

# Hypergroup Structures with Regular Multiplications

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## Abstract

In Banach algebras, the group algebra  $L(G)$  is Arens regular if and only if  $G$  is finite. In this paper, the researcher has obtained a hypergroup structure (in the sense of Dunkl) whose measure algebra has regular multiplication. The most interesting result was that if  $L(X)$  is Arens regular then the convolution is Arens regular as a bilinear map. The condition obtained gives regularity of multiplication in the Hypergroup, which  $X$  is not finite.

## Introduction

The regularity of a bounded bilinear mapping was defined by Arens (see [1]). For some important Banach algebras, the first and the second Arens product, on their second dual, are different. Therefore, these algebras are not Arens regular. A number of Banach algebras, commonly occurring in functional and harmonic analysis, are not Arens regular. The group algebra  $L(G)$  of a locally compact Hausdorff group is Arens regular if and only if  $G$  is finite (see [4], [13], [14]). In [9], the researcher has shown that the hypergroup algebra  $L(X)$ , where  $X$  is a locally compact Hausdorff space, can be Arens regular without  $X$  being finite. Also, in [10], for a general measure algebra  $\mathcal{L}$ , in  $M(X)$ , if  $e \in X$  is not isolated in  $\text{supp } \mathcal{L}$ , and that  $\delta_e$  acts as an identity for  $\mathcal{L}$ , then  $\mathcal{L}$  is not Arens regular. These are not the only ways to construct the regular or irregular multiplications. In [8], for the circle group  $T$ , two multiplications have been constructed on  $M(T)$ , one of which is regular and the other is irregular. In the present note, we obtain a hypergroup structure whose measure algebra has regular multiplication. Some related results can be

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found in [8],[9],[10]. Also, basic facts about measure algebras on hypergroups can be found in [6], [7], [11].

The researcher will begin with the Arens multiplications and the hypergroup structures.

## 1- Other Versions of Arens Multiplications

In [1], Arens showed how the multiplication of a Banach algebra could be extended to a multiplication on the second dual. His method was essentially algebraic, and this is indeed the easiest way to prove that the construction works. However, we shall describe the results. See [12] for details.

Let  $A$  be a set with a multiplication  $(x, y) \rightarrow x.y$ . Let  $B$  be a set with  $A \subseteq B$ , under its topology,  $A$  is dense in  $B$ . For  $x, y$  in  $B$ , take  $(\tau_\alpha), (s_\beta)$  in  $A$  such that  $\lim_\alpha \tau_\alpha = x, \lim_\beta s_\beta = y$ . Then, the first extension of multiplication is given by

$$\tau_\alpha.y = \lim_\beta \tau_\alpha.s_\beta, \quad x.y = \lim_\alpha \lim_\beta \tau_\alpha.s_\beta,$$

while the second extension is given by

$$x \odot s_\beta = \lim_\alpha \tau_\alpha.s_\beta, \quad x \odot y = \lim_\alpha \tau_\alpha.s_\beta.$$

The set  $A$  is Arens regular (or, briefly, regular) if  $x.y = x \odot y$ . By the above definition for  $a$  and  $b$  in  $B$ , the product  $a.b$  (resp.  $a \odot b$ ) is continuous in the  $a$  (resp.  $b$ ) variable for each fixed  $b$  (resp.  $a$ ) in  $B$ . Generally,  $a.b$  (resp.  $a \odot b$ ) is not continuous in  $b$  (resp.  $a$ ) when  $a \notin A$  (resp.  $b \notin A$ ). This suggests the first result about regularity, which is entirely elementary.

**Proposition 1.1.** Let  $A \subseteq B$  and let  $A$  be dense in  $B$ . Then.

- (i) if  $A$  is commutative then  $A$  is Arens regular if and only if  $B$  is commutative.
- (ii)  $A$  is Arens regular if and only if the multiplication in  $B$  is continuous in each variable (without any restriction on the other).
- (iii) Let  $B$  be compact. Then,  $A$  is Arens regular if and only if the multiplication in  $B$  separately sequentially continuous.

**Proof.** See [5] and [12].

**Regularity of Banach algebras 1.2.** Let  $A$  be a normed space over  $K$  ( $K=\mathbb{R}$  or  $K=\mathbb{C}$ ). The dual space  $A^*$ ; i.e.  $A^*$ , is the vector space  $\Lambda(A^*, K)$  equipped with the norm

$$\|f\| = \sup\{|f(x)| : x \in A, \|x\| \leq 1\}.$$

Thus,  $A^*$  is a Banach space. Let  $(A^*)^*$  be the dual space of  $A^*$ ,  $(A^*)^* = \Lambda(A^*, K)$ . Since  $A^*$  is itself a Banach space, it is susceptible to the same construct; i.e. one can form  $(A^*)^* = A^{**}$ ; this is also a Banach space, called the dual or bidual of  $A$ , and denoted by  $A^{**}$ . This can go on.

For each  $x \in A$ , the value of an element  $x^{**} \in A^{**}$  is defined by  $x^{**}(f) = f(x)$  for all  $f \in A^*$ . So  $x^{**}$  is linear on  $A^*$ , and  $\|x^{**}\| = \|x\|$ . Thus, the canonical embedding mapping  $x \rightarrow x^{**}$  preserves norms and an isometric from  $A$  into its second dual  $A^{**}$ . Therefore, we can regard  $A$  as a subspace of  $A^{**}$ .

Let  $\sigma(A^{**}, A^*)$  be the weak\*-topology on  $A^{**}$ . By [3],  $A$  is weak\*-dense in  $A^{**}$ . So, for  $F \in A^{**}$ ,  $G \in A^{**}$ , we can find two bounded nets  $(\mu_\alpha), (v_\beta)$  in  $A$  with  $F = \omega^* - \lim_\alpha \mu_\alpha$ ,  $G = \omega^* - \lim_\beta v_\beta$ . The topological extension of first and second Arens product are given by

$$FG = \omega^* - \lim_\alpha \omega^* - \lim_\beta \mu_\alpha v_\beta, \quad F \circ G = \omega^* - \lim_\beta \omega^* - \lim_\alpha \mu_\alpha v_\beta.$$

Thus, the order in which the limits are taken distinguishes between the extensions. Moreover, the first Arens product is characterized by the two properties:

- (i) for each  $G \in A^{**}$ , the map  $F \rightarrow FG$  is weak\*-continuous on  $A^{**}$ .
- (ii) For each  $\mu \in A$ , the map  $G \rightarrow \mu G$  is weak\*-continuous on  $A^{**}$ .

The second Arens product is defined similarly. Therefore, the second dual  $A^{**}$  of  $A$  can be given the Banach algebra structure by means of the first (or second) Arens product.

Now, we want to describe Arens products as an algebraic extension. Indeed, for  $F, G \in A^{**}$ ,  $f \in A^*$ , and  $\mu, v \in A$ , one can find  $FG, F \circ G$  successively as follows:

$$\begin{aligned} \langle FG, f \rangle &= \langle F, Gf \rangle, \langle Gf, \mu \rangle = \langle G, f\mu \rangle, \langle f\mu, v \rangle = \langle f, \mu v \rangle, \\ \langle F \circ G, f \rangle &= \langle G, f \circ F \rangle, \langle f \circ F, \mu \rangle = \langle F, \mu \circ f \rangle, \langle \mu \circ f, v \rangle = \langle f, v\mu \rangle. \end{aligned}$$

So, a Banach algebra is said to have regular multiplication if  $FG = F \circ G$ .

$A^{**}$  is not compact in the weak\*-topology. But the closed unit ball of  $A^{**}$  is weak-compact [3]. So, by definition or [5], we have:

**Proposition 1.3.** Let  $A$  be commutative.  $A$  is Arens regular if and only if the first or second Arens product in  $A^{**}$  is weak\*-continuous in each variable.

**Proof.** For  $F, G \in A^{**}$ , There are two nets  $(\mu_\alpha)$  and  $(\nu_\beta)$  in  $A$  which weak\*-converge to  $F$  and  $G$ . So,  $A$  is Arens regular if and only if  $FG = F \circ G$ . It is equivalent to this fact; for all  $f \in A^*$ ,

$$\lim_{\alpha} \lim_{\beta} f(\mu_\alpha \nu_\beta) = \lim_{\beta} \lim_{\alpha} f(\mu_\alpha \nu_\beta).$$

## 2- Hypergroup Structures

Let  $X$  be a locally compact Hausdorff space and  $M(X)$  denotes the set of all bounded, regular, complex Borel measures on  $X$ . For each  $\mu$  and  $\nu$  in  $M(X)$ ,  $\mu * \nu$  denotes the convolution of  $\mu$  and  $\nu$ . Let  $\delta_r$  be the unit mass at  $r$ . The product formulas of the type.

$$\mu * \nu(f) = \int_x \int_x (\delta_x * \delta_y)(f) d\mu(x) d\nu(y)$$

On  $M(X)$  becomes a Banach algebra. Dunkl (1972) and Jewett. (1975) have shown how one defines a product on  $M(X)$ , which makes it a Banach algebra. In some cases, an investigation begins with a convolution algebra of measures as the primitive object, upon which to build a theory; this is the case of the analysis of the objects called hypergroups which are generalizations of the convolution algebra of Borel measures on a group. One of the objects of this paper will be the introduction of a large class of new convolution structures, many of which are new hypergroups.

Let  $C_b(X)$ ,  $C_0(X)$  and  $C_c(X)$  denote the spaces of continuous functions on  $X$  which are bounded, those which vanish at infinity and those having compact support respectively. By  $M(X)$  and  $M_p(X)$ , we abbreviate the space of Radon measures and probability measures on  $X$ .

**Definition 2.1.** A hypergroup  $(X, *)$  is a Banach algebra of the Borel measures  $M(X)$  on a locally compact Hausdorff space  $X$  with product  $*$  called convolution it satisfies the following axioms:

- (i) There is a map  $\lambda: X \times X \rightarrow M_p(X)$  with for every  $x, y \in X$ , the measures  $\lambda_{(x,y)}$  have compact supports and  $\lambda_{(x,y)} = \lambda_{(y,x)}$ ;

- (ii) for each  $f \in C_c(X)$ , the map  $(x, y) \rightarrow \lambda_{(x,y)}(\int)$  is in  $C_b(X \times X)$  and  $r \rightarrow \lambda_{(x,y)}(\int)$  is in  $C_c(X)$ , for every  $y \in X$ ;
- (iii) the convolution  $(\mu, \nu) \rightarrow \mu * \nu$  (or  $\mu \nu$ ) of measures defined by
- $$\mu * \nu(f) = \int_X \int_X \lambda_{(x,y)}(f) d\mu(x) d\nu(y), \quad (\mu, \nu \in M(X), \int \in C_o(X))$$
- is associative (and clearly  $\lambda_{(x,y)} = \delta_x * \delta_y$ )
- (iv) there is a unique  $e \in X$  such that  $\lambda_{(x,y)} = \delta_x$  for all  $x \in X$ .

In [7], Ghahramani and Medghalchi have constructed and studied the subalgebra of  $M(X)$  which is determined in the following way:

$$L(X) = \{\mu \in M(X) : x \rightarrow |\mu| * \delta_x, x \rightarrow \delta_x * |\mu| \text{ are norm-continuous}\}.$$

This algebra generalizes the algebra  $L(G)$  of  $M(G)$  for the locally compact topological groups  $G$ . They have shown that  $L(X)$  is a Banach subalgebra of  $M(X)$  and it has a bounded approximate identity of norm 1. Therefore,  $L(X)$  can be regarded as a subspace of  $L(X)^{**}$  and then  $L(X)$  is weak\*-dense in  $L(X)^{**}$  in [11], Medghalchi studied the second dual of  $L(X)$ .

Let  $A, B, C$ , be disjoint. And  $e, z$  be single points not in  $A \cup B \cup C$ . Write  $X = \{e\} \cup A \cup B \cup C \cup \{z\}$ . Let  $X$  be a compact Hausdorff space with  $e$  and  $z$  as isolated points. Each  $\mu \in M(X)$  can be written in the unique form

$$\mu_e \delta_e + \mu_B + \mu_C + \mu_z \delta_z$$

where,  $\mu_A, \mu_B, \mu_C$  are the restriction of  $\mu$  to  $A, B, C$ , respectively, and  $\mu_e$  and  $\mu_z$  are scalars.

The following Theorem gives the structure of a hypergroup on the locally compact space  $X$ .

**Theorem 2.2.** Let  $\lambda : A \times B \rightarrow (M_y(C), \text{weak}^*)$  be continuous and the map  $x \rightarrow \lambda_{(x,y)}(f)$  is in  $C_c(C)$  for every  $x, y \in A \cup B, f \in C(C)$  and  $\lambda(a,b) = \lambda(b,a)$  ( $a \in A, b \in B$ ). There is a hypergroup structure on  $X$  such that  $M(X - \{e\}) = L(X)$  if and only if  $M(A \cup B) \subseteq L(X)$ .

**Proof.** We define a multiplication on  $M(X)$  by the following way:

take a map  $\tilde{\lambda} : X \times X \rightarrow M_p(X)$  with two properties.

- (i)  $\tilde{\lambda}_{(x,e)} = \tilde{\lambda}_{(e,x)} = \delta_x$  ( $x \in X$ );

$$(ii) \quad \tilde{\lambda}_{(x,y)} = \begin{cases} \lambda_{(x,y)} & (x \in A, y \in B) \\ \delta_z & \text{otherwise} \end{cases}$$

it is clear that  $\|\tilde{\lambda}_{(x,y)}\| = 1$  for all  $x, y \in X$ .

Then, we define a multiplication  $\mu\nu$ , for  $\mu, \nu \in M(X)$ , by

$$\mu\nu = \int_X \int_X \tilde{\lambda}_{(x,y)} d\mu(x) d\nu(y)$$

It is easy to show that the multiplications is commutative and is the identity. Now, we prove that it is associative.

If one of the  $x, y, z$ , (in  $X$ ) is equal to  $e$ , then

$$(\delta_x * \delta_y) * \delta_z = \delta_x * (\delta_y * \delta_z)$$

Now, suppose that  $\mu, \nu, \xi$  are in  $M(X - \{e\})$ . Therefore,

$$\begin{aligned} \mu &= \mu_A + \mu_B + \mu_C + \mu_z \delta_z, \\ \nu &= \nu_A + \nu_B + \nu_C + \nu_z \delta_z, \\ \xi &= \xi_A + \xi_B + \xi_C + \xi_z \delta_z. \end{aligned}$$

Suppose  $Y = A \cup C \cup \{z\}$ . We have

$$\begin{aligned} \mu_A \nu_Y &= \int_X \int_X \tilde{\lambda}(x, y) d\mu_A(x) d\nu_Y(y) \\ &= \int_X \int_X \delta_z d\mu_A(x) d\nu_Y(y) \\ &= \mu_A(I) \nu_Y(I) \delta_z. \end{aligned}$$

Similarly  $\mu_Y \nu_A = \mu_Y(I) \nu_A(I) \delta_z$  and if  $T = B \cup C \cup \{z\}$  then

$$\mu_T \nu_B = \mu_T(I) \nu_B(I) \delta_z, \mu_B \nu_T = \mu_B(I) \nu_T(I) \delta_z.$$

Let “\*” be the convolution arising from the multiplication in  $A$  and  $B$  (or in  $B$  and  $A$ ) i.e.,

$$\begin{aligned} \mu_A * \nu_B &= \mu_A \nu_B = \int_X \int_X \tilde{\lambda}(x, y) d\mu_A(y) d\nu_B(x), \\ \mu_A * \nu_B(f) &= \int_A \int_B \int_C f(t) d\lambda_{(x,y)}(t) d\nu_B(x) d\mu_A(y). \end{aligned}$$

Hence,  $\text{supp}(\mu_A * \nu_B) \subseteq C$ . Therefore, we have

$$\begin{aligned} \mu\nu &= (\mu_A + \mu_B + \mu_C + \mu_z \delta_z)(\nu_A + \nu_B + \nu_C + \nu_z \delta_z) \\ &= \mu_A \nu_Y + \mu_A * \nu_B + \mu_B \nu_T + \mu_B * \nu_A + \mu_z \delta_z \nu \\ &= \mu_A(I) \nu_Y(I) \delta_z + \mu_B(I) \nu_T(I) \delta_z + \mu_z \delta_z(I) \nu(I) \delta_z + \mu_A * \nu_B + \mu_B * \nu_A \\ &= [\mu(I) \nu(I) - \mu_A(I) \nu_B(I) - \mu_B(I) \nu_A(I)] \delta_z + \mu_A * \nu_B + \mu_B * \nu_A \end{aligned}$$

On the other hand,

$$\begin{aligned}\mu_A * \nu_B(1) &= \int_A \int_B \int_C d\lambda_{(x,y)}(t) d\nu_B(x) d\mu_A(y). \\ &= \int_A \int_B d\nu_B(x) d\mu_A(y) = \mu_A(1) \nu_B(1).\end{aligned}$$

Hence,

$$\begin{aligned}(\mu \nu) \psi &= [\mu(1) \nu(1) - \mu_A(1) \nu_B(1) - \mu_B(1) \nu_A(1)] \delta_z \psi + (\mu_A * \nu_A + \mu_B * \nu_B) \psi \\ &= \mu(1) \nu(1) \psi(1) \delta_z.\end{aligned}$$

Similarly  $\mu(\nu \psi) = \mu(1) \nu(1) \psi(1) \delta_z$ . So that, the multiplication is associative

Now, let  $f \in C_c(X)$ , then we have

$$\begin{aligned}\int_X f(t) d\tilde{\lambda}_{(x,e)}(t) &= f(x), \\ \int_X f(t) d\tilde{\lambda}_{(x,y)}(t) &= \begin{cases} d\lambda_{(x,y)}(f) & (x \in A, y \in B) \\ f(z) & \text{otherwise.} \end{cases}\end{aligned}$$

Then the map  $(x, y) \rightarrow \tilde{\lambda}_{(x,e)}(f)$  is in  $C_b(X \times X)$  and the map  $x \rightarrow \tilde{\lambda}_{(x,y)}(f)$  is in  $C_c(X)$  for all  $y \in X$ . Therefore,  $X$  is a hypergroup.

We now prove that  $L(X) = M(X - \{e\})$ . Since  $X$  is not discrete,  $\delta_e \notin L(X)$  (Theorem 2, [9]). Let  $\mu \in M(X - \{e\})$ . Then

$$\delta_x |\mu| = \begin{cases} |\mu| & (x = e) \\ |\mu|(1) \delta_z & (x = z). \end{cases}$$

Now, if  $x \in X - \{e, z\}$  then,

$$\delta_x |\mu| = \begin{cases} |\mu|(1) - |\mu_B|(1) \delta_z + \delta_x * |\mu_B| & (x = A) \\ |\mu|(1) - |\mu_A|(1) \delta_z + \delta_x * |\mu_A| & (x = B). \end{cases}$$

Hence, for  $x=y=z$  or  $x=y=e$ ,

$$\|\delta_x |\mu| - \delta_y |\mu|\| = 0$$

Otherwise,

$$\|\delta_x |\mu| - \delta_y |\mu|\| = \begin{cases} \|\delta_x |\mu_B| - \delta_y |\mu_B|\| & (x, y \in A) \\ \|\delta_x |\mu_A| - \delta_y |\mu_A|\| & (x, y \in B). \end{cases}$$

Therefore,  $\mu \in L(X)$  if and only if  $\mu_A \in L(X), \mu_B \in L(X)$ . This statement is equivalent to,  $\mu \in L(X)$  if and only if  $M(A \cup B) \subseteq L(X)$ . So, the conclusion holds.

Let  $M(X)$  be the space of bounded regular Borel measures. We shall say that  $M(X)$  has a general measure multiplication, if there exists a bilinear associative map

$\phi: M(X) \times M(X) \rightarrow M(X)$  such that

$$\phi(M_p(X) \times M_p(X)) \subseteq M_p(X).$$

Also, we shall say that  $\phi$  is Arens regular, if for every two nets  $(\mu_\alpha), (v_\beta)$  in  $M(X)$ ,

$$w^* - \lim_{\alpha} w^* - \lim_{\beta} \phi(\mu_\alpha, v_\beta) = w^* - \lim_{\beta} w^* - \lim_{\alpha} \phi(\mu_\alpha, v_\beta),$$

when, both exist (see introduction [10]).

**Theorem 2.3.** Let  $M(A \cup B) \subset L(X)$ . Then,  $L(X)$  is Arens regular if and only if there is a bilinear map  $\phi: M(A) \times M(B) \rightarrow M(C)$  which is Arens regular.

**Proof.** By theorem 1,  $L(X) = M(X - \{e\})$  and (by the lemma, 4, [9]),

$$L(X) = M(A) \oplus M(B) \oplus M(C) \oplus \square \delta_z,$$

$$L(X)^{**} = M(A)^{**} \oplus M(B)^{**} \oplus M(C)^{**} \oplus \square \delta_z.$$

So, each  $\mu \in L(X)$ ,  $F \in L(X)^{**}$ , can be written uniquely in the form

$$\mu = \mu_A + \mu_B + \mu_C + \mu_z \delta_z,$$

$$F = F_A + F_B + F_C + \mu_z \delta_z,$$

where,  $\mu_A$  is the restriction of  $\mu$  to  $A$  and  $F_A$  is the restriction of  $F$  to  $M(A)^*$  and so on.

Let  $\mu, v \in M(X)$ . We define  $\Phi: M(X) \times M(X) \rightarrow M(X)$  by

$$\Phi(\mu, v) = \mu v = \int_X \int_X \tilde{\lambda}(x, y) d\mu(x) dv^*(y).$$

If  $\mu, v \in M_p(X)$  then  $\phi(\mu, v) \in M_p(X)$ . Thus, the multiplication  $\Phi$  maps probability measures to probability measures.

First, suppose that  $L(X)$  is Arens regular. So, for every nets  $(\mu_\alpha) \subset M(A)$ ,  $(v_\beta) \subset M(B)$ , if

$$w^* - \lim_{\alpha} w^* - \lim_{\beta} \Phi(\mu_\alpha, v_\beta), w^* - \lim_{\beta} w^* - \lim_{\alpha} \Phi(\mu_\alpha, v_\beta)$$

exist, and then they are equal (by Theorem 1, [5]).

Conversely, let  $\Phi$  be Arens regular and  $F, G \in L(X)^{**}$ . There are two nets  $(\mu_\alpha)$  and  $(v_\beta)$  in  $L(X)$  whit

$$w^* - \lim_{\alpha} \mu_\alpha = F, w^* - \lim_{\beta} \mu_\beta = G,$$

$$\text{supp } \mu_\alpha \subseteq X - \{e\}, \text{supp } \mu_\beta \subseteq X - \{e\}$$

The multiplication  $\Phi$  is Arens regular. Therefore,

$$w^* - \lim_{\alpha} w^* - \lim_{\beta} \phi((\mu_A)_\alpha, (v_B)_\beta) = w^* - \lim_{\beta} w^* - \lim_{\alpha} \phi((\mu_A)_\alpha, (v_B)_\beta),$$

$$w^* - \lim_{\alpha} w^* - \lim_{\beta} \phi((v_B)_\alpha, (\mu_A)_\alpha) = w^* - \lim_{\beta} w^* - \lim_{\alpha} \phi((v_B)_\alpha, (\mu_A)_\alpha).$$



Combining the above equalities, we have

$$\begin{aligned} FG &= w^* - \lim_{\alpha} w^* - \lim_{\beta} \mu_{\alpha} \nu_{\beta} \\ &= w^* - \lim_{\alpha} w^* - \lim_{\beta} [(\mu_{\alpha}(1) \nu_{\beta}(1) - (\mu_A)_{\alpha}(1) (\nu_B)_{\beta}(1) - (\mu_B)_{\alpha}(1) (\nu_A)_{\alpha}(1)) \delta_z + \\ &\quad \phi((\mu_A)_{\alpha} (\nu_B)_{\beta}) + \phi((\mu_B)_{\alpha} (\nu_A)_{\beta})] = w^* - \lim_{\beta} w^* - \lim_{\alpha} \mu_{\alpha} \nu_{\beta} = GF. \end{aligned}$$

Thus,  $FG=GF$ . By (Propositon1, [5]),  $L(X)$  is Arens regular.

Now, let  $A$  and  $C$  be disjoint,  $X=\{e\} \cup A \cup C \cup \{z\}$  and  $e, z \notin A \cup C$ . With the topology of  $X$ ,  $A$  and  $C$  are compact subspaces of  $X$  and  $e, z$  are isolated points.

**Theorem2.4.** If  $\lambda: A \times A \rightarrow M_p(X)$  is weak\*-continuous and symmetric, then there exists a hypergroup structure on  $X$  so that

(i)  $M(A) \subseteq L(X)$  if and only if  $M(X-\{e\})=L(X)$ ;

(ii) Let  $M(A) \subseteq L(X)$ . Then there exists a bilinear associative map

$\Phi: M(A) \subseteq M(A) \rightarrow M(C)$  which maps probability measures to probability measures and  $L(X)$  is Arens regular if and only if  $\Phi$  is Arens regular.

**Proof.** Define  $\tilde{\lambda}: X \times X \rightarrow M_p(X)$  with the following equations:

$$\begin{aligned} \text{(i)} \quad \tilde{\lambda}_{(x,e)} &= \tilde{\lambda}_{(e,x)} = \delta_x; \\ \text{(ii)} \quad \tilde{\lambda}_{(x,y)} &= \begin{cases} \lambda^{(x,y)} & (x, y \in A) \\ \delta_z & (\text{otherwise}). \end{cases} \end{aligned}$$

Then, we define a convolution  $\mu\nu$  for  $\mu, \nu \in M(X)$  by

$$\mu\nu = \int_X \int_X \tilde{\lambda}_{(x,y)} d\mu(x) d\nu(y).$$

It is clear that, if  $\mu, \nu \in M(X-\{e\})$  and then

$$\mu\nu = (\mu(1)\nu(1) - \mu_A(1)\nu_A(1))\delta_z + \mu_A * \nu_A.$$

The rest of the proof is the same as the proof of last theorem.

Let  $G$  be an arbitrary locally compact Hausdorff group and  $\mu$  be a right invariant Haar measure on  $G$ . The space  $L^1(G)$  of integrable functions on  $G$ , with the convolution taken as product, is a Banach Algebra. P. Civin and B. Yood [4] have shown that; if  $G$  is commutative and infinite set then the Banach Algebra  $L^1(G)$  is not Arens regular. N.Young [14] has extended this result to non-commutative case. A.Ulger [13] has

presented a very simple proof of the Theorem, which says that the group algebra  $L^1(G)$  is Arens regular if and only if  $G$  is finite. In this paper we present a Theorem, which shows that the Young's result does not hold in a hypergroup. This theorem is an application of Theorem 2.2.

**Theorem 2.4.** There is a hypergroup algebra  $M(X)$  which has regular multiplication and  $L(X)$  is Arens regular but  $X$  is not finite.

**Construction.** Let  $A=[a_1, b_1]$ ,  $B=[a_2, b_2]$ ,  $C=\{a, b\}$  be subsets of an ordered set in the ordered topology and  $A, B, C$  are disjoint. Let  $X=\{e\} \cup A \cup B \cup C \cup \{z\}$ , with the topology in which  $e, a, b, z$  are isolated points and supposed  $\phi: X \rightarrow [0, 1]$  is a continuous function with  $\text{supp } \phi = X$ . Define  $\lambda: A \times B \rightarrow M_p(C)$  by

$$\lambda_{(x,y)} = \phi(x)\phi(y)\delta_a + (1-\phi(x)\phi(y))\delta_b.$$

So,  $\lambda_{(x,y)} = \lambda_{(y,x)}$ . If  $f \in C_o(X) \setminus C_c(X)$  then,

$$\lambda_{(x,y)}(f) = \int_X f(t) d\lambda_{(x,y)}(t) = \phi(x)\phi(y)f(a) + (1-\phi(x)\phi(y))f(b)$$

Hence, if  $\{(x_n, y_n)\}$  is sequence converge to  $(x, y)$  then

$$\lim_n \lambda_{(x_n, y_n)}(f) = \lambda_{(x,y)}(f).$$

Therefore,  $\lambda: A \times B \rightarrow (M_p(C), \text{Weak}^*)$  is continuous. By Theorem 2.2  $X$  is a hypergroup.

To prove  $M(A \cup B) \subseteq L(X)$ , let  $\mu \in M(A \cup B)$ . Then,  $\mu = \mu_A + \mu_B$ . Suppose that  $x \in B$ , then,

$$\begin{aligned} \delta_x |\mu_A| &= \int_X \int_X \lambda_{(u,v)} d\delta_x(v) d|\mu_A|(u) \\ &= \int_X \lambda_{(u,x)} d|\mu_A|(u), \end{aligned}$$

So, for  $x, y \in B$ ,

$$\begin{aligned} \|\delta_x |\mu_A| - \delta_y |\mu_A|\| &= \left\| \int_X \lambda_{(u,x)} - \lambda_{(u,y)} d|\mu_A|(u) \right\| \\ &= \|\phi(x) - \phi(y)\| \left\| \int_X \phi(u) d|\mu_A|(u) \delta_a - \int_X \phi(u) d|\mu_A|(u) \delta_b \right\| \\ &\leq 2\|\phi(x) - \phi(y)\| \|\mu_A\|(\phi), \end{aligned}$$

$\phi$  is continuous, so  $\mu_A \in L(X)$ . Similarly,  $\mu_B \in L(X)$ . Then  $L(X) = M(X \setminus \{e\})$ .

We now prove that  $\phi: M(A) \times M(B) \rightarrow M(C)$  is Arens regular. First suppose that  $\mu_A \in M(A)$ ,  $\nu_B \in M(B)$ . So,

$$\begin{aligned}\phi(\mu_A, \nu_B) &= \mu_A * \nu_B = \int_X \int_X \lambda_{(x,y)} d\mu_A(x) d\nu_B(y) \\ &= \int_X \int_X [\phi(x)\phi(y)\delta_a + (1-\phi(x)\phi(y)\delta_b)] d\mu_A(x) d\nu_B(y) \\ &= \mu_A(\phi)\nu_B(\phi)\delta_a + (\mu_A(1)\nu_B(1) - \mu_A(\phi)\nu_B(\phi))\delta_b.\end{aligned}$$

Now, suppose  $\{(\mu_A)_n\}, \{(\nu_B)_m\}$  are two sequences such that

$w^* - \lim_n w^* - \lim_m \phi((\mu_A)_n, (\nu_B)_m), w^* - \lim_m w^* - \lim_n \phi((\mu_A)_n, (\nu_B)_m)$  exist, then

$$\begin{aligned}w^* - \lim_n w^* - \lim_m \phi((\mu_A)_n, (\nu_B)_m) &= w^* - \lim_n w^* - \lim_m [(\mu_A)_n(\phi)(\nu_B)_m(\phi)\delta_a \\ &\quad + ((\mu_A)_n(I)(\phi)_m(1) - (\mu_A)_n(\phi)(\nu_B)_m(\phi)\delta_b)] \\ &= w^* - \lim_m w^* - \lim_n \phi((\mu_A)_n, (\nu_B)_m).\end{aligned}$$

Hence,  $\phi$  is Arens regular. By Theorem 2.3,  $L(X)$  is Arens regular, but  $X$  is not finite.

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