# Contractibility and idempotents in Banach algebras

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### Abstract

Let A be a Banach algebra. It is shown that a contractible ideal of a Banach algebra is complemented by its annihilator. Then, it is proved the existence of minimal central idempotents in a contractible Banach algebra with a nonzero character. Moreover, the notion of b-contractibility and one of its equivalent forms are introduced. Through an example, it is shown that b-contractibility is strictly weaker than contractibility.

# Introduction

Taylor in [13, Theorem 5.11] showed that a contractible Banach algebra with bounded approximation property is finite dimensional. Johnson in [6, Proposition 8.1] showed that a contractible commutative semisimple Banach algebra is finite dimensional. Curtis and Loy [1, Theorem 6.2] extended this result by dropping the semisimplicity assumption. But the question for noncommutative case has remained open. For more results of this type see [4],[5], [8], [10], [13].

This paper is organized as follows. In the second section, we show that a contractible ideal of a Banach algebra is controlled by its commutant and annihilator. Then, we prove the existence of minimal central idempotents in a contractible Banach algebra with a nonzero character. In the third section, we introduce a weaker version of contractibility which we call b-contractibility. We give a characterization of b-contractibility analog to that of contractibility given by Taylor. Also, we show that b-contractibility is strictly weaker than contractibility.

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First we recall some terminology. Throughout this paper, A is a Banach algebra and A-module means Banach A-bimodule. For a subset E of A, E' is the commutant of E. If for every A-bimodule X every bounded derivation from A into X is inner, then A is called *contractible*. Also, the term "semisimple" means Jacobson semisimple. An idempotent  $e \in A$  is called *minimial* if eAe is a division ring. If e and f are idempotents in A, we write  $e \leq f$  if fe = ef = e holds. A nonzero idempotent  $e \in A$  is called *primitive* if  $0 \leq f \leq e$  imlies that f = 0 or f = e. Also, two idempotents e and f are said to be *orthogonal* if they satisfy ef = fe = 0. Let S be a subset of A. The *right annihilator* of S in A which we denote by ran(S) is the set

$$ran(S) = \{a \in A : ba = 0 \text{ for } b \in S\}.$$

The left annihilator lan(S) is defined semilarly. The *annihilator* of S is the set  $Ann(S) = ran(S) \cap lan(S)$ .

## Contractibility

**Theorem 2.1.** Let A be a contractible Banach algebra which is an ideal in a Banach algebra B. Then A + A' = B.

*Proof.* If  $A + A' \neq B$ , then we can choose  $b \in B - (A + A')$ . Now define

$$D: A \to A, x \mapsto xb - bx.$$

Clearly *D* is a derivation on *A*. By assumption there exists an  $a \in A$  such that D(x) = xa - ax for all  $x \in A$ . The latter result implies that  $b - a \in A'$  or equivalently  $b \in A + A'$  which contradicts the selection of *b*. Therefore A + A' = B.

**Theorem 2.2.** Let A be a contractible Banach algebra which is an ideal in a Banach algebra B. Then  $B = A \oplus Ann(A)$ .

*Proof.* Since A is contractible then  $M_2(A)$  with  $l^1$ -norm is contarctible, where  $M_2(A)$  is the algebra of  $2 \times 2$  matrices with the enteries from A. On the other hand  $M_2(A)$  is an ideal in  $M_2(B)$  and by Theorem 2.1 we have the equality  $M_2(B) = M_2(A) + M_2(A)'$ . One can easily observe that

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$$M_2(A)' = \begin{bmatrix} A' & Ann(A) \\ Ann(A) & A' \end{bmatrix}.$$

Thus B = A + Ann(A). But  $A \cap Ann(A) = 0$ , because A is unital. Therefore the identity  $B = A \oplus Ann(A)$  holds.

**Remark.** In Theorems 2.1 and 2.2, A and B are related only algebrically. Indeed if there exists an infinite dimensional contractible Banach algebra A which is an ideal in a Banach algebra B, then the norm topology of A could be different from the relative norm topology of A which inherits from B.

**Theorem 2.3.** Let *A* be a contractible Banach algebra which admits a nonzero multiplicative linear functional *f*. Then *A* contains a central minimal idempotent. *Proof.* Let  $d = \sum_{n=1}^{\infty} a_n \otimes b_n$  be a diagonal for *A* and define

$$T: A, \to a \mapsto \sum_{n=1}^{\infty} < f, aa_n > b_n$$

Since  $\sum_{n} a_n b_n = 1$ , then

$$< f, T(1) >=< f, \sum_{n} < f, a_{n} > b_{n} >= \sum_{n} < f, a_{n} >< f, b_{n} >$$
  
=  $\sum_{n} < f, a_{n}b_{n} >=< f, \sum_{n} a_{n}b_{n} >=< f, 1 >= 1.$ 

Thus T(1)  $\neq 0$ . Moreover for every  $a \in A$  and  $g, h \in A^*$  we have

$$< h, \sum_{n} < g, aa_{n} > b_{n} >= \sum_{n} < g, aa_{n} > < h, b_{n} >$$
$$= < g \otimes h, \sum_{n} aa_{n} \otimes b_{n} >$$
$$= < g \otimes h, \sum_{n} a_{n} \otimes b_{n} a >$$
$$= \sum_{n} < g, a_{n} > < h, b_{n} a >$$
$$= < h, \sum_{n} < g, a_{n} > b_{n} a >.$$

This implies that

$$\sum_{n} \langle g, aa_{n} \rangle b_{n} = \sum_{n} \langle g, a_{n} \rangle b_{n}a.$$

Thus we assume that

T(1)=e, then we have  $T(a) = \sum_{n} \langle f, aa_{n} \rangle b_{n} = \sum_{n} \langle f, a_{n} \rangle b_{n} a = ea$ . On the other hand we have  $T(a) = \sum_{n} \langle f, aa_{n} \rangle b_{n} = \langle f, a \rangle \sum_{n} \langle f, a_{n} \rangle b_{n} = \langle f, a \rangle e$ . Hence T is an operator of rank one and  $e^{2} = T(e) = \langle f, e \rangle e = e$ . Now define

$$T_1: A \to A, a \mapsto \sum_n a_n < f, aa_n > 0$$

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With a similar argument we can show that

$$T_1(a) = ae' = < f, a > e' \quad a \in A$$

where  $e' = T_1(1)$ . Also we have  $e'^2 = e'$  and  $\langle f, e' \rangle = 1$ . Now the identities

$$ee' = \langle f, e' \rangle e = e, \qquad ee' = \langle f, e \rangle e' = e'$$

imply that e = e' and for every  $a \in A$  we have

$$ea = < f, a > e = < f, a > e' = ae' = ae.$$

Therefore *e* is a central idempotent. In addition since *T* is a rank one operator and ranT = eAe, then eA = eAe = Ce is a division ring. Therefore *e* is a minimal idempotent.

## **b**-Contractibility

**Definition.** Let A be a Banach algebra and  $\pi$  be the natural map,

$$\pi : A \otimes A \longrightarrow A, \quad \pi (\sum_{n} a_{n} \otimes b_{n}) \rightarrow \sum_{n} a_{n} b_{n}.$$

Let  $b \in A$  and X be an A-module. We say that a derivation  $\delta A \longrightarrow X$  is a *b*derivation if there exists another derivation  $\delta' A \longrightarrow X$  such that  $\delta = b\delta'$ , where  $(b\delta')(a) = b\delta'(a)$ . Also we say that A is *b*-contractible if for every A-module X, every bounded *b*-derivation from A into X is inner. We call  $d \in A \otimes A$  a *b*-diagonal if  $\pi(d) = b$  and a.d = d.a for all  $a \in A$ .

**Theorem 3.1.** Let A be a unital Banach algebra and  $b \in A' - \{0\}$ . Then A is b-contractible if and only if A has a b-diagonal.

*Proof.* First suppose A is b -contractible and  $\pi$  is defined as above. Clearly ker $\pi$  is an A -module and if we define

$$\delta : A \to \ker \pi, a \mapsto ab \otimes 1 - b \otimes a$$

then it is easy to see that  $\delta$  is a *b*-derivation. Indeed  $\delta = b\delta'$  where

$$\delta' : A \to \ker \pi, a \mapsto a \otimes 1 - 1 \otimes a$$

ince A is b-contractible, then three exists an element  $\sum_{n} c_n \otimes d_n \in \ker \pi$  such that

$$\delta(a) = \sum_{n} ac_{n} \otimes d_{n} - \sum_{n} c_{n} \otimes d_{n} a \quad a \in A.$$

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Let  $d = b \otimes 1 - \sum_{n} c_{n} \otimes d_{n}$ . The above identities show that  $\pi(d) = b$  and a.d = d.afor all  $a \in A$ . Therefore, d is a b-diagonal for A. Conversely suppose  $d = \sum_{n} a_{n} \otimes b_{n}$  is a b-diagonal for A, X is an A-module and

 $\delta : A \longrightarrow X$  is a bounded derivation. Clearly the map

$$\psi : A \times A \to X, (a,c) \mapsto a\delta(c)$$

is a bounded bilinear map. So by the universal property of projective tensor product there is a linear map  $\varphi : A \otimes A \longrightarrow X$  such that  $\varphi \circ \otimes = \psi$  that is  $\varphi(a \otimes c) = a\delta(c)$ . In particular using the fact that *d* is a *b*-diagonal for *A*, we get

$$\sum aa_n \delta(b_n) = \varphi(a.d) = \varphi(d.a) = \sum a_n \delta(b_n a), \quad a \in$$

Now if  $x = \sum_{n} a_n \delta(b_n)$ , then for every  $a \in A$  we have

$$ax - xa = \sum_{n} aa_{n}\delta(b_{n}) - \sum_{n} a_{n}\delta(b_{n})a = \sum_{n} aa_{n}\delta(b_{n}) + b\delta(a) - \sum_{n} a_{n}\delta(b_{n}a).$$

Thus the identity  $ax - xa = b\delta(a)$  holds for every  $a \in A$ . Therefore every b-derivation is inner.

**Example 3.2.** Let A be the Banach algebra  $l_1(N)$  with pointwise multiplication and  $\{e_n\}$  be the standard basis for A. Then for every positive integer n, A is  $e_n$ -contractible. Indeed  $e_n \otimes e_n$  is an  $e_n$ -diagonal for A. But A is not contractible, since it is not unital. Therefore b -contractibility dose not imply contractibility.

**Remark.** If A is contractible, then it is unital and one can easily observe that A is bcontractible for every  $b \in A - \{0\}$ . However the above example shows that for nonunital Banach algebras the converse is not true. We do not know whether this is true for unital Banach algebras or not.

**Problem.** Does there exist a unital Banach algebra which is b-contactible for some nonzero central idempotent b, but is not contractible?

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