^{t}(M)} \to \frac{Im(f)}{\boldsymbol{a}Im(f)} \to 0.$$

So, $\frac{f_{\boldsymbol{a}}^{t}(M)}{\boldsymbol{a}f_{\boldsymbol{a}}^{t}(M)} \cong \frac{Im(f)}{\boldsymbol{a}Im(f)}$ because $x \in \boldsymbol{a}$. Consequently, $\frac{f_{\boldsymbol{a}}^{t}(M)}{\boldsymbol{a}f_{\boldsymbol{a}}^{t}(M)} \in \mathcal{S}$.

Proposition 3.8. For a finitely generated R- module M,

$$f.cd_{\mathcal{S}}(\boldsymbol{a}, M) = max\,\{f.cd_{\mathcal{S}}\left(\boldsymbol{a}, \frac{R}{P}\right) | P \in Ass_{R}(M)\}.$$

Proof. Set $N := \bigoplus_{P \in Ass_{R}(M)} \frac{R}{P}$. Then $Supp_{R}(M) = Supp_{R}(N)$. So, by Theorem 3.2 and Corollary 3.5, $f.cd_{\mathcal{S}}(\boldsymbol{a}, M) = f.cd_{\mathcal{S}}(\boldsymbol{a}, N) = max\,\{f.cd_{\mathcal{S}}\left(\boldsymbol{a}, \frac{R}{P}\right) | P \in Ass_{R}(M)\}$.

Proposition 3.9. Assume that $\boldsymbol{a}$ is an ideal of the local ring $(R, \boldsymbol{m})$. Then $Hom_{R}(\frac{R}{\boldsymbol{m}}, f_{\boldsymbol{a}}^{0}(M)) \in \mathcal{S}$ if and only if. $Hom_{R}(\frac{R}{\boldsymbol{m}}, \widehat{M}^{\boldsymbol{a}}) \in \mathcal{S}$.

Proof. It is enough to consider the following isomorphisms

$$Hom_{R}\left(\frac{R}{\boldsymbol{m}}, f_{\boldsymbol{a}}^{0}(M)\right) \cong \varprojlim_{n \in \mathbb{N}} Hom_{R}\left(\frac{R}{\boldsymbol{m}}, H_{\boldsymbol{m}}^{0}\left(\frac{M}{\boldsymbol{a}^{n}M}\right)\right) \cong \varprojlim_{n \in \mathbb{N}} Hom_{R}\left(\frac{R}{\boldsymbol{m}}, \frac{M}{\boldsymbol{a}^{n}M}\right)$$

$$\cong Hom_{R}\left(\frac{R}{\boldsymbol{m}}, \varprojlim_{n \in \mathbb{N}} \frac{M}{\boldsymbol{a}^{n}M}\right) \cong Hom_{R}\left(\frac{R}{\boldsymbol{m}}, \widehat{M}^{\boldsymbol{a}}\right).$$

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