

# On Weak McCoy Rings

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## Abstract

In this note we introduce the notion of weak McCoy rings as a generalization of McCoy rings, and investigate their properties. Also we show that, if  $R$  is a semi-commutative ring, then  $R$  is weak McCoy if and only if  $R[x]$  is weak McCoy.

## 1. Introduction

Throughout this paper, all rings are associative with identity. For a commutative ring  $R$ , McCoy [10] obtained the following result: If  $f(x)g(x) = 0$  for some non-zero polynomials  $f(x), g(x) \in R[x]$ , then  $f(x)c = 0$  for some non-zero  $c \in R$ . According to Nielsen [12], a ring  $R$  is called *right McCoy* whenever polynomials  $f(x), g(x) \in R[x] - \{0\}$  satisfy  $f(x)g(x) = 0$ , there exists a non-zero  $r \in R$  such that  $f(x)r = 0$ . Left McCoy rings are defined similarly. If a ring is both left and right McCoy, we say that the ring is a *McCoy ring*. It is well known that commutative rings are always McCoy rings [10], but it is not true for non-commutative rings (see [12]).

Recall that a ring  $R$  is called:

*reduced* if  $a^2 = 0 \Rightarrow a = 0$ , for all  $a \in R$ ,

*reversible* if  $ab = 0 \Rightarrow ba = 0$ , for all  $a, b \in R$ ,

*symmetric* if  $abc = 0 \Rightarrow acb = 0$ , for all  $a, b, c \in R$ ,

*semi-commutative* if  $ab = 0 \Rightarrow aRb = 0$ , for all  $a, b \in R$ .

The following implications hold:

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reduced  $\Rightarrow$  symmetric  $\Rightarrow$  reversible  $\Rightarrow$  semi-commutative.

Reversible rings are McCoy rings by [12]. But the converse is not true; there exists a non-reversible McCoy ring (see [12]).

Motivated by the above, as a generalization of McCoy rings, in this paper we introduce the notion of weak McCoy rings and investigate their properties and extend several known results relating to McCoy rings to a general setting.

For a ring  $R$ , we denote by  $nil(R)$  the set of all nilpotent elements of  $R$ , by  $N_*(R)$  the prime radical of  $R$  and by  $M_n(R)$ ,  $U_n(R)$  and  $L_n(R)$  the  $n \times n$  matrix ring over  $R$ , the  $n \times n$  upper and lower triangular matrix rings over  $R$  respectively.

## 2. On Weak McCoy rings

**Definition 2.1.** We say  $R$  is a *weak McCoy ring* if  $f(x)g(x) \in nil(R[x])$  implies  $f(x)c \in nil(R[x])$ , for some non-zero  $c \in R$ , where  $f(x)$  and  $g(x)$  are non-zero polynomials in  $R[x]$ .

**Remark 2.2.** Since  $ab$  is nilpotent if and only if  $ba$  is nilpotent in a ring, hence the definition of weak McCoy rings is left-right symmetric.

**Proposition 2.3.** McCoy rings are weak McCoy.

**Proof.** Let  $R$  be a McCoy ring and  $f(x)g(x) \in nil(R[x])$  for non-zero polynomials  $f(x), g(x) \in R[x]$ . Then there exists  $m, n \geq 1$ , such that  $(f(x)g(x))^n = (g(x)f(x))^m = 0$ , and  $(f(x)g(x))^{n-1}, (g(x)f(x))^{m-1} \neq 0$ . If  $f(x)g(x) = 0$  or  $g(x)f(x) = 0$ , then the result follows from the definition of McCoy rings. Assume  $f(x)g(x) \neq 0 \neq g(x)f(x)$  and  $0 = (f(x)g(x))^n = f(x)(g(x)f(x) \dots f(x)g(x)) = f(x)h(x)$ .

If  $h(x) = g(x)f(x) \dots f(x)g(x) \neq 0$ , then  $f(x)c = 0$  for some non-zero  $c \in R$ , since  $R$  is McCoy.

Let  $h(x) = g(x)(f(x)g(x) \dots f(x)g(x)) = g(x)(f(x)g(x))^{n-1} = 0$ . Since  $(f(x)g(x))^{n-1} \neq 0$  and  $R$  is McCoy, there exists  $0 \neq d \in R$  such that  $g(x)d = 0$ . Therefore  $f(x)c = 0$  or

$g(x)d = 0$  for some non-zero  $c, d \in R$ . Hence  $f(x)c \in \text{nil}(R[x])$  or  $dg(x) \in \text{nil}(R[x])$  for some non-zero  $c, d \in R$ . Therefore  $R$  is weak McCoy.

**Proposition 2.4.** Let  $R$  be a ring. Then  $U_n(R)$  and  $L_n(R)$  are weak McCoy for each  $n \geq 2$ .

**Proof.** Clearly  $U_n(R)[x] \cong U_n(R[x])$  and for each  $A = \begin{bmatrix} 0 & f_{12} & \cdots & f_{1n} \\ 0 & 0 & \cdots & f_{2n} \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \in U_n(R[x]),$

$A^n = 0$ . Let  $0 \neq A = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ 0 & f_{22} & \cdots & f_{2n} \\ 0 & 0 & \ddots & \cdots \\ 0 & 0 & \cdots & f_{nn} \end{bmatrix} \in U_n(R[x]).$  Then

$$A \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \ddots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} 0 & g_{12} & \cdots & g_{1n} \\ 0 & 0 & \cdots & g_{2n} \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \quad \text{and} \quad \left( A \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 1 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \right)^n = 0. \quad \text{Hence}$$

$U_n(R)$  is weak McCoy. By a similar argument one can show that  $L_n(R)$  is weak McCoy.

**Proposition 2.5.** Let  $R$  and  $S$  be rings and  ${}_R M_S$  a bimodule. Then  $\begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$  is a weak McCoy ring.

**Proof.** Similarly, as used in Proposition 2.4 one can prove it.

The following example shows that  $U_n(R)$  and  $M_n(R)$  are neither left nor right McCoy for some  $n \geq 2$ .

**Example 2.6.** Let  $R$  be a ring. We show that  $U_4(R)$  and  $M_4(R)$  are neither right nor

left McCoy. Let  $f(x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x$  and

$$g(x) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x \in U_4(R)[x] \subseteq M_4(R)[x]. \text{ Then } f(x)g(x) = 0.$$

$$\text{If } f(x)A = 0, \text{ for some } A = [a_{ij}] \in M_4(R), \text{ then } 0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{and } 0 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} A = \begin{bmatrix} -a_{21} & -a_{22} & -a_{23} & -a_{24} \\ 0 & 0 & 0 & 0 \\ -a_{41} & -a_{42} & -a_{43} & -a_{44} \\ 0 & 0 & 0 & 0 \end{bmatrix}. \text{ Hence } A = 0 \text{ and } U_4(R)$$

and  $M_4(R)$  are not right McCoy. If  $Bg(x) = 0$  for some  $B \in M_4(R)$ , then by a similar way as above, we can show  $B = 0$ . Therefore  $U_4(R)$  and  $M_4(R)$  are not left McCoy.

**Definition 2.7.** A ring  $R$  is called *right Ore* if given  $a, b \in R$  with  $b$  regular there exist  $a_1, b_1 \in R$  with  $b_1$  regular such that  $ab_1 = ba_1$ . It is well-known that  $R$  is a right Ore ring if and only if the classical right quotient ring of  $R$  exists. We use  $C(R)$  to denote the set of all regular elements in  $R$ .

**Theorem 2.8.** Let  $R$  be a right Ore ring with its classical right quotient ring  $Q$ . If  $R$  is weak McCoy then  $Q$  is weak McCoy.

**Proof.** Let  $0 \neq F(x) = \sum_{i=0}^m a_i u^{-1} x^i$  and  $0 \neq G(x) = \sum_{j=0}^n b_j v^{-1} x^j$  with  $a_i, b_j \in R, u, v \in C(R)$

such that  $F(x)G(x) \in \text{nil}(Q[x])$ .

**Case1.**  $F(x)G(x) = 0$  or  $G(x)F(x) = 0$ . Assume that  $F(x)G(x) = 0$ . Since  $R$  is right Ore, there exists  $b'_j \in R$  and  $u_1 \in C(R)$  such that  $u^{-1}b_j = b'_j u_1^{-1}$  for  $j = 1, \dots, n$ . Let

$f(x) = \sum_{i=0}^m a_i x^i$  and  $g(x) = \sum_{j=0}^n b'_j x^j$ . Then  $f(x)g(x) = 0$ . Since  $R$  is weak McCoy, there exists

$0 \neq c \in R$  with  $f(x)c \in \text{nil}(R[x]) \subseteq \text{nil}(Q[x])$ . Hence  $F(x)uc = f(x)u^{-1}uc = f(x)c \in \text{nil}(Q[x])$ .

If  $G(x)F(x) = 0$ , then by a similar argument we can show that  $G(x)vd \in \text{nil}(Q[x])$  for some non-zero  $d \in R$ .

**Case2.**  $F(x)G(x) \neq 0$  and  $G(x)F(x) \neq 0$ . Since  $F(x)G(x) \in \text{nil}(Q[x])$ , there exists  $n \geq 2$  such that  $(F(x)G(x))^n = 0$  and  $(F(x)G(x))^{n-1} \neq 0$ . Let  $(F(x)G(x))^n = F(x)H(x)$ . If  $H(x) \neq 0$ , then by a similar argument as above there exists  $\alpha \in C(R)$ ,  $r \in R$  such that  $F(x)\alpha r \in \text{nil}(Q[x])$ . Now assume  $H(x) = G(x)\underbrace{F(x)G(x)\dots F(x)G(x)}_{n-1} = 0$ . Since  $(F(x)G(x))^{n-1} \neq 0$  and  $R$  is weak McCoy, then by Case 1, there exists  $\beta \in C(R)$ ,  $s \in R$  such that  $G(x)\beta s = 0$ . Therefore  $Q$  is weak McCoy.

According to Bell [2], a ring  $R$  is called semi-commutative if  $ab = 0$  implies  $aRb = 0$ . We say an ideal  $I$  is a *semi-commutative ideal*, if  $R/I$  is a semi-commutative ring.

**Lemma 2.9.** Let  $R$  be a semi-commutative ring. If  $c_1c_2\cdots c_k = 0$  for some  $c_i \in R$ , then  $c_1Rc_2Rc_3\cdots Rc_k = 0$ .

**Proof.** By induction, let  $c'_{k-1} = c_{k-1}c_k$ . Then  $c_1c_2\cdots c'_{k-1} = 0$  and by induction assumption, we have  $0 = c_1Rc_2Rc_3\cdots Rc'_{k-1} = c_1Rc_2Rc_3\cdots Rc_{k-1}c_k$ . Hence, for all  $x \in c_1Rc_2Rc_3\cdots Rc_{k-1}$ , we have  $xc_k = 0$ . It follows by hypothesis that  $xRc_k = 0$ . Thus  $c_1Rc_2Rc_3\cdots Rc_k = 0$ , as desired.

**Lemma 2.10** (4, Lemma 2.5). Let  $R$  be a semi-commutative ring. Then  $\text{nil}(R)$  is a semi-commutative ideal of  $R$ .

**Proof.** Let  $a, b \in \text{nil}(R)$ . Then  $a^n = 0 = b^m$  for some  $m, n \geq 0$ . Each term of the expansion of  $(a+b)^{m+n+1}$  has the form  $x := (a^{i_1}b^{j_1})\cdots(a^{i_{m+n+1}}b^{j_{m+n+1}})$  where  $i_r, j_s \in N \cup \{0\}$ . Since  $(i_1 + j_1) + (i_2 + j_2) + \dots + (i_{m+n+1} + j_{m+n+1}) = \sum_{r=1}^n i_r + \sum_{s=1}^m j_s = m+n+1$ , either  $\sum_{r=1}^n i_r \geq n$  or  $\sum_{s=1}^m j_s \geq m$ . If  $\sum_{r=1}^n i_r \geq n$ , then  $a^{i_1}a^{i_2}\cdots a^{i_{m+n+1}} = 0$ . Thus  $(a^{i_1}b^{j_1})\cdots(a^{i_{m+n+1}}b^{j_{m+n+1}}) = 0$ , by Lemma 2.9. If  $\sum_{r=1}^n i_r < n$ , then  $\sum_{s=1}^m j_s \geq m$ . Thus  $b^{j_1}b^{j_2}\cdots b^{j_{m+n+1}} = 0$  and so  $(a^{i_1}b^{j_1})\cdots(a^{i_{m+n+1}}b^{j_{m+n+1}}) = 0$ , by Lemma 2.9. Hence  $(a+b)^{m+n+1} = 0$ .

Now suppose that  $a^n = 0$  and  $r \in R$ . Then  $(ar)^n = 0 = (ra)^n$ , by Lemma 2.9. Thus  $\text{nil}(R)$  is an ideal of  $R$ .

Since  $R/\text{nil}(R)$  is a reduced ring, hence it is a semi-commutative ring. Therefore  $\text{nil}(R)$  is a semi-commutative ideal of  $R$ .

**Lemma 2.11.** Let  $R$  be a semi-commutative ring. Then  $\text{nil}(R[x]) = \text{nil}(R)[x]$ .

**Proof.** Let  $f(x) = a_0 + \dots + a_n x^n \in \text{nil}(R[x])$ . Then  $f(x)^k = 0$ , for some integer  $k \geq 0$ . Hence  $a_n^k = 0$ , and that  $a_n \in \text{nil}(R)$ . There exists  $g(x), h(x) \in R[x]$  such that  $f(x)^k = (a_0 + \dots + a_{n-1} x^{n-1})^k + a_n g(x) + h(x) a_n$ . Since  $\text{nil}(R)[x]$  is an ideal of  $R[x]$  and  $a_n g(x), h(x) a_n, f(x)^k \in \text{nil}(R)[x]$ , we have  $(a_0 + \dots + a_{n-1} x^{n-1})^k \in \text{nil}(R)[x]$ . Hence  $a_{n-1}^k \in \text{nil}(R)$  and that  $a_{n-1} \in \text{nil}(R)$ . Continuing this process yields  $a_0, \dots, a_n \in \text{nil}(R)$ . Therefore  $\text{nil}(R[x]) \subseteq \text{nil}(R)[x]$ .

Now, let  $f(x) = a_0 + \dots + a_n x^n \in \text{nil}(R)[x]$ . Then  $a_i^{m_i} = 0$ , for some positive integer  $m_i$ . Let  $k = m_0 + \dots + m_n + 1$ . Then  $(f(x))^k = \sum (a_0^{i_{01}} (a_1 x)^{i_{11}} \dots (a_n x^n)^{i_{n1}}) \dots (a_0^{i_{0k}} (a_1 x)^{i_{1k}} \dots (a_n x^n)^{i_{nk}})$ , where  $i_{0r} + \dots + i_{nr} = 1$ , for  $r = 1, \dots, k$  and  $0 \leq i_{rs} \leq 1$ . Each coefficient of  $f(x)^k$  is a sum of such elements  $\gamma = (a_0^{i_{01}} \dots (a_n)^{i_{n1}}) \dots (a_0^{i_{0k}} \dots (a_n)^{i_{nk}})$ , where  $i_{0r} + \dots + i_{nr} = 1$ .

It can be easily checked that there exists  $a_k \in \{a_0, \dots, a_n\}$  such that  $i_{t1} + \dots + i_{tk} \geq m_t$ . Since  $a_t^{m_t} = 0$  and  $R$  is semi-commutative,  $\gamma = 0$ . Thus  $(f(x))^k = 0$  and  $\text{nil}(R)[x] \subseteq \text{nil}(R[x])$ . Therefore  $\text{nil}(R[x]) = \text{nil}(R)[x]$ .

**Lemma 2.12.** Let  $R$  be a semi-commutative ring. Then  $\text{nil}(R[x][y]) = \text{nil}(R[x])[y]$ .

**Proof.** By Lemma 2.11,  $\text{nil}(R[x])$  is an ideal of  $R[x]$ . Since  $R[x]/\text{nil}(R[x])$  is a reduced ring, hence  $\text{nil}(R[x])$  is a semi-commutative ideal of  $R[x]$ , and that  $\text{nil}(R[x])[y] \subseteq \text{nil}(R[x][y])$ .

Now, let  $F(y) = \sum_{i=0}^m f_i y^i \in \text{nil}(R[x][y])$ , where  $f_i = \sum_{s=0}^{p_i} a_{is} x^s \in R[x]$ . Then  $F(y)^n = 0$ , for some positive integers  $n$ . As in the proof of [1], let  $k = n \sum \deg f_i$ , where the degree is as polynomial in  $x$  and the degree of zero polynomial is taken to be 0. Then  $(F(x^k))^n = 0$  and the set of coefficients of  $F(x^k)$  is equal to the set of all coefficients of  $f_i$ ,  $0 \leq i \leq m$ . Hence by Lemma 2.11,  $a_{ij} \in \text{nil}(R)$  for all  $i, j$  and that  $f_i \in \text{nil}(R[x])$ , for each  $i$ . Thus  $F(y) \in \text{nil}(R[x])[y]$ . Therefore  $\text{nil}(R[x][y]) = \text{nil}(R[x])[y]$ .

If  $R$  is semi-commutative, then  $R[x]$  may not be semi-commutative, by [5, Example 2]). Here we will show that if  $R$  is semi-commutative, then  $R$  is weak McCoy if and only if  $R[x]$  is weak McCoy.

**Theorem 2.13.** If  $R$  is a semi-commutative ring, then  $R[x]$  is a weak McCoy ring if and only if  $R$  is weak McCoy.

**Proof.** Suppose that  $R$  is a weak McCoy ring. Let  $F(t) = \sum_{i=0}^m f_i t^i$ ,  $G(t) = \sum_{j=0}^n g_j t^j$  be non-zero polynomials in  $R[x][t]$  such that  $F(t)G(t) \in \text{nil}(R[x][t])$ , where  $f_i = \sum_{s=0}^{p_i} a_{is} x^s$ ,  $g_j = \sum_{t=0}^{q_j} b_{jt} x^t \in R[x]$ . As in the proof of [1], let  $k = \sum \deg f_i + \sum \deg g_j$ , where the degree is as polynomial in  $x$  and the degree of zero polynomial is taken to be 0. Then  $F(x^k) = \sum_{i=0}^m f_i x^{ik}$ ,  $G(x^k) = \sum_{j=0}^n g_j x^{jk} \in R[x]$ , and the set of coefficients of the  $F(x^k)$  is (respectively  $G(x^k)$ ) equal to the set of all coefficients of  $f_i$ ,  $0 \leq i \leq m$  (respectively  $g_j$ ,  $0 \leq j \leq n$ ). Since  $(F(t)G(t))^p = 0$ , for some  $p \geq 1$ , and  $x$  commutes with elements of  $R$ ,  $(F(x^k)G(x^k))^p = 0$ . Since  $R$  is weak McCoy, there is  $0 \neq r \in R$  such that  $F(x^k)r \in \text{nil}(R[x])$  and  $a_{is}r \in \text{nil}(R)$ ,  $f_i r \in \text{nil}(R[x])$  for  $0 \leq i \leq m$ ,  $0 \leq s \leq p_i$  by Lemma 2.11. Hence  $F(t)r \in \text{nil}(R[x][t])$ , by Lemma 2.12. Therefore  $R[x]$  is weak McCoy.

Now suppose  $R[x]$  is a weak McCoy ring and  $f(t)g(t) \in \text{nil}(R[t]) \subseteq \text{nil}(R[x][t])$ . Since  $R[x]$  is weak McCoy, there exists  $0 \neq h(x) \in R[x]$  such that  $f(t)h(x) \in \text{nil}(R[x][t])$ . Let  $h(x) = a_0 + \dots + a_n x^n \in R[x]$  ( $a_0 \neq 0$ ). Then  $f(t)a_0 \in \text{nil}(R[t])$ , since  $(f(t)h(x))^k = (f(t)a_0)^k + k_1 x + \dots + k_{nk} x^{nk}$  with  $k_1, \dots, k_{nk} \in R[t]$ . Therefore  $R$  is weak McCoy.

**Theorem 2.14.** Let  $R$  be a ring and  $\Delta$  a multiplicatively closed subset of  $R$  consisting of central regular elements. Then  $R$  is weak McCoy if and only if  $\Delta^{-1}R$  is weak McCoy.

**Proof.** If  $R$  is a weak McCoy ring, then by a similar way as used in Theorem 2.8, one can show that  $\Delta^{-1}R$  is weak McCoy.

Conversely, let  $\Delta^{-1}R$  be a weak McCoy ring. Let  $f(x) = \sum_{i=0}^m a_i x^i$  and  $g(x) = \sum_{j=0}^n b_j x^j$  be non-zero polynomials of  $R[x]$  such that  $f(x)g(x) \in \text{nil}(R[x])$ . Since  $\Delta^{-1}R$  is weak McCoy,  $f(x)(c\alpha^{-1}) \in \text{nil}((\Delta^{-1}R)[x])$  for some non-zero  $c\alpha^{-1} \in \Delta^{-1}R$ . Thus  $f(x)c \in \text{nil}(R[x])$  and  $R$  is weak McCoy.

**Corollary 2.15.** Let  $R$  be a ring. Then  $R[x]$  is weak McCoy if and only if  $R[x, x^{-1}]$  is weak McCoy.

**Proof.** Clearly  $\Delta = \{1, x, x^2, \dots\}$  is a multiplicatively closed subset of  $R[x]$  consisting of central regular elements and  $\Delta^{-1}R[x] = R[x, x^{-1}]$ . Hence the proof follows from Theorem 2.14.

**Theorem 2.16.** The classes of weak McCoy rings are closed under direct limits.

**Proof.** Let  $A = \{R_i, \alpha_{ij}\}$  be a direct system of weak McCoy rings  $R_i$  for  $i \in I$  and ring homomorphisms  $\alpha_{ij} : R_i \rightarrow R_j$  for each  $i \leq j$  with  $\alpha_{ij}(1) = 1$ , where  $I$  is a directed partially ordered set. Let  $R = \varinjlim R_i$  be the direct limit of  $A$  with  $\ell_i : R_i \rightarrow R$  and  $\ell_j \alpha_{ij} = \ell_i$ . We show that  $R$  is weak McCoy ring. Let  $a, b \in R$ . Then  $a = \ell_i(a_i)$ ,  $b = \ell_j(b_j)$  for some  $i, j \in I$  and there is  $k \in I$  such that  $i \leq k, j \leq k$ . Define  $a + b = \ell_k(\alpha_{ik}(a_i) + \alpha_{jk}(b_j))$  and  $ab = \ell_k(\alpha_{ik}(a_i)\alpha_{jk}(b_j))$ , where  $\alpha_{ik}(a_i), \alpha_{jk}(b_j) \in R_k$ . Then  $R$  forms a ring with  $0 = \ell_i(0)$  and  $1 = \ell_i(1)$ . Let  $f, g \in R[x]$  be non-zero polynomials such that  $fg \in \text{nil}(R[x])$ . There is  $k \in I$  such that  $f, g \in R_k[x]$ . Hence  $fg \in \text{nil}(R_k[x])$ . Since  $R_k$  is weak McCoy, there exists  $0 \neq c_k \in R_k$  such that  $fc_k \in \text{nil}(R_k[x])$ . If  $c = \ell_k(c_k)$ , then  $fc \in \text{nil}(R[x])$  with non-zero  $c$ . Therefore  $R$  is weak McCoy.

**Proposition 2.17.** (1) Let  $R$  be a ring. If there exists a non-zero ideal  $I$  of  $R$  such that  $I[x] \subseteq \text{nil}(R[x])$ , then  $R$  is weak McCoy.



(2) Every non-semiprime ring is weak McCoy.

(3) Let  $R$  be a ring with a non-zero nilpotent ideal. Then  $Mat_n(R)$  ( $n \geq 2$ ) is weak McCoy.

**Proof.** (1) Let  $0 \neq f \in R[x]$ . If  $f \in I[x]$ , then  $fr \in nil(R[x])$  for all  $r \in R$ . If  $f \notin I[x]$  then  $fs \in I[x] \subseteq nil(R[x])$  for all non-zero  $s \in I$ . Thus  $R$  is weak McCoy.

(2) Let  $R$  be a ring with  $N_*(R) \neq 0$ . Since  $0 \neq N_*(R)[x] = N_*(R[x]) \subseteq nil(R[x])$ ,  $R$  is weak McCoy by (1).

(3) Since  $Mat_n(R)$  is non-semiprime, hence by (1)  $Mat_n(R)$  is weak McCoy.

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