A Characterization of Commutative Rings Satisfying Dccr

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Abstract

The modules (rings) satisfying acc on certain submodules investigated in [2] and various important properties of Noetherian modules and rings can be generalized to modules and rings of this class. The present author introduced and developed the concept of modules (rings) satisfying dcc on certain submodules in [5] and [6]. In this paper we present a new characterization of rings satisfying dcc on certain submodules.

1. Introduction

All rings considered in this paper are assumed to be commutative with non-zero identidy. Let M be a module over a ring A. M is said to satisfy accr (resp. accr*) if the ascending chain of submodules of the form $N:I \subseteq N:I^2 \subseteq \ldots$ terminates for every submodule N of M and every finitely generated (resp. principal) ideal I of A. The class of such modules investigated in [2].

The author introduced and developed the dual concept, i.e., the concept of modules (rings) satisfying dcc on certain submodules [5]. The A-module M is said to satisfy dccr (resp. dccr*) if the descending chain of submodules of M of the form $IN \supseteq I^2N \supseteq \ldots$ terminates for every submodule N of M and every finitely generated (resp. principal) ideal I of A. A ring satisfies dccr (dccr*) if it does as a modules over itself. Satisfying dccr and dccr* are equivalent properties of modules (rings) [5, Theorem]. In [5] and [6] among the other things, characterization of modules (resp. rings) satisfying dccr was given, see [5, Theorem] (res. [6. Corollary 1.6]).

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In this note we present a further characterization of rings satisfying dccr. We recall the following definition of [1]. The A-module M is said to be semi-Hopfian (resp. semi-co Hopfian) if for any a∊A, the endomorphism of M given by multiplication by a is an isomorphism, provided that it is surjective (resp. injective).

We establish the following characterization of rings satisfying dccr which is the main result of this note.

Theorem 1.1. The following are equivalent (A is a ring)

(i) A satisfies dccr,

(ii) Every A- module is semi-co Hopfian,

(iii) Every A-module is semi-Hofian,

(iv) Every A- module satisfies accr,

(v) Every A- module satisfies dccr,

(vi) A is zero dimentional,

(vii) if A is Noetherian, then (i)-(vi) are equivalent to

(viii) Every non-zero A-module of finite Goldie dimension is co-Laskerian.

2. The proof of theorem 1.1

Let us recall some definitions.

Definition 2.1.

(i) A module M over a commutative Noetherian ring is said to have finite Goldie dimension if M does not contain an infinite sum of non-zero submodules, or equivalenty, the injective hull E (M) if M decomposes as a finite direct sum of indecomposable injective submodules.

(ii) A non-zero A- module S is said to be secondary, if for any a ϵ A, the map induced by multiplication by a is either surjective or nilpotent. We say the A-module M is representable, if there are secondary submodules $N_1,\ldots,N_k$ such that $M=N_1+\ldots+N_k$. For more details about secondary representations see [3].

(iii) The non-zero A-module M is said to be co-Laskerian if every non-zero submodule of M is representable.

Lemma 2.2.

(i) Every A-module which satisfies dccr is semi-co-Hopfion,

(ii) Every A-module which satisfies dccr is semi-Hopfian.

Proof. (i) Let $a\in A$ and the endomorphism $M \xrightarrow{a} M$ be injective. By assumption there is positive integer t such that $a^tM= a^{t+1} M = \ldots$. Let $x\in M$. then $a^tx = a^{t+1} y$ for some $y\in M$. Hence $a^t (x-ay) = 0$, or $x = ay$, i.e., $M= aM$.

(i) Let $a\in A$ and the endomorphism $M \xrightarrow{a} M$ be surjective. Suppose $m\in M$ is such that $am = 0$.By assumption, there is positive integer t such that $(0:_M a^t) = (0:_M a^{t+1})$. Since $M= a^tM$, there is $m' \in M$ such that $m= a^t m'$ and so $m' \in (0:_M a^t)$. Hence $m = a^t m' =0$.

Proof of Theorem 1.1.

(i) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi) by [6,Corollary 1,6].

(ii) $\Rightarrow$ (iii) Let M be an A-module and D(.): = Hom_A (.,E) where E is an injective cogenerator of A. Let $a\in A$ be such that the map $M \xrightarrow{a} M$ is surjective. Then the map $D(M) \xrightarrow{a} D(M)$ is injective and this map is also surjective since D (M) is semi-co Hopfian by our assumption. But the functor D (.) is faithfully exact and so M is semi-Hopfian.

(iii) $\Rightarrow$ (ii) The prof is similar to (ii) $\Rightarrow$ (iii).

(ii) $\Rightarrow$ (vi) Let $P \in spec(A)$ and $m \in max(A)$ be such that $p\subseteq m$. Let $a \in m\backslash p$. then $\frac{A}{p} \xrightarrow{\alpha} \frac{A}{p}$ is injective and so surjective and this is a contradiction.

Now suppose A is Noetherian and we show (v) $\Rightarrow$ (vii) and (vii) $\Rightarrow$ (i).

(v) $\Leftrightarrow$ (vii) Follows from [5; Proposition 3].

Corollary 2.3. Let M be an A-module such that $\frac{A}{Ann_A M}$ satisfies dccr. Then M satisfies both accr and dccr.

Proof. Let $\overline{A} := \frac{A}{Ann_A M}$. Then M has a natural structure as $\overline{A}$ - module. Futther a subset N of M is an A- submodule of M it and only if it is an $\overline{A}$ submodule. Thus M satisfies accr (resp. dccr) as A- module if and only if it satisfies accr (resp. dccr) as $\overline{A}$ -module. Hence the result follows from Theorem 1.1.

For the next corollary, we recall from [4, section 3] that the A- module M is called special if for each x∈ M and every element α of any maximal ideal m, there exists n∈ N and c∈A\m such that $ca^n x = 0$. Whenever M is special and finitely generated then the ring $\frac{A}{Ann_A M}$ is zero dimensional. To see this, let $M=Ax_2 + \ldots + Ax_r$ and I= {r∈ A|rx_i = 0}. Then I= Ann M. Let p∈ spec (A) such that p ⊇ I. then there exists m∈ max (A) such that p ⊆ m. Let a∈m. by hypothesis, there is n∈ N, c∈A\m such that ca^n∈I ⊆ p. Hence a∈p and so P=m. So by Theorem 1.1 and corollary 2.3 get:

Corollary2.4. Let M be a finitely generated special A- module. Then M satisfies both accr and dccr.

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